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Hadamard states for linearized gravity on asymptotically flat spacetimes

Tesi per la Laurea Magistrale di
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Abstract

Goal of this thesis is the construction of a Hadamard state for linearized gravity on asymptotically flat, globally hyperbolic spacetimes. To this end, first of all we characterize the symplectic space of classical solutions for the linearization of Einstein equations. Afterwards we construct via standard methods the associated algebra of fields fulfilling the canonical commutation relations. At this stage we employ the so-called bulk-to-boundary mechanism: We project each solution and hence each element of the algebra of observables into a suitable counterpart which is intrinsically defined on null infinity, the conformal boundary. The net advantage of such procedure is the following: Each state constructed for the algebra on the boundary yields via pull-back a bulk counterpart. In between all boundary states we can identify a distinguished choice by requiring invariance under the group of asymptotic symmetries. The bulk counterpart turns out to be of Hadamard form, invariant under all isometries and, thus, coincident with the Poincaré vacuum in Minkowski spacetime.

Abstract

L'obiettivo di questa tesi è la costruzione dello stato di Hadamard per la gravità linearizzata su spaziotempi asintoticamente piatti e globalmente iperbolici. Per questo fine, prima si caratterizza lo spazio simplettico delle soluzioni classiche dell'equazioni di Einstein. Successivamente tramite i metodi standard si realizza un'algebra dei campi che soddisfa le canoniche relazioni di commutazione. In questo modo si può ricorrere all'uso della corrispondenza bulk-bordo: si proietta ogni soluzione e quindi ogni elemento dell'algebra dell'osservabili in una adeguata controparte che è intrisicamente definita nell'infinito di tipo temporali, il bordo conforme. Il grande vantaggio di questa procedura è il seguente: ogni stato costruito per l'algebra del bordo produce via pull-back una controparte nel bulk. Tra tutti i possibili stati definiti sull'algebra del bordo si può identificarne uno richiedendo l'invarianza sotto il gruppo delle simmetrie asintotiche. La controparte nel bulk è quindi della forma di Hadamard, invariante sotto tutte le isometrie e, così, coincide con il vuoto di Poincaré nello spaziotempo di Minkowski.

≪Nevertheless,
due to the inneratomic movement of electrons,
atoms would have to radiate not only electromagnetic
but also gravitational energy,
if only in tiny amounts.
As this is hardly true in Nature,
it appears that quantum theory would have to modify
not only Maxwellian electrodynamics
but also the new theory of gravitation.≫

A. Einstein

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Introduction

A key ingredient is the usual approach to quantum field theory on Minkowski spacetime is the Poincaré group of isometries. The latter group is used in particular to select a distinguished state, called the vacuum, which is mostly notable for enjoying a uniqueness property [1]. However, even the slightest perturbation of the background spacetime can cause this picture to break down. Moreover, in a curved spacetime one may find specific effects which cannot be described by Minkowskian quantum field theory like the renown Hawking radiation [2] and the Unruh effect [3]. It is hence appropriate to ask ourselves if Minkowskian quantum field theory is a valid approximation even if quantum effects of gravity are negligible. Since we do not question the success of quantum field theory in general, as it confirms observations with an unprecedented accuracy [4], we strive to improve upon what quantum field theory has taught us in the past.
Following our argument according to which we should never neglect gravitational effects but might disregard quantum counterpart at least in a first approximation due to the weakness of the gravitational coupling, we can focus only on quantum fields living on a fixed classical background. In contrast to what a full-fledged theory of quantum gravity will need to accomplish, the background spacetime in quantum field theory on curved spacetime is fixed by hand or by need. Since this spacetime will, in general, not even have a timelike Killing fields, we cannot perform the standard construction to identify a global vacuum state [5]. Therefore, the notion of a quantum field has to be formulated without referring to a preferred state. This can be accomplished in the algebraic approach to quantum field theory in which one starts with an abstract algebra of local observables encoding the dynamics of the quantum field [1, 5].
Nevertheless, a state is still needed to obtain any concrete result which can then be understood in the usual probabilistic interpretation of quantum theories. Yet not all possible states have physically reasonable properties. For a free field theory one demands the Hadamard condition: It is the natural generalisation of the energy-positivity condition of the Minkowski vacuum state which encodes the UV properties of physical states in QFT [6, 7, 8]. It guarantees moreover the possibility to regularise quantum observables building a generalization of Wick polynomials, used to cope with interactions at a perturbative level. It was later found that Hadamard states can be characterized in terms of their wavefront set of their associated two-point functions [9, 10]. This discovery lead to an improved understanding

of Hadamard states and opened the doors for the development of a rigorous perturbation theory on curved spacetimes [11, 12, 13, 14]. Although the existence of Hadamard states was proven for various quantum fields on globally hyperbolic spacetimes using deformation arguments, an explicit construction is often notoriously difficult [8, 15, 16].

In [17] Dappiaggi, Moretti and Pinamonti suggested a construction which yields a boundary state on the conformal boundary of asymptotically flat spacetimes that is invariant under the action of the symmetry group of the boundary manifold, the Bondi-Metzner-Sachs group. Such state can be interpreted as an asymptotic vacuum as shown in [18]. Subsequently it was proven that pulling it back to the bulk one obtains indeed a Hadamard state. This construction, called the bulk to boundary correspondence, was initially performed for a conformally coupled massless scalar field and later also applied to the Dirac [19, 20] and the electromagnetic field [21].

In this thesis we construct a Hadamard state for a quantum field theory of linearized gravity, which models quantum fluctuations of the gravitational field at low energy. At a classical level linearized gravity perturbation is described by linearized Einstein equations. The class of asymptotically flat spacetimes is particularly interesting from the cosmological point of view, where linearized gravitons can induce fluctuations in the cosmic microwave background [22]. These arise in a discussion of inflation and tensor fluctuations. For these reasons, the quantum field theory of linearized gravity has been studied on several spacetimes [23, 24]. Though such topic has been previously considered by numerous authors, in this thesis we propose to employ algebraic quantum field theory. The first step consists of the construction of a classical phase space: One introduces a space of solutions of the linearized Einstein equations and endows it with a weakly non-degenerate symplectic product. This can be achieved for the case when the background spacetime admits a compact Cauchy surface. The resulting symplectic space is quantized using Dirac prescription, which permits the construction of an algebra of observables consisting of gauge-invariant smeared fields. It is shown that this algebra satisfies a time-slice condition. Before using the bulk to boundary correspondence to construct a Hadamard state it is shown how to associate to each solution of the linearized Einstein equations a field defined on the conformal boundary. In this sense, we have identified an injection between the space of solution in the bulk and the space of solution on the boundary. This translates to the existence of an injective $*$-homomorphism between the algebra of observables in the bulk and a suitable boundary counterparts. Hereupon we identify a distinguished state which, via pull back yields a counterpart in the bulk of Hadamard form.

An outline of the thesis in due course:

- **chapter I**: We describe the geometrical preliminaries necessary for a formulation quantum field theories on curved spacetimes: In particular Riemannian geometry of rather generic, Weyl manifolds, asymptotically flat spacetimes, distributions on manifolds and wave equations are discussed.
- **chapter II**: We lay down the foundation to the formulation of quantum theories on curved backgrounds following the algebraic approach. We introduce the algebra of observables and the Haag-Kaslter axioms, and, after a review on the category formalism, we discuss the local covariance of a theory. In the last part of chapter we introduce the Hadamard states and a method to construct them on asymptotically flat spacetime.
- **chapter III**: We start our study of linearized Einstein equations. After constructing an algebra of observables starting from the phase space associated to the solutions of linearized Einstein equations, we find a correspondence between these solutions and associated fields on the boundary. This allows us to apply the bulk to boundary correpondence and to construct a Hadamard state .

Chapter 1

Mathematical structures

Contents

In theoretical physics the mathematical representation of spacetime would presumably be based upon a suitable set $\mathcal{M}$ whose elements represent "spacetime points". Yet a set has no structure other than the elements it contains, and the question arises naturally, therefore, which other mathematical structures should be imposed in $\mathcal{M}$. In practice, the idea has arisen that $\mathcal{M}$ should also be a topological space whose open sets represent certain privileged regions in spacetime. The use of such familiar mathematical model has many important implications, not the least of which is that *differentiation* can be defined: This leads to the very fruitful idea that the dynamical evolution of a physical system can be modelled by differential equations defined on spacetime itself.

The topological spaces that have actually been used historically to represent spacetime (or space and time separately) are however of very special type: In Newtonian physics, the three-dimensional physical space is represented mathematically by the Euclidean $\mathbb{R}^3$, and one-dimensional time by $\mathbb{R}$; in special relativity, the combined notion of "spacetime" is represented by Minkowskian space $\mathbb{R}^4$.

One of Einstein major contributions to physics was his realisation that it is possible to generalise the mathematical model of spacetime: In General Relativity a spacetime is modelled by a "*differentiable manifold*", of which Minkowski space $\mathbb{R}^4$ is just a "special" example.

1.1 Riemannian Geometry

In this first section we introduce Riemannian Geometry, a powerful tool in the formulation of General Relativity and Quantum Field Theory on curved spacetimes. Our treatment is based on [25], [26], [27] and [28] but for more information it is possible to consult [29], [30] and [31].

1.1.1 Differentiable manifolds

A manifold is one of the most fundamental concepts in mathematics and physics. In fact it summarized the idea of a space which may be intrinsically curved and have a complicated topology, but in local regions looks just like $\mathbb{R}^n$. (What we mean with "just like" is specified in the definition below).

Definition 1.1.1. A *topology* on a set $\mathcal{X}$ is a collection τ of subsets of $\mathcal{X}$ such that:

(i) $\emptyset$ and $\mathcal{X}$ are in τ

(ii) Any union of elements of τ lies in τ

(iii) The intersection of any finite collection of elements in τ lies in τ

A set $\mathcal{X}$ for which a topology τ has been specified is called a *topological space*.

Properly speaking, a topological space is a pair $(\mathcal{X}$,$\tau)$ consisting of a set $\mathcal{X}$ and a topology τ on $\mathcal{X}$, but we often omit specific mention of τ unless any source of confusion will arise.
If $\mathcal{X}$ is a topological space with topology τ, we say that a subset $\mathcal{U}$ of $\mathcal{X}$ is an *open set* of $\mathcal{X}$ if $\mathcal{U}$ belongs to the collection τ. Using this terminology, one can say that a topological space is a set $\mathcal{X}$ together with a collection of subsets of $\mathcal{X}$, called open sets, such that $\emptyset$ and $\mathcal{X}$ are both open, and such that arbitrary unions and finite intersections of open sets are open.
The open sets of $\mathbb{R}$, defined by unions of open intervals $a < x < b$ and the null set $\emptyset$, satisfy the above definition. This topology is called the *standard topology* on $\mathbb{R}$.

Definition 1.1.2. A *differentiable manifold* of dimension n is a topological space $\mathcal{M}$ and a family of injective mappings $\mathbf{x}_\alpha : \mathcal{U}_\alpha \subset \mathcal{M} \to \mathbb{R}^n$ of open sets $\mathcal{U}_\alpha$ of $\mathcal{M}$ into $\mathbb{R}^n$ such that:

(i) $\bigcup_\alpha U_\alpha = \mathcal{M}$;

(ii) for any pair α, β with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta = \mathcal{W} \neq \emptyset$, the sets $\mathbf{x}_\alpha(\mathcal{W})$ and $\mathbf{x}_\beta(\mathcal{W})$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$ are differentiable; (Fig. 1.1)

(iii) The family $\{(\mathcal{U}_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions (i) and (ii).

The pair $(\mathcal{U}_\alpha, \mathbf{x}_\alpha)$ with $p \in \mathcal{U}_\alpha$ is called a *chart* (or *system of coordinates*) of $\mathcal{M}$ at p; $\mathcal{U}_\alpha$ is then called a *coordinate neighbourhood* at p. A family $\{(\mathcal{U}_\alpha, \mathbf{x}_\alpha)\}$ satisfying (i) and (ii) is called a *differentiable structure* on $\mathcal{M}$.

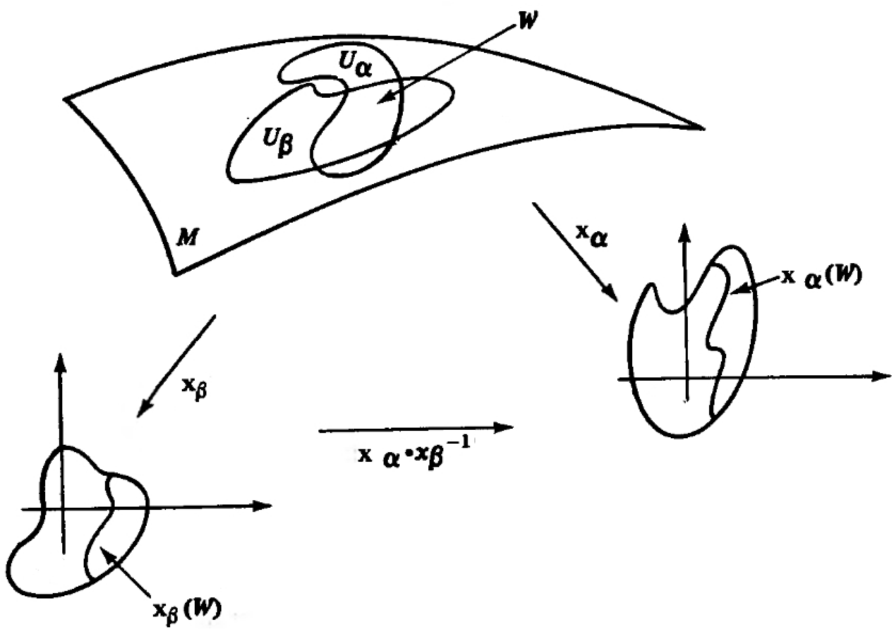

Figure 1.1

Example 1.1. The Euclidean space $\mathbb{R}^n$, with the differentiable structure given by the identity, is a trivial example of a differentiable manifold.

Example 1.2. Consider the space which consists of two closed half lines $(-\infty, O]$ and one open half line (O, ∞) together with a basis of open neighbourhoods O_i, $i = 1, 2$ (Fig. 1.2) of the form $(-a_i, O_i] \cup (O, b)$. This space is locally euclidean; it is not Hausdorff because any two open sets containing O_1 e O_2 will contain a segment (O, b).

In order to avoid a potential pathology like the presence of a spacetime with "two origins" we shall require that the following property is always fulfilled:

Definition 1.1.3. A topological manifold is *Hausdorff (separated)* if any two distinct points lies in disjoint neighbourhoods.

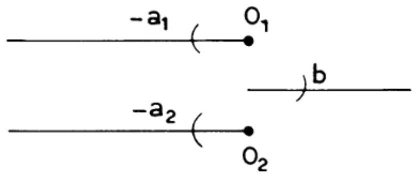

Figure 1.2

The differentiability of a function on a vector space can be defined with the help of charts it is possible to extend the concept of differentiability of a function to a manifold. Consider a function f on $\mathcal{M}$,

$$\begin{aligned} f: \quad & \mathcal{M} \to \mathbb{R} \\ & p \mapsto f(p). \end{aligned}$$

Let $(\mathcal{U}_\alpha, \mathbf{x}_\alpha)$ be a chart at $p \in \mathcal{M}$, then $f \circ \ \mathbf{x}_\alpha^{-\mathbf{1}}$ is a mapping from $\mathcal{U}_\alpha$ into $\mathbb{R}$.

Definition 1.1.4. The function f is *differentiable* at p on a differentiable manifold $\mathcal{M}$ if, in a chart at p, $f \circ \mathbf{x}_\alpha^{-\mathbf{1}}$ is differentiable at p.

The definition does not depend on the chart: if $f \circ \ \mathbf{x}_\alpha^{-\mathbf{1}}$ is differentiable at p for a chart $(\mathcal{U}_\alpha, \mathbf{x}_\alpha)$ at p, then $f \circ \ \mathbf{x}_\beta^{-\mathbf{1}}$ is differentiable at p for every chart $(\mathcal{U}_\beta, \mathbf{x}_\beta)$ at p because

$$f \circ \ \mathbf{x}_\beta^{-\mathbf{1}} = (f \circ \ \mathbf{x}_\alpha^{-\mathbf{1}}) \circ (\mathbf{x}_\alpha \circ \mathbf{x}_\beta^{-\mathbf{1}});$$

the proposition results from the differentiability of a composite mapping. From now we omit specific mention of α if non confusion will arise.
Let us extend the idea of differentiability to mappings between manifolds.

Definition 1.1.5. Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. A mapping $\varphi : \mathcal{M} \to \mathcal{N}$ is *differentiable at* $p \in \mathcal{M}$ if given a chart $\mathbf{y} : \mathcal{V} \subset \mathbb{R}^n \to \mathcal{N}$ at $\varphi(p)$ there exists a chart $\mathbf{x} : \mathcal{U} \subset \mathbb{R}^m \to \mathcal{M}$ at p such that $\varphi(\mathbf{x}(\mathcal{U})) \subset \mathbf{y}(\mathcal{V})$ and the mapping $\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : \mathcal{U} \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{x}^{-1}(p)$. (Fig. 1.3)

Definition 1.1.6. Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. A differentiable mapping $\varphi : \mathcal{M} \to \mathcal{N}$ is said to be an *immersion* if $d\varphi_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$ is injective for all $p \in \mathcal{M}$. If, in addition, φ is a homeomorphism onto $\varphi(\mathcal{M}) \subset \mathcal{N}$, where $\varphi(\mathcal{M})$ has the subspace topology induced from $\mathcal{N}$, we say that φ is an *embedding*. If $\mathcal{M} \subset \mathcal{N}$ and the inclusion $i : \mathcal{M} \hookrightarrow \mathcal{N}$ is an embedding, we say that $\mathcal{M}$ is a *submanifold* of $\mathcal{N}$.

Example 1.3. The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|)$ is not a differentiable map at t=0.

Example 1.4. The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3, t^2)$ is a differentiable map but it is not an immersion. Indeed, the condition for the map to be an immersion

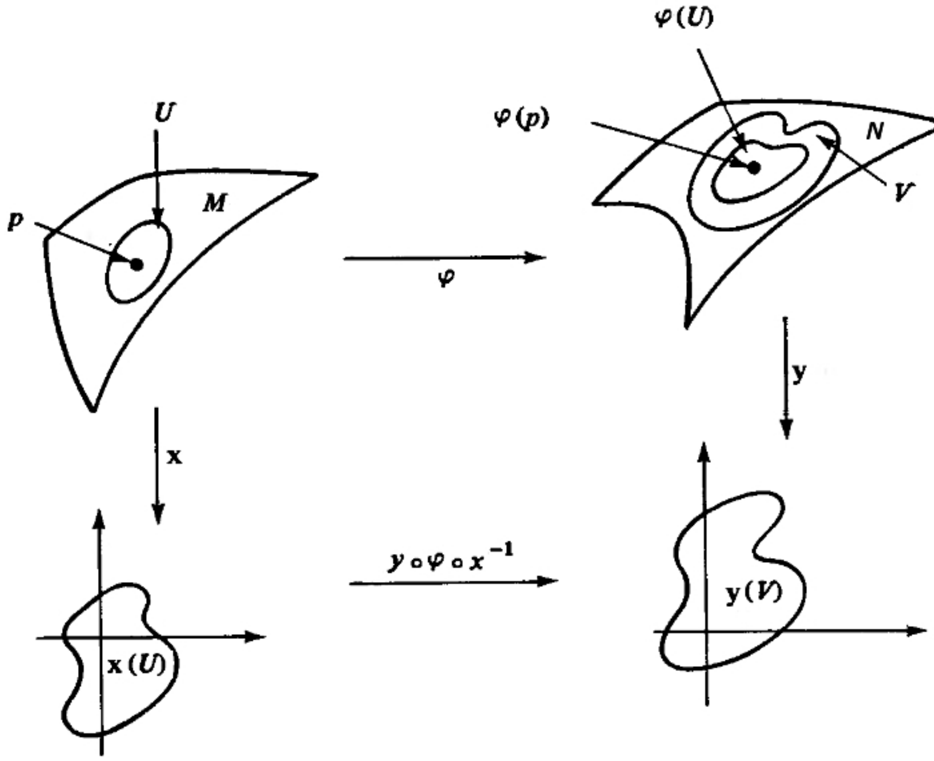

Figure 1.3

in this case is equivalent to the requirement that $\dot{\alpha}(t) \neq 0$, which does not hold true at $t = 0$.

Example 1.5. The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3 - 4t, t^2 - 4)$ is an immersion which has a self-intersection for $t = 2$ and $t = -2$. Therefore, α is not an embedding.

Example 1.6. The curve $\alpha : (-3, 0) \to \mathbb{R}^2$ given by:

$$\alpha(t) : \begin{cases} (0, -(t+2)) & \text{se } t \in (-3, -1), \\ \text{regular curve (see Fig. 1.4)} & \text{se } t \in (-1, -\frac{1}{\pi}), \\ (-t, -\sin\frac{1}{t}) & \text{se } t \in (-\frac{1}{\pi}, 0). \end{cases}$$

is an immersion without self-interaction. Nevertheless, α is not an embedding. Indeed, a neighbourhood of a point p, in the vertical part of the curve (Fig. 1.4) consists of an infinite number of connected components in the topology induced from $\mathbb{R}^2$. On the other hand, a neighbourhood of such a point in the topology "induced" from α (that is the topology of the line) is an open interval, hence a

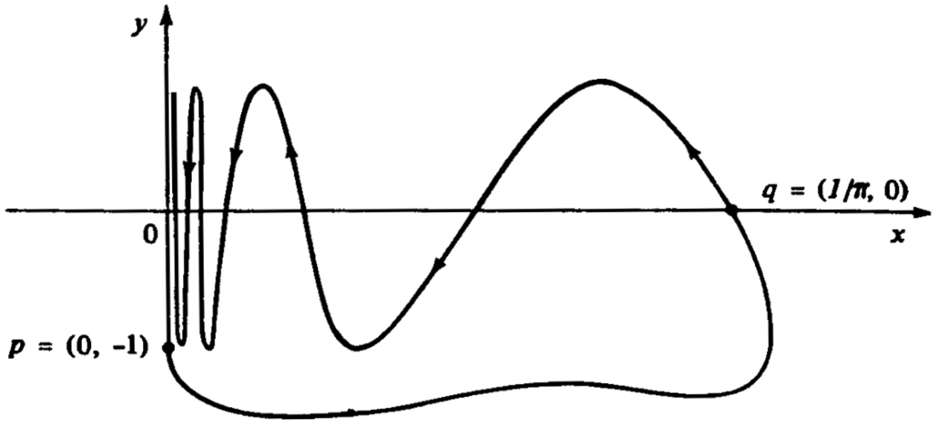

Figure 1.4

connected set.

Example 1.7. It is clear that a regular surface $S \subset \mathbb{R}^3$ has a differentiable structure given by it chart $\mathbf{x} : \mathcal{U} \to S$. With such a structure, the maps $\mathbf{x}_\alpha$ are differentiable and, indeed, embeddings of $\mathcal{U}$ into S; thus is a consequence of the definition of regular surface in $\mathbb{R}^3$.

1.1.2 Tangent and cotangent space

We would like to extend the idea of tangent vector to differentiable manifolds. It is convenient to use our experience with regular surfaces S in $\mathbb{R}^3$. A tangent vector at a point $p \in S$ is defined as the "velocity" in $\mathbb{R}^3$ of a curve in the surface passing through p. Since we do not have at our disposal the support of the ambient space, we will have to find a characteristic property of the tangent vector which will replace the idea of velocity. The next considerations will motivate the definition that we are going to present below.
Let $\sigma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$, with $\sigma(0) = p$. Write

$$\sigma(t) = (x^1(t), \ldots, x^n(t)), \quad t \in (-\varepsilon, \varepsilon), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then

$$\dot{\sigma}(t) = (\dot{x}^1(0), \ldots, \dot{x}^n(0)) = X \in \mathbb{R}^n.$$

Now let f be a differentiable function defined in a neighbourhood of p. We can restrict f to the curve σ and express the directional derivative with respect to the vector $X \in \mathbb{R}^n$ as

$$\left.\frac{d(f \circ \sigma)}{dt}\right|_{t=0} = \sum_{\mu=1}^{n} \left.\frac{\partial f}{\partial x^\mu}\right|_{t=0} \left.\frac{dx^\mu}{dt}\right|_{t=0} = \left(\sum_{\mu=1}^{n} \dot{x}^\mu(0) \frac{\partial}{\partial x^\mu} \right) f.$$

Therefore, the directional derivative with respect to the vector X is an operator on differentiable functions that depends uniquely on X. This is the characteristic property that we are going to use to define tangent vectors on a manifold.

Definition 1.1.7. Let $\mathcal{M}$ be a differentiable manifold and $\sigma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ a differentiable *curve*. Suppose that $\alpha(0) = p \in \mathcal{M}$, and let $\mathcal{D}$ be the set of functions on $\mathcal{M}$ that are differentiable at p. The *tangent vector to the curve* σ at $t = 0$ is the function $\dot{\sigma}(0) : \mathcal{D} \to \mathbb{R}$ given by

$$\dot{\sigma}(0)f = \left.\frac{d(f \circ \sigma)}{dt}\right|_{t=0}, \quad f \in \mathcal{D}.$$

A *tangent vector* at p is the tangent vector at $t = 0$ of a curve $\sigma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $\sigma(0) = p$. The set of all tangent vectors to $\mathcal{M}$ at p will be indicated by $T_p\mathcal{M}$.

If we choose a local chart $\mathbf{x} : \mathcal{U} \subset \mathcal{M} \to \mathbb{R}^n$ at p, we can express the function f and the curve σ in this parametrization as

$$f \circ \mathbf{x}^{-1}(q), \quad q = (x_1, \ldots, x_n) \in \mathcal{U}$$

and

$$\mathbf{x} \circ \sigma(t) = (x^1(t), \ldots, x^n(t)),$$

respectively. Therefore, restricting f to σ, we obtain

$$\begin{aligned}\dot{\sigma}(0)f &= \left.\frac{d}{dt}(f \circ \sigma)\right|_{t=0} = \left.\frac{d}{dt} f(x^1(t), \ldots, x^n(t))\right|_{t=0} = \\ &= \sum_{\mu=1}^{n} \dot{x}^\mu(0) \left(\frac{\partial f}{\partial x^\mu}\right) = \left(\sum_{\mu=1}^{n} \dot{x}^\mu(0) \left(\frac{\partial}{\partial x^\mu}\right)\right) f.\end{aligned}$$

In other words, the vector $\dot{\sigma}(0)$ can be expressed in the chart $\mathbf{x}$ by

$$\dot{\sigma}(0) = \sum_{\mu=1}^{n} \dot{x}^\mu(0) \left(\frac{\partial}{\partial x^\mu}\right).$$

Observe that $\frac{\partial}{\partial x^\mu}$ is the tangent vector at p of the "μ-th coordinate curve" (Fig. 1.5) To avoid confusion, from now we employ *Einstein summation convention*: if the same index appears twice, once as a superscript and once as a subscript, then the index is summed over all possible values. For example, if μ runs from 1 to m, we have

$$A^\mu B_\mu = \sum_{\mu=1}^{m} A^\mu B_\mu.$$

Definition 1.1.8. A *vector field* X on a differentiable manifold $\mathcal{M}$ is a correspondence that associates to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p\mathcal{M}$. Considering a

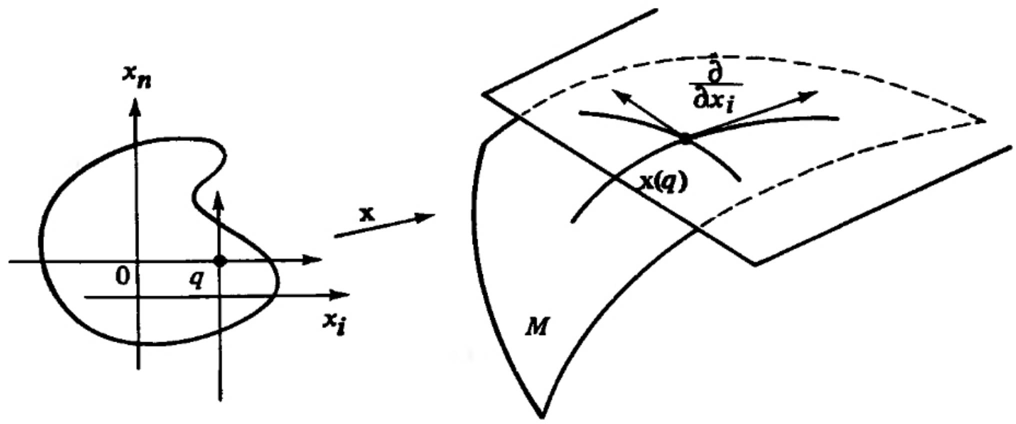

Figure 1.5

chart $\mathbf{x} : \mathcal{U} \subset \mathcal{M} \to \mathbb{R}^n$ we can define a differentiable vector field as:

$$\begin{aligned} X: \quad & \mathcal{M} \to T_p\mathcal{M} \\ & p \mapsto X(p) \doteq x^\mu \partial_\mu \end{aligned}$$

where each $x^\mu : \mathcal{U} \to \mathbb{R}$ is function on $\mathcal{U}$ and $\{\partial_\mu\} \doteq \{\frac{\partial}{\partial x^\mu}\}$ is the basis associated to $\mathbf{x}(\mathcal{U})$, $\mu = 1, \ldots, dim\mathcal{M}$.

This is not the unique definition that can be given.

Definition 1.1.9. For a generic manifold $\mathcal{M}$, a *tangent vector* at a point $p \in \mathcal{M}$ is a linear function $X : \mathcal{D}(\mathcal{M}) \to \mathbb{R}$ which satisfies the following two properties:

(i) $X(af + bg) = aX(f) + bX(g)$ for all $f, g \in \mathcal{D}(\mathcal{M})$ and for all $a, b \in \mathbb{R}$

(ii) $X(fg) = f(p)X(g) + X(f)g(p)$ for all $f, g \in \mathcal{D}(\mathcal{M})$.

An important consequence of this definition of tangent space is the following.

Proposition 1.1.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds and $\varphi \in C^\infty(\mathcal{M}, \mathcal{N})$. Then φ induces a natural homomorphism $\varphi_* : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$, called push-forward, such that*

$$(\varphi_* X)f \doteq X(f \circ \varphi), \qquad \forall f \in \mathcal{D}(\mathcal{N}) \quad \textit{and} \quad \forall X(p) \in T_p\mathcal{M}$$

Proof. We need to verify that $\varphi_* X$ is indeed a vector at $\varphi(p)$ according to Definition 1.1.9. Let us start with linearity, the simplest condition. Let $f, g \in \mathcal{D}(\mathcal{N})$ and let $a, b \in \mathbb{R}$, then

$$\begin{aligned} (\varphi_* X)(af + bg) &= X[(af + bg) \circ \varphi] = X[af \circ \varphi + bg \circ \varphi] = \\ &= aX(f \circ \varphi) + bX(g \circ \varphi) = a(\varphi_* f)X + b(\varphi_* g)X \end{aligned}$$

where, in the various equalities, we exploited only the definition of φ_* and the linearity property of X.
To show the Leibniz rule, let us consider $f, g \in \mathcal{D}(\mathcal{N})$, then

$$\begin{aligned}(\varphi_* X)(fg) &= X(f \circ \varphi g \circ \varphi) = \\ &= (f \circ \varphi)(p) X(g \circ \varphi)(p) + X(f \circ \varphi)(p)(g \circ \varphi)(p) = \\ &= f(\varphi(p))(\varphi_* X)(g) + (\varphi_* X)(f) g(\varphi(p))\end{aligned}$$

where, besides the definition of φ_*, we used that the product fg is meant pointwisely: $(fg)(x) = f(x)g(x)$ for all $x \in \mathcal{N}$. □

Since $T_p\mathcal{M}$ is a vector space, there exists a dual vector space to $T_p\mathcal{M}$, whose elements are the linear functions from $T_p\mathcal{M}$ to $\mathbb{R}$. The dual space is called *cotangent space* at p, denoted by $T_p^*\mathcal{M}$. An element $w : T_p\mathcal{M} \to \mathbb{R}$ of $T_p^*\mathcal{M}$ is called a *dual vector* or *cotangent vector* or, in the context of differential forms, a *one-form*. The simplest example of a one-form is the differential df of a function $f \in \mathcal{D}(\mathcal{M})$. The action of a vector X on f is $X(f) = x^\mu \partial_\mu f \in \mathbb{R}$. The action of $df \in T_p^*\mathcal{M}$ on $X \in T_p\mathcal{M}$ is defined by

$$\langle df, X \rangle \doteq X(f) = x^\mu \partial_\mu f \in \mathbb{R}.$$

$\langle df, X \rangle$ is $\mathbb{R}$-linear in both X and f.
Noting that df is expressed in terms of the coordinates at $x = \mathbf{x}(p)$ as $df = (\partial_\mu f) dx^\mu$, it is natural to regard $\{dx^\mu\}$ as a basis of $T_p^*\mathcal{M}$. This is a dual basis, since

$$\langle dx^\mu, \partial_\nu \rangle = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu.$$

Taken an arbitrary one-form w and a vector X, written respectively as $w = w_\mu dx^\mu$ and $X = x^\mu \partial_\mu$, it's possible to define the *inner product*

$$\begin{aligned}\langle\ ,\ \rangle :\quad & T_p^*\mathcal{M} \otimes T_p\mathcal{M} \to \mathbb{R} \\ & w, X \mapsto \langle w, X \rangle = w^\mu x_\nu \delta^\mu_\nu.\end{aligned}$$

The definition of the cotangent space has an important consequence. A map $\varphi : \mathcal{M} \to \mathcal{N}$ induces a map:

$$\varphi^* : T^*_{\varphi(p)}\mathcal{N} \to T_p^*\mathcal{M}.$$

Note that φ_* goes in the same direction as φ, while φ^* goes backward, hence the name *pull-back*. If we take $X \in T_p\mathcal{M}$ and $w \in T^*_{\varphi(p)}\mathcal{N}$, the pull-back of w by φ^* is defined as

$$\langle \varphi^* w, X \rangle \doteq \langle w, \varphi_* X \rangle.$$

An important special case of the pull-back on a one-form w on a manifold $\mathcal{M}$ occurs in the context of a vector field X with an associated one-parameter group of local diffeomorphism (flow). Let $X, Y \in \mathfrak{X}(\mathcal{M})$ and $\sigma(s, x)$ be a flow generated by the vector field X

$$\frac{d\sigma^\mu(s,p)}{ds} = X^\mu(\sigma(s,p)).$$

Let us evaluate the change of the vector field Y along $\sigma(s,p)$. To do this, we have to compare the vector Y at a point $p \in \mathcal{M}$ with that at a nearby point $q = \sigma_\varepsilon(p)$. We cannot however simply take the difference between the components of Y at two points since they belong to different tangent spaces $T_p\mathcal{M}$ and $T_q\mathcal{M}$; the naive difference between vectors at different points is ill-defined. To define a sensible derivative, we first map $Y|_{\sigma_\varepsilon(p)}$ to $T_p\mathcal{M}$ by $(\sigma_{-\varepsilon})_* : T_{\sigma_\varepsilon(p)}\mathcal{M} \to T_p\mathcal{M}$, after which we take a difference between two vectors $(\sigma_{-\varepsilon})_* \, Y|_{\sigma_\varepsilon(p)}$ and $Y|_p$ both of which lie in $T_p\mathcal{M}$.

Definition 1.1.11. The *Lie derivative* of a field Y along the flow σ of X is defined as

$$\mathcal{L}_X Y \doteq \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* \, Y|_{\sigma_\varepsilon(p)} - Y|_p].$$

Let $(\mathcal{U}, \mathbf{x})$ be a chart and let $X = x^\mu\partial_\mu$ and $Y = y^\mu\partial_\mu$ we find

$$\mathcal{L}_X Y = (x^\mu\partial_\mu y^\nu - y^\mu\partial_\mu x^\nu)\partial_\nu.$$

Geometrically the Lie derivative is a measure of the non-commutativity of the two flows.

In order to proceed from tangent and cotangent vectors to general tensors and n-forms it was necessary to introduce the idea of the tensor product $T_p^*\mathcal{M} \otimes T_p\mathcal{M}$. Applying this idea at the elements of the tangent and the cotangent space leads to the following definition.

Definition 1.1.12. A *tensor* of type (r,s) at a point $p \in \mathcal{M}$ is an element of the tensor product space

$$T_p^{r,s}\mathcal{M} \doteq \left[\bigotimes^r T_p^*\mathcal{M}\right] \otimes \left[\bigotimes^s T_p\mathcal{M}\right].$$

As for a vector field, we can define a *tensor field* of type (r,s) by a smooth assignment of an element of $T_p^{r,s}(\mathcal{M})$ at each point $p \in \mathcal{M}$.

An important example of tensor is given by the metric.

Definition 1.1.13. Let $\mathcal{M}$ be a differentiable manifold. A *Riemannian metric* g on $\mathcal{M}$ is a type $(0,2)$ tensor field on $\mathcal{M}$ which satisfies the following conditions at each point $p \in \mathcal{M}$:

(i) $g_p(X,Y) = g_p(Y,X) \; \forall X, Y \in T_p\mathcal{M}$,

(ii) $g_p(X,X) \geq 0 \; \forall X, Y \in T_p\mathcal{M}$ where the equality holds only when $X = 0$.

Before the Definition 1.1.12, we have introduced the paring between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$. If there exists a metric g, we define an additional inner product between two vectors $X, Y \in T_p\mathcal{M}$ as $g_p(X,Y)$. Since g_p is a map $T_p\mathcal{M} \otimes T_p\mathcal{M} \to \mathbb{R}$ for all $X \in T_p\mathcal{M}$, we obtain a linear map $g_p(X,\ ) : T_p\mathcal{M} \to \mathbb{R}$ by $Y \mapsto g_p(X,Y)$. We can identify $g_p(X,\ )$ with a one-form $w_Y \in T_p^*\mathcal{M}$. Similarly $w \in T_p^*\mathcal{M}$ induces $X_w \in T_p\mathcal{M}$ by $\langle w, Y\rangle = g_p(X_w, Y)$. The metric g_p gives thus rise to an isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$.
Let $(\mathcal{U}, \mathbf{x})$ be a chart in $\mathcal{M}$ and $\{x_i\}$ the coordinates. Since $g \in \bigotimes_s^2 T^*\mathcal{M}$, it can be expanded in terms of a basis $dx^i \otimes dx^j$ of $\bigotimes_s^2 T^*\mathcal{M}$ as

$$g_p = g_{ij}(p) dx^i \otimes dx^j.$$

It is easily checked that

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ji}(p).$$

We usually omit p in g_{ij} unless it causes confusion. The isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$ is now expressed as:

$$w_\mu = g_{\mu\nu} x^\nu \qquad x^\mu = g^{\mu\nu} w_\nu$$

Since $g_{\mu\nu}$ is a symmetric tensor, the eigenvalues are real. If g is Riemannian, all eigenvalues are strictly positive and if g is pseudo-Riemannian, some of them may be negative. If there are ν positive and μ negative eigenvalues, the pair (μ, ν) is called the *index* of the metric. If $\mu = 1$, the metric is called *Lorentzian*. Once a metric is diagonalised at any point by an appropriate orthogonal matrix, all the diagonal elements are possible to reduce to ± 1 by a suitable scaling of basis vector with positive number. If we start with a Riemannian metric we end up with the *Euclidean metric* $\delta = diag(1, \ldots, 1)$ whereas, if we start with a Lorentz metric, we get the Minkowski metric $\eta = (-1, 1, \ldots, 1)$. In the case $(\mathcal{M}, g)$ is Lorentzian, the elements of $T_p\mathcal{M}$ are divided in three classes:

(i) $g(X,X) > 0 \to X$ is *space-like*,

(ii) $g(X,X) = 0 \to X$ is *light-like* (or *null*),

(iii) $g(X,X) < 0 \to X$ is *time-like*.

Lorentz manifolds are of special interest in the theory of relativity.
Now, let $\mathcal{M}$ be an m-dimensional submanifold of an n-dimensional Riemannian manifold $\mathcal{N}$ with the metric $g_\mathcal{N}$. If $f : \mathcal{M} \to \mathcal{N}$ is the embedding which induces the submanifold structures on $\mathcal{M}$, the pull-back map f^* induces the natural metric $g_\mathcal{N} = f^* g_\mathcal{N}$ on $\mathcal{M}$. The components of $g_\mathcal{M}$ are given by

$$g_{\mathcal{M}\mu\nu}(p) = g_{\mathcal{N}\alpha\beta}(f(p)) \partial_\mu f^\alpha \partial_\nu f^\beta,$$

where f^α denotes the coordinates of $f(p)$. For example, consider the metric of the unit sphere embedded in $(\mathbb{R}^3, \delta)$. Let (ϑ, φ) be the polar coordinates of S^2 and define f by the usual inclusion

$$f:\ (\vartheta, \varphi) \mapsto (\sin\vartheta\cos\vartheta, \sin\vartheta\sin\varphi, \cos\vartheta),$$

from which we obtain the induced metric

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi.$$

When a manifold $\mathcal{N}$ is pseudo-Riemannian, its submanifold $f : \mathcal{M} \to \mathcal{N}$ has a metric $g_{\mathcal{M}} = f^* g_{\mathcal{N}}$. The tensor $f^* g_{\mathcal{N}}$ is a metric only when it has a fixed index on $\mathcal{M}$.

1.1.3 Isometries and conformal transformations

A particular class of diffeomorphisms between two manifolds which plays a central role in the construction of Hadamard states in asymptotically flat spacetimes are conformal transformations. Before introducing it, it is due time to recall what are isometries and Killing vectors.

Definition 1.1.14. Let $\mathcal{M}$ and $\mathcal{N}$ be (pseudo-) Riemannian manifolds endowed with a metric $g_{\mathcal{M}}$ and $g_{\mathcal{N}}$ respectively. A smooth map $\varphi : \mathcal{M} \to \mathcal{N}$ is an *isometry* if

$$\varphi^* g_{\mathcal{N}} = g_{\mathcal{M}}.$$

Consider a point $p \in \mathcal{M}$; for all $X \in T_p\mathcal{M}$ we call *length of a vector* the quantity $g_{p,\mathcal{M}}(X, X)$. Let φ be an isometry

$$\begin{aligned} \varphi :\ & \mathcal{M} \to \mathcal{N} \\ & p \mapsto q = \varphi(p) \end{aligned}$$

and

$$\begin{aligned} \varphi_* :\ & T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N} \\ & X \mapsto Y \doteq \varphi_*(X) \end{aligned}$$

Since φ is an isometry $\varphi^* g_{\mathcal{N}} = g_{\mathcal{M}}$, that is $g_{p,\mathcal{M}}(X, X) = \varphi^* g_{\varphi(p),\mathcal{N}}(X, X) \doteq g_{\varphi(p),\mathcal{N}}(\varphi_* X, \varphi_* X) = g_{\varphi(p),\mathcal{N}}(Y, Y)$ for all $X \in T_p\mathcal{M}$. In other words an isometry maps a vector $X \in T_p\mathcal{M}$ in $\varphi_* X = Y \in T_q\mathcal{N}$, keeping the length invariant. An isometry may be regarded as a rigid motion.

Let $p \in \mathcal{M}, \mathcal{U} \subset \mathcal{M}$ be a neighbourhood of p, $\mathfrak{X}(\mathcal{M})$ be the set of smooth vector field defined in $\mathcal{M}$ and $X \in \mathfrak{X}(\mathcal{M})$. Let $\varphi : (-\varepsilon, \varepsilon) \times \mathcal{U} \to \mathcal{M}$ be a differentiable map such that, for any $q \in \mathcal{U}$, the curve $t \mapsto \varphi(t, q)$ is an integral curve of X passing through q at $t = 0$ ($\mathcal{U}$ and φ are given by the fundamental theorem for ordinary differential equation). X is called a *Killing field* if, for each $t_0 \in (-\varepsilon, \varepsilon)$ the mapping

$\varphi(t_0,\ ) : \mathcal{U} \subset \mathcal{M} \to \mathcal{M}$ is an isometry. Necessary and sufficient condition fod this to happen is that X satisfies *Killing equations*

$$x^\xi \partial_\xi g_{\mu\nu} + \partial_\mu x^\alpha g_{\alpha\nu} + \partial_\nu x^\beta g_{\mu\beta} = 0.$$

From the definition of Lie derivative, this is tantamount to

$$\mathcal{L}_X g_{\mu\nu} = 0. \tag{1.1.1}$$

Equation (1.1.1) shows that the local geometry does not change as we move along φ. In this sense, the Killing vector fields represent the direction of the symmetries of a manifold.

A map that preserves a metric up to a scale factor is a conformal map.

Definition 1.1.15. Let $\mathcal{M}$ and $\mathcal{N}$ be a (pseudo-) Riemannian manifolds with the metric $g_\mathcal{M}$ and $g_\mathcal{N}$ respectively and $\Omega \in \mathcal{D}(\mathcal{M})$. A diffeomorphism

$$\begin{aligned} \varphi: \quad & \mathcal{M} \to \mathcal{N} \\ & p \mapsto q = \varphi(p) \end{aligned}$$

is a *conformal transformation* if

$$\varphi^* g_\mathcal{N} = \Omega^2 g_\mathcal{M}$$

The set of conformal transformations on $\mathcal{M}$ is a group, the *conformal group* denoted by $Conf(\mathcal{M})$. Let us define the angle ϑ between two vectors $X, Y \in T_p(\mathcal{M})$ by

$$\cos\vartheta \doteq \frac{g_{p,\mathcal{M}}(X,Y)}{[g_{p,\mathcal{M}}(X,X) g_{p,\mathcal{M}}(Y,Y)]^{\frac{1}{2}}} = \frac{g_{\mu\nu}X^\mu Y^\nu}{[g_{\alpha\beta}X^\alpha X^\beta g_{\gamma\lambda}Y^\lambda Y^\gamma]^{\frac{1}{2}}}.$$

If φ is a conformal transformation, the angle Θ between $\varphi_* X, \varphi_* Y \in T_{\varphi(p)}\mathcal{N}$ is

$$\cos\Theta = \frac{\Omega^2 g_{\mu\nu}X^\mu Y^\nu}{[\Omega^2 g_{\alpha\beta}X^\alpha X^\beta \Omega^2 g_{\gamma\lambda}Y^\lambda Y^\gamma]^{\frac{1}{2}}} = \cos\vartheta.$$

Hence φ preserves the angle.
An important consequence on a Lorentzian manifold is that φ_* preserves the local light cone structure, namely

$$\varphi_* : \begin{cases} \text{space-like vector} & \mapsto \quad \text{space-like vector} \\ \text{light-like vector} & \mapsto \quad \text{light-like vector} \\ \text{time-like vector} & \mapsto \quad \text{time-like vector} \end{cases}$$

1.1.4 Affine connection

A vector X is a directional derivative acting on $f \in \mathcal{D}(\mathcal{M})$ as $X : f \mapsto X(f)$. However, there is no directional derivative acting on a tensor field of type (p, q), which arises naturally from the differentiable structure of $\mathcal{M}$. What we need is an extra structure called *connection*, which specifies how tensors are transported along a curve.

Definition 1.1.16. An *affine connection* ∇ on a differentiable manifold $\mathcal{M}$ is a map

$$\begin{aligned} \nabla : &\mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \\ &X, Y \mapsto \nabla_X Y \end{aligned}$$

which satisfies the following properties:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X fZ = f\nabla_X Z + X(f)Z$,

in which $X, Y, Z \in \mathfrak{X}(\mathcal{M})$ and $f, g \in \mathcal{D}(\mathcal{M})$

This definition is not as transparent as that of tangent vector. The following proposition should clarify the situation a little.

Proposition 1.1.17. *Let $\mathcal{M}$ be a differentiable manifold with an affine connection ∇. There exists a unique correspondence which associates to a vector field X along the differentiable curve $\sigma : I \to \mathcal{M}$ another vector field $\frac{DX}{dt}$ along σ called the covariant derivative of X along σ, such that:*

(i) $\frac{D}{dt}(X + Y) = \frac{DX}{dt} + \frac{DY}{dt}$,

(ii) $\frac{D}{dt}(fX) = \frac{df}{dt}X + f\frac{DX}{dt}$,

(iii) if $X(t) = Y(\sigma(t))$, *then* $\frac{DX}{dt} = \nabla_{\frac{d\sigma}{dt}} Y$,

where $X, Y \in \mathfrak{X}(\mathcal{M})$ *and* $f \in \mathcal{D}$.

Proof. Let us suppose initially that there exists a correspondence satisfying (i), (ii) and (iii). Let $\mathbf{x} : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{M}$ be a system of coordinates with $\sigma(I) \cap \mathbf{x}(\mathcal{U}) \neq \emptyset$ and let $(x^1(t), \ldots, x^n(t))$ be the local expression of $\sigma(t)$, with $t \in I$. Let $e_\mu = \partial_\mu$. Then we can express the field X locally as $X = x^\mu e_\mu$, $\mu = 1, \ldots, dim\mathcal{M}$, where $x^\mu \doteq x^\mu(t)$ and $e_\mu \doteq e_\mu(\sigma(t))$.
By (i) and (ii), we have

$$\frac{DX}{dt} = \frac{dx^\mu}{dt} e_\mu + x^\mu \frac{De_\mu}{dt}.$$

By (iii) and (i) of the Definition 1.1.16,

$$\frac{De_\mu}{dt} = \nabla_{\frac{d\sigma}{dt}} e_\mu = \nabla_{\left(\frac{dx^\nu}{dt} e_\nu\right)} e_\mu = \frac{dx^\nu}{dt} \nabla_{e_\nu} e_\mu .$$

Therefore,

$$\frac{DX}{dt} = \frac{dx^\mu}{dt} e_\mu + x^\mu \frac{dx^\nu}{dt} \nabla_{e_\nu} e_\mu . \tag{1.1.2}$$

Equation (1.1.2) shows us that if there exists a correspondence satisfying the condition of Proposition 1.1.17, then it is unique.
To show existence, define $\frac{DX}{dt}$ in $\mathbf{x}(\mathcal{U})$ by (1.1.2). It is possible to verify that (1.1.2) possesses the desired properties. If $\mathbf{y}(\mathcal{V})$ is another coordinate neighbourhood, with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) \neq \emptyset$ and we define $\frac{DX}{dt}$ in $\mathbf{y}(\mathcal{V})$ by (1.1.2), the definitions agree in $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$, by uniqueness of $\frac{DX}{dt}$ in $\mathbf{x}(\mathcal{U})$. It follows that the definition can be extended over all of $\mathcal{M}$, and this concludes the proof. □

Part (iii) of Definition 1.1.16 allows us to show that the notion of affine connection is actually a local notion. Choosing a system of coordinates $(\mathcal{U}, \mathbf{x})$ about $p \in \mathcal{M}$ and writing

$$X = x^\mu e_\mu \qquad Y = y^\nu e_\nu$$

where $e_\mu = \partial_\mu$, we have

$$\nabla_X Y = x^\mu \nabla_{e_\mu} (y^\nu \nabla_{e_\nu}) = x^\mu y^\nu \nabla_{e_\mu} e_\nu + x^\mu e_\mu (y^\nu) e_\nu .$$

Setting $\nabla_{e_\mu} e_\nu = \Gamma^k{}_{\mu\nu} e_k$, we conclude that all $\Gamma^k{}_{\mu\nu}$ are differentiable functions and that

$$\nabla_X Y = \left(x^\mu y^\nu \Gamma^k{}_{\mu\nu} + x^\mu e_\mu (y^k) \right) e_k ,$$

which proves that $\nabla_X Y(p)$ depends on $x^\mu(p)$ and $y^\nu(p)$ and the derivatives $x^\mu e_\mu(y^k) = X(y^k)$ of y^k by X.
The concept of *parallelism* and *geodesic* follows in a natural way.

Definition 1.1.18. Let $\mathcal{M}$ be a differentiable manifold with an affine connection ∇. A vector X along a curve $\sigma : I \to \mathcal{M}$ is called *parallel* when $\frac{DX}{dt} = 0$, for all $t \in I$ or, equivalently, when $\nabla_X X = 0$. The curve $\sigma(t)$ is called *geodesic*.

An important result on affine connections is the *Levi-Civita theorem*. We need to recall be forehand a few useful definition.

Definition 1.1.19. Let $\mathcal{M}$ be a differentiable manifold with an affine connection ∇ and a Riemannian metric g. A connection is said to be *compatible* with the metric g when for any smooth curve σ and any pair of parallel vector field X and Y along σ, we have $g(X, Y) = \text{constant}$.

Proposition 1.1.20. *Let $\mathcal{M}$ be a differentiable manifold with an affine connection ∇ and a Riemannian metric g. A connection is said to be compatible with the metric* *if and only if for any vector field X and Y along the differentiable curve $\sigma : I \to \mathcal{M}$ we have*

$$\frac{d}{dt} g(X,Y) = g\left(\frac{DX}{dt}, Y\right) + g\left(X, \frac{DY}{dt}\right), \quad t \in I \tag{1.1.3}$$

Proof. Equation (1.1.3) entails that ∇ is compatible with g. Let us prove the converse. Choose an orthonormal basis $\{P_1(t_0), \ldots, P_n(t_0)\}$ of $T_{x(t_0)}\mathcal{M}$, $t \in I$. We can extend the vectors $P_i(t_0)$ along the curve σ by parallel transport. Since ∇ is compatible with the metric, $\{P_1(t), \ldots, P_n(t)\}$ is an orthonormal basis of $T_{\sigma(t)}\mathcal{M}$, for any $t \in I$. We can write, therefore,

$$X = x^\mu P_\mu, \qquad Y = y^\mu P_\mu$$

where x^μ and y^μ are differentiable functions on I. It follows that

$$\frac{DX}{dt} = \frac{dx^\mu}{dt} P_\mu, \qquad \frac{DY}{dt} = \frac{dy^\mu}{dt} P_\mu,$$

It descends,

$$g\left(\frac{DX}{dt}, Y\right) + g\left(X, \frac{DY}{dt}\right) = \left(\frac{dx^\mu}{dt} y^\mu + \frac{dy^\mu}{dt} x^\mu\right) = \\ = \frac{d}{dt}\left(x^\mu y^\mu\right) = \frac{d}{dt} g(X,Y)$$

□

Proposition 1.1.21. *Let $\mathcal{M}$ be a differentiable manifold with an affine connection ∇ and a Riemannian metric g. A connection is compatible with the metric if and only if*

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \mathfrak{X}(\mathcal{M}). \tag{1.1.4}$$

Proof. Suppose that ∇ is compatible with the metric. Let $p \in \mathcal{M}$ and let $\sigma : I \to \mathcal{M}$ be a differentiable curve with $\sigma(t_0) = p$, $t_0 \in I$, and with $\frac{d\sigma}{dt}\big|_{t=t_0} = X(p)$. Then

$$X(p)g(Y,Z) = \frac{d}{dt} g(Y,Z)\Big|_{t=t_0} = g(\nabla_{X(p)} Y, Z) + g(Y, \nabla_{X(p)} Z)$$

Since p is arbitrary, equation (1.1.4) follows. The converse is obvious. □

Definition 1.1.22. An affine connection ∇ on a smooth manifold $\mathcal{M}$ is said *symmetric* when $\nabla_X Y - \nabla_Y X = [X,Y] = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

We state Levi-Civita theorem.

Theorem 1.1.23 (Levi-Civita)**.** *For any a Riemannian manifold $\mathcal{M}$, there exists a unique affine connection ∇ on $\mathcal{M}$ satisfying the conditions:*

(i) ∇ *is symmetric,*

(ii) ∇ is compatible with the Riemannian metric.

Proof. Suppose ∇ exists. Then

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{1.1.5}$$

$$Yg(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \tag{1.1.6}$$

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{1.1.7}$$

By summing equation (1.1.5) and (1.1.6) and subtracting (1.1.7) it holds on account of the symmetry of ∇, that

$$\begin{aligned} &Xg(Y,Z) + Yg(Z,Y) - Zg(X,Y) = \\ &= g([X,Z],Y) + g([Y,Z],X) + g([X,Y],Z) + 2g(Z,\nabla_Y X). \end{aligned}$$

Therefore

$$\begin{aligned} g(Z,\nabla_Y X) =& \frac{1}{2}\left(Xg(Y,Z) + Yg(Z,Y) - Zg(X,Y) + \right. \\ & \left. -g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\right). \end{aligned} \tag{1.1.8}$$

Formula (1.1.8) shows that ∇ is uniquely determined from the metric g. Hence, if there exists, it has to be unique.
To prove existence, define ∇ by (1.1.8). One can readily verify that ∇ is well-defined and that it satisfies the desired conditions. □

The connection given by the theorem will be referred to, from now on, as the *Levi-Civita connection* on $\mathcal{M}$. Taken a coordinate system $(\mathcal{U}, \mathbf{x})$, the functions $\Gamma^\kappa{}_{\mu\nu}$, defined on $\mathcal{U}$ by $\nabla_{e_\mu} e_\nu = \Gamma^\kappa{}_{\mu\nu} e_\kappa$, are called *Christoffel symbols of the connection* ∇. From equation (1.1.8) it follows that

$$\Gamma^\alpha{}_{\mu\nu} g_{\alpha\kappa} = \frac{1}{2}\left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right),$$

where $g_{\mu\nu} \doteq g(e_\mu, e_\nu)$.
Since $g_{\mu\nu}$ admits inverse $g^{\mu\nu}$, we obtain

$$\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) g^{\alpha\kappa}.$$

In the terms of Christoffel symbols, the covariant derivative has the classical expression

$$\frac{DX}{dt} = \left(\frac{dx^\kappa}{dt} + \Gamma^\kappa{}_{\mu\nu} x^\nu \frac{dx^\mu}{dt}\right) e_\kappa.$$

Observe that $\frac{DX}{dt}$ differs from the usual derivative in Euclidean space by terms which involve the Christoffel symbols. Since for the Euclidean space $\mathbb{R}^n$, we have $\Gamma^\kappa{}_{\mu\nu} = 0$, the covariant derivative coincides with the usual derivative.

1.1.5 Curvature and torsion

Since Γ is not a tensor, it cannot have an intrinsic geometrical meaning as a measure of how much a manifold is curved. For example, in normal coordinates, at a point $p \in \mathcal{M}$ Christoffel symbols of the connection vanish at p. As intrinsic objects, we defined the *torsion tensor* and the *Riemann curvature tensor* (or simply *Riemann tensor*

Definition 1.1.24. The *torsion tensor* T of a Riemannian manifold $\mathcal{M}$ is

$$\begin{aligned} T: \quad & \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \\ & X, Y \mapsto T(X,Y) \doteq \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

Definition 1.1.25. The *Riemann curvature tensor Riem* of a Riemannian manifold $\mathcal{M}$ is map

$$\begin{aligned} Riem: \quad & \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \\ & X, Y, Z \mapsto R(X,Y,Z) \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned}$$

Since T and *Riem* are tensors, their actions on vector fields is obtained upon linearity from the counterpart on the basis vectors. With respect to both a coordinate basis $\{e_\mu\}$ and to dual basis $\{dx^\mu\}$, the components of these tensors are

$$\begin{aligned} T^\kappa_{\mu\nu} &= \langle dx^\kappa, T(e_\mu, e_\nu) \rangle = \langle dx^\kappa, \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu \rangle = \\ &= \langle dx^\kappa, \Gamma^\alpha{}_{\mu\nu} e_\alpha - \Gamma^\alpha{}_{\nu\mu} e_\alpha \rangle = \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} \end{aligned}$$

and

$$\begin{aligned} R^\kappa_{\mu\nu\gamma} &= \langle dx^\kappa, R(e_\mu, e_\nu) e_\gamma \rangle = \langle dx^\kappa, \nabla_{e_\mu} \nabla_{e_\nu} e_\gamma - \nabla_{e_\nu} \nabla_{e_\mu} e_\gamma \rangle = \\ &= \langle dx^\kappa, \nabla_{e_\mu} \Gamma^\alpha{}_{\nu\gamma} e_\alpha - \nabla_{e_\nu} \Gamma^\alpha{}_{\mu\gamma} e_\alpha \rangle = \\ &= \langle dx^\kappa, (\partial_\mu \Gamma^\alpha{}_{\nu\gamma}) e_\alpha + \Gamma^\alpha{}_{\nu\gamma} \Gamma^\beta{}_{\mu\alpha} e_\beta - (\partial_\nu \Gamma^\alpha{}_{\mu\gamma}) e_\alpha + \Gamma^\alpha{}_{\mu\gamma} \Gamma^\beta{}_{\nu\alpha} e_\beta \rangle = \\ &= \partial_\mu \Gamma^\kappa{}_{\nu\gamma} - \partial_\nu \Gamma^\kappa{}_{\mu\gamma} + \Gamma^\alpha{}_{\nu\gamma} \Gamma^\kappa{}_{\mu\alpha} - \Gamma^\alpha{}_{\mu\gamma} \Gamma^\kappa{}_{\nu\alpha}. \end{aligned}$$

From the Riemann curvature tensor, we construct new tensors by contracting the indices. The *Ricci tensor Ric* is of type $(0,2)$:

$$Ric(X,Y) \doteq \langle dx^\mu, R(e_\mu, Y, X) \rangle$$

the components are

$$R_{\mu\nu} = R(e_\mu, e_\nu, \alpha) = R^\gamma_{\mu\gamma\nu}.$$

Contracting once more, we obtain the *scalar curvature* R

$$R \doteq g^{\mu\nu} Ric(e_\mu, e_\nu) = g^{\mu\nu} R_{\mu\nu}.$$

The Riemann tensor satisfy two important identities.

$$R^\kappa{}_{\mu\nu\gamma} + R^\kappa{}_{\nu\gamma\mu} + R^\kappa{}_{\gamma\mu\nu} = 0, \qquad \textit{(first Bianchi identity)}$$

$$\nabla_{e_\lambda} R^{\kappa}{}_{\mu\nu\gamma} + \nabla_{e_\mu} R^{\kappa}{}_{\nu\lambda\gamma} + \nabla_{e_\nu} R^{\kappa}_{\lambda\mu\gamma} = 0. \qquad \textit{(second Bianchi identity)} \tag{1.1.9}$$

Contracting the indices k and ν of the second identity and using $\nabla_{e_\lambda} \doteq \nabla_\lambda$

$$\nabla_\lambda R_{\mu\gamma} - \nabla_\mu R_{\lambda\gamma} + \nabla_k R^{\kappa}{}_{\lambda\mu\gamma} = 0.$$

If the indices γ and μ are further contracted, we obtain

$$\nabla_\mu (R - 2R^{\mu}{}_{\nu}) = 0$$

or equivalently,

$$\nabla_\mu G^{\mu\nu} = 0,$$

where $G^{\mu\nu}$ is the *Einstein tensor* defined as

$$G^{\mu\nu} \doteq R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R. \tag{1.1.10}$$

1.2 Weyl Manifolds

In 1918 H. Weyl introduced a generalization of Riemannian geometry in his attempt to formulate a unified field theory. Weyl theory failed for physical reasons, but it remains a beautiful piece of mathematics, and it provides an instructive example of non-Riemannian connections. The physical motivation for Weyl ideas is as follows: In the general theory of relativity, Einstein used Riemannian geometry as a model for physical space. However, the Universe is not a Riemannian manifold, for there is no absolute measure of length. To wit, instead of assuming a scalar product on the tangent space at each point, we are given a scalar product determined pointwisely only up to a positive factor. This fact does not produce essential changes in the geometry provided that a measure of length at one point uniquely induces a counterpart globally (*i.e.*, if it makes sense to compare the size of two tangent vectors at two distinct points). We make these ideas precise. Our treatment is based on [32].
$\mathcal{M}$ will always denote an n-dimensional smooth manifold, $T_p\mathcal{M}$ the tangent space of $\mathcal{M}$ at p and $\Lambda^1(\mathcal{M})$ the space of one-forms on $\mathcal{M}$.

Definition 1.2.1. Two Riemannian metrics g and $\widetilde{g}$ on $\mathcal{M}$ are said to be *equivalent* if and only if $\widetilde{g} = \Omega^2 g$ where $\Omega \in \mathcal{D}(\mathcal{M})$ and $\Omega \neq 0$.

Definition 1.2.2. A *conformal structure* on $\mathcal{M}$ is an equivalence class G of Riemannian metrics on $\mathcal{M}$. A manifold with a conformal structure is called a *conformal manifold.*

Note that in a conformal manifold one can speak of the angle between two vectors at a point, or of the ratio of their length, although their absolute lengths are not defined. Also the notions of a symmetric or skew-symmetric transformation of vector fields make sense.

Definition 1.2.3. A *Weyl structure* on $\mathcal{M}$ is a map $F : G \mapsto \Lambda^1(\mathcal{M})$ satisfying $F(\Omega^2 g) = F(g) - \partial_\mu \ln \Omega$, where g and $\Omega^2 g$ are two arbitrary representation of the same equivalence class in G. A manifold with a Weyl structure is called a *Weyl manifold*

The main facts about Weyl manifolds, that we shall not prove, though one can refer to [32], are the following.

Theorem 1.2.4. *Let g be an arbitrary element of G, and denote $F(g)$ by φ.*

(i) A linear connection on a Weyl manifold $\mathcal{M}$ is said to be compatible if and only if $\partial_k g_{\mu\nu} + g_{\mu\nu}\varphi_k = 0$

(ii) On every Weyl manifold there exists a unique torsion-free compatible linear connection, the Weyl connection $\widetilde{\nabla}$; its components $\widetilde{\Gamma}^\kappa_{\mu\nu}$ with respect to a chart $(\mathcal{U}_\alpha, \mathbf{x}_\alpha)$ are:

$$
\begin{aligned}
\widetilde{\Gamma}^\kappa_{\mu\nu} &= \frac{1}{2} g^{ik}(\partial_\mu g_{i\nu} + \partial_\nu g_{\mu i} - \partial_i g_{\mu\nu}) + \frac{1}{2}(\delta^\kappa_\mu \varphi_\nu + \delta^\kappa_\nu \varphi_\mu - g_{\mu\nu} g^{ik} \varphi_i) = \\
&= \Gamma^\kappa_{\mu\nu} + \frac{1}{2}(\delta^\kappa_\mu \varphi_\nu + \delta^\kappa_\nu \varphi_\mu - g_{\mu\nu} g^{ik} \varphi_i) = \\
&= \Gamma^\kappa_{\mu\nu} + \varphi^\kappa_{\mu\nu}
\end{aligned}
$$

where $\varphi^\kappa_{\mu\nu} \doteq \frac{1}{2}(\delta^\kappa_\mu \varphi_\nu + \delta^\kappa_\nu \varphi_\mu - g_{\mu\nu} g^{ik} \varphi_i)$

(iii) Conversely, a torsion-free linear connection on a manifold $\mathcal{M}$ for which there exist a Riemannian metric g and a one-form φ satisfying (ii) is the induced connection of the Weyl structure determined by g and φ

As a consequence, the Weyl connection, is conformally invariant: If $\widetilde{g}$ and g are equivalent, namely $\widetilde{g} = \Omega^2$, the following property holds

$$
\widetilde{\Gamma}^\kappa_{\mu\nu}(\Omega^2 g) = \widetilde{\Gamma}^\kappa_{\mu\nu}(g) \tag{1.2.1}
$$

In place of $\widetilde{\nabla}$ that is not metric compatible, one can use the Weyl covariant derivative defined via

$$
D_\mu T^{\alpha\dots}_{\beta\dots} = (\widetilde{\nabla}_\mu + \omega\varphi_\mu) T^{\alpha\dots}_{\beta\dots} = (\nabla_\mu + \omega\varphi_\mu) T^{\alpha\dots}_{\beta\dots} + \varphi^\alpha_{i\mu} T^{i\dots}_{\beta\dots} - \varphi^i_{\beta\mu} T^{\alpha\dots}_{i\dots} \tag{1.2.2}
$$

where T is a given conformal tensor field of rank (r,s) and conformal weight $\omega = \omega(T)$. Under conformal transformations, conformal tensors are mapped as

$$
T \mapsto \Omega^\omega T.
$$

The Weyl covariant derivative is metric compatible.
Before introducing the "The Weyl-to-Riemann method" let us define the Weyl-

invariant Riemannian tensor

$$\begin{aligned}\widetilde{R}^{\varrho}{}_{\sigma\mu\nu} &= \partial_\mu \widetilde{\Gamma}^{\varrho}_{\nu\sigma} - \partial_\nu \widetilde{\Gamma}^{\varrho}_{\mu\sigma} + \widetilde{\Gamma}^{\varrho}_{\mu\lambda}\widetilde{\Gamma}^{\lambda}_{\nu\sigma} - \widetilde{\Gamma}^{\varrho}_{\nu\lambda}\widetilde{\Gamma}^{\lambda}_{\mu\sigma} = \\ &= R^{\varrho}_{\sigma\mu\nu} + \nabla_\mu \varphi^{\varrho}_{\nu\sigma} - \nabla_\nu \varphi^{\varrho}_{\mu\sigma} + \varphi^{\varrho}_{\mu\lambda}\varphi^{\lambda}_{\nu\sigma} - \varphi^{\varrho}_{\nu\lambda}\varphi^{\lambda}_{\mu\sigma} = \\ &= R^{\varrho}_{\sigma\mu\nu} + \widehat{R}^{\varrho}_{\sigma\mu\nu}.\end{aligned}$$

1.2.1 The Weyl-to-Riemann method

We present a simple, systematic and practical algorithm which translates the problems of constructing conformally invariant equations in arbitrary Riemannian spaces into a set of algebraic equations. This method, called "Weyl-to-Riemann", is based on two features of Weyl geometry:

(i) Using the Weyl covariant derivative and the Weyl geometrical tensor, all homogeneous equations written in Weyl space are conformally invariant.

(ii) When the Weyl structure satisfies $\varphi_\mu = \partial_\mu \ln \Omega$, the Weyl space is called Weyl Integrable Space (WIS) and the Weyl connection reduces to $\widetilde{\Gamma}^{\kappa}_{\mu\nu} = \overline{\Gamma}^{\kappa}_{\mu\nu}$, where $\overline{\Gamma}^{\kappa}_{\mu\nu}$ is the Christoffel connection of the metric tensor $\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

In practice, to construct a Riemannian conformally invariant n^{th}-order equation for a tensor field T of weight ω, one starts by writing the most general homogeneous equation in Weyl space, say

$$a_1\widetilde{D}_1 + a_2\widetilde{D}_2 + \cdots + b_1\widetilde{U}_1 + b_2\widetilde{U}_2 + \cdots = 0,$$

where $\widetilde{D}_i$ are derivative terms acting on T, $\widetilde{U}_i$ are geometrical terms whereas a_i, b_i are free parameters. As stated above, this equation is conformally invariant in a Weyl space. The game consists in looking for combinations of ω, a_i, b_i in order to obtain, when reducing to a Riemannian space imposing $\varphi_\mu = \partial_\mu \ln \Omega$, an equation

$$a_1 D_1 + a_2 D_2 + \cdots + b_1 U_1 + b_2 U_2 + \cdots = 0,$$

which is conformally invariant.

Example 1.8. This example[1] is taken by [33]. The most general second order scalar field equation in a Weyl space involves the second-rank geometrical tensors. It reads

$$g^{\mu\nu}(D_\mu D_\nu + b_1\widetilde{R}_{\mu\nu} + b_2\widetilde{R}g_{\mu\nu})\psi = (D^2 + \alpha\widetilde{R})\psi = 0,$$

[1] The author employs the signature $(+,-,-,-)$. In this way the Weyl covariant derivative is

$$D_\mu T^{\alpha\dots}_{\beta\dots} = (\nabla_\mu - \omega\varphi_\mu)T^{\alpha\dots}_{\beta\dots} - \varphi^{\alpha}_{i\mu}T^{i\dots}_{\beta\dots} + \varphi^{i}_{\beta\mu}T^{\alpha\dots}_{i\dots}$$

whereas the Ricci tensor is

$$\widetilde{R} = R - 6(\nabla^\mu\varphi_\mu - \varphi^\mu\varphi_\mu)$$

where $\alpha = b_1 + 4b_2$. This equation is expanded as

$$\left(\Box + \alpha R - 2(1+\omega)\varphi_\mu \nabla^\mu - (6\alpha+\omega)\nabla^\mu \varphi_\mu + (\omega^2 + 2\omega + 6\alpha)\varphi^\mu \varphi_\mu\right) \psi = 0, \tag{1.2.3}$$

where $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$ and all other contractions are performed using $g_{\mu\nu}$. We can eliminate φ from this equations if and only if

$$\begin{aligned} 1 + \omega &= 0 \\ 6\alpha + \omega &= 0 \\ \omega^2 + 2\omega + 6\alpha &= 0 \end{aligned}$$

This system has a unique solution

$$\begin{aligned} \omega &= -1 \\ \alpha &= \frac{1}{6}. \end{aligned}$$

Equation (1.2.3) is nothing but the usual conformally invariant scalar field equation in $4D$ Riemannian spaces,

$$(\Box + \frac{1}{6}R)\psi = 0.$$

1.3 Asymptotically flat spacetimes

A spacetime is a four-dimensional smooth, Hausdorff, second countable, time oriented, manifold $\mathcal{M}$ equipped with a Lorentzian metric g assumed to be everywhere smooth.

Definition 1.3.1. A *vacuum spacetime* is a spacetime satisfying vacuum Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{1.3.1}$$

Contracting with $g^{\mu\nu}$ we obtain

$$g^{\mu\nu} G_{\mu\nu} = -R = 0$$

In this manner, a vacuum spacetime satisfies

$$R_{\mu\nu} = 0$$

We shall give a brief introduction of the notion that a spacetime be is "causally well behaved". Our treatment is based on [34], [35] and [36].

All spacetimes in General Relativity have only *locally* the same causal structure as in the special relativity. Globally they can be have very significant differences. For example, we can construct a (flat) spacetime with the topology $S^1 \times \mathbb{R}^3$ by identifying the hyperplanes $t = 0$ and $t = 1$ of Minkowski spacetime, as illustrated in Figure 1.6

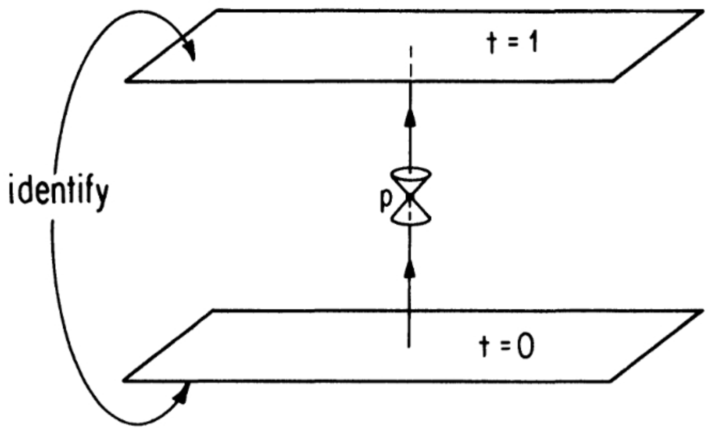

Figure 1.6

The integral curves of $(\partial_t)^a$ in this spacetime will be closed and timelike. In addition, in this spacetime with closed causal curves, severe consistency conditions may exist on the solutions of equations describing the propagation of physical fields. Spacetimes with closed causal curves can not be blamed entirely on "artificial" topological identifications as in Figure 1.6, since examples with the topology $\mathbb{R}^4$ can be constructed by "twisting" the light cones, as illustrated in Figure 1.7.

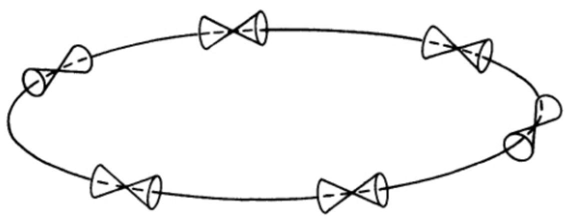

Figure 1.7

It is generally believed that spacetimes with closed causal curves are not physically realistic. It is thus very useful to formulate precise conditions which characterize this type of behaviour. One such characterization is strongly causality.

Definition 1.3.2. A spacetime $(\mathcal{M}, g)$ is said to be *strongly causal* if, for all $p \in \mathcal{M}$ and every neighbourhood O of p, there exists a neighbourhood V of p contained O such that no causal curve intersects V more than once.

Unfortunately this requirement does not suffice neither to single out the possibility that a perturbation of the metric around two or more points leads to closed causal curves nor to guarantee the existence of a splitting of M which singles out a family of hypersurfaces where to assign Cauchy initial data for a suitable field equation. Let us introduce a few characterizations:

Definition 1.3.3. Let $\mathcal{M}$ be a smooth time-orientable manifold endowed with a smooth metric g. Then we call:

- *chronological future* of $p \in \mathcal{M}$, the set $I^+(p)$ of points $q \in \mathcal{M}$ such that there exists $\sigma : I \subseteq \mathbb{R} \to \mathcal{M}$ connecting p and q and such that the tangent vector of σ is everywhere timelike and future directed.

- *causal future* of $p \in \mathcal{M}$, the set $J^+(p)$ of points $q \in \mathcal{M}$ such that there exists $\sigma : I \subseteq \mathbb{R} \to \mathcal{M}$ connecting p and q and such that the tangent vector of σ is everywhere either timelike or lightlike and future directed.

An analogous definition holds for the chronological and causal past of p, indicated as $I^-(p)$ and $J^-(p)$ respectively.

Definition 1.3.4. A subset $\Sigma \subset \mathcal{M}$ is said to be *achronal* if there do not exist $p, q \in S$ such that $q \in I^+(p)$, *i.e.*, if $I^+(S) \cap \Sigma = \emptyset$, where $I^+(p)$ is defined as the set of events that can be reached by future directed timelike curve starting from p. Futhermore we call

- the *edge* of Σ, the set of $p \in \Sigma$ such that, for all $\mathcal{O} \subset \mathcal{M}$ containing p, there exist $q \in \mathcal{O} \cap I^+(p)$ and $r \in \mathcal{O} \cap I^-(p)$ and a timelike curve which joins q and r without intersecting Σ,

- the domain of dependence of Σ, $D(\Sigma)$, the set of all points $p \in \mathcal{M}$ such that every inextensible curve through p intersects Σ.

From this definition, it descends a proposition whose proof is available in chapter 8 of [34]

Proposition 1.3.5. *Any non-empty closed achronal set $\Sigma \subset \mathcal{M}$ with empty edge is an embedded C^0 submanifold of $\mathcal{M}$ of codimension 1.*

This proposition leads us to define globally hyperbolic spacetimes

Definition 1.3.6. A time-oriented spacetime $(\mathcal{M}, g)$ is called *globally hyperbolic* if and only if it contains a non-empty closed achronal set Σ with empty edge and $D(\Sigma) = \mathcal{M}$. Furthermore Σ is called a Cauchy (hyper)surface of $(\mathcal{M}, g)$.

We introduce a other class of manifolds asymptotically flat spacetime. For more information see [37] and [38].

Definition 1.3.7. A vacuum spacetime $(\mathcal{M}, g)$ is called to be *asymptotically flat at null and spatial infinity* if there exist $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\widetilde{g}$ everywhere smooth except possibly at a point i^0 where is at least C^0 and a conformal isometry $\varphi : \mathcal{M} \to \varphi[\mathcal{M}] \subset \widetilde{\mathcal{M}}$ with conformal factor Ω satisfying the following conditions :

(i) $\overline{J^+(i^0)}\bigcup\overline{J^-(i^0)} = \widetilde{\mathcal{M}} - \mathcal{M}$. In other words, i^0 is spacelike related to all points in $\mathcal{M}$ and the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ consists of the union of i^0, $\mathscr{I}^+ \equiv \partial\overline{J^+(i^0)} - i^0$ and $\mathscr{I}^- \equiv \partial\overline{J^-(i^0)} - i^0$.

(ii) There exists an open neighbourhood V of $\partial\mathcal{M} = i^0\bigcup\mathscr{I}^+\bigcup\mathscr{I}^-$ such that spacetime $(V, \widetilde{g})$ is strongly causal.

(iii) Ω can be extended to a function on all of $\widetilde{\mathcal{M}}$ which is C^2 at i^0 and C^∞ elsewhere.

(iv) (a) On $\mathscr{I}^+$ and $\mathscr{I}^-$ we have $\Omega = 0$ and $\widetilde{\nabla}_\mu\Omega \neq 0$.
(b) We have $\Omega(i^0) = 0$, $lim_{i^0}\widetilde{\nabla}_\mu\Omega = 0$ and $lim_{i^0}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\Omega = 2\widetilde{g}_{\mu\nu}(i^0)$

(v) The map of null directions at i^0 into the space of integral curves of $n^\mu \equiv \widetilde{g}^{\mu\nu}\widetilde{\nabla}_\mu\Omega$ on $\mathscr{I}^+$ and $\mathscr{I}^-$ is a diffeomorphism.

Note that we have defined asymptotic flatness only for the case of vacuum spacetimes, *i.e.* $R_{\mu\nu} = 0$. However, since only properties "near infinity" will play role in our analysis, we required only $R_{\mu\nu} = 0$ in $V\bigcap\mathcal{M}$. The physical Ricci tensor $R_{\mu\nu} = 0$ is related to the unphysical Ricci tensor $\widetilde{R}_{\mu\nu}$ by

$$R_{\mu\nu} = \widetilde{R}_{\mu\nu} + 2\Omega^{-1}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\Omega + \widetilde{g}_{\mu\nu}\widetilde{g}^{\varrho\sigma}(\Omega^{-1}\widetilde{\nabla}_\varrho\widetilde{\nabla}_\sigma\Omega - 3\Omega^{-2}\widetilde{\nabla}_\varrho\Omega\widetilde{\nabla}_\sigma\Omega) \tag{1.3.2}$$

In equation (1.3.2), the right-hand side is the vacuum Einstein field equation expressed in terms of unphysical variables.
It should be emphasized that there exist a considerable arbitrariness in the association of an unphysical spacetime $(\widetilde{\mathcal{M}}, \widetilde{g})$ with an asymptotically flat physical spacetime $(\mathcal{M}, g)$. In fact, $(\widetilde{\mathcal{M}}, \widetilde{g}, \Omega)$ is defined up to a rescaling transformation

$$(\widetilde{\mathcal{M}}, \widetilde{g}, \Omega) \mapsto (\widetilde{\mathcal{M}}, \omega^2\widetilde{g}, \omega\Omega) \tag{1.3.3}$$

Thus, there exists a considerable gauge freedom in the choice of the unphysical metric. In other words, it is possible to implement condition five of Definition 1.3.7: For the smooth function, ω on $\widetilde{\mathcal{M}} - i^0$ with $\omega > 0$ on $\mathcal{M}\bigcup\mathscr{I}^+\bigcup\mathscr{I}^-$ which satisfies $\widetilde{\nabla}_\mu(\omega^4 n^\mu) = 0$ on $\mathscr{I}^+\bigcup\mathscr{I}^-$, the vector field $\omega^{-1}n^\mu$ is complete on $\mathscr{I}^+\bigcup\mathscr{I}^-$.

Let us now discuss the meaning of all five conditions appearing in definition (1.3.7). The first three imply that $(\widetilde{\mathcal{M}}, \widetilde{g})$ possesses the basic properties of the conformal completion of Minkowski spacetime. The first one states that i^0 represent spatial infinity, the second assures that no causal pathologies occur near infinity and the third fixes the behaviour of Ω near infinity. An important role is played by condition (iv) . The requirement that Ω vanishes at $\mathscr{I}^+$, $\mathscr{I}^-$ and i^0 implies that $\mathscr{I}^+$, $\mathscr{I}^-$ and i^0 represents "infinity" for the physical spacetime. Furthermore, the requirement on the derivatives of Ω imply that the metric g is asymptotically Minkowskian as one approaches spatial infinity. Finally, condition (v) fixes the topology of $\mathscr{I}^+$ and $\mathscr{I}^-$ to be $S^2 \times \mathbb{R}$.
Note that the five conditions define asymptotic flatness at both spatial and null

infinity. From now with asymptotically flat spacetimes we would indicate asymptotically flat vacuum spacetimes at future null infinity.

We conclude this section by mentioning an important property of asymptotically flat spacetimes, which is related to the notion of asymptotic symmetries. Minkowski spacetime $(\mathbb{R}^4, \eta)$ has a 10-parameter group of isometries, the Poincaré group. Such isometries play an important role in the analysis of the behaviour of physical fields on Minkowski spacetime, in particular in the proof of conservation laws. On a curved spacetime one would not expect any exact isometries to be present. However, in an asymptotically flat spacetime, one might expect to recover a notion of asymptotic symmetry. The asymptotic group, however, is not the Poincaré one, but rather an infinite dimensional group known as the Bondi-Metzner-Sachs (BMS) group. Let us introduce it.
In the unphysical spacetime, the unphysical metric $\widetilde{g}$ induces a degenerate metric q on the null hypersurface $\mathscr{I}^+$. Futhermore, since the vector field $n^\mu = \widetilde{g}^{\mu\nu}\widetilde{\nabla}_\nu\Omega$ is tangent to $\mathscr{I}^+$, n^μ may be viewed as a vector field on $\mathscr{I}^+$. Under a gauge transformation of the form (1.3.3) the triple $(\mathscr{I}^+, q, n)$ behaves as

$$\mathscr{I}^+ \mapsto \mathscr{I}^+, \qquad q \mapsto \omega^2 q, \qquad n \mapsto \omega^{-1} n. \tag{1.3.4}$$

If C denotes the class containing all triples $(\mathscr{I}^+, q, n)$ transforming as (1.3.4) for a fixed asymptotically flat spacetime, the gauge freedom does not allow to select a preferred element in C. If C_1 and C_2 are the classes of triples associated to $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ respectively, there exists a diffeomorphism

$$\varphi : \mathscr{I}_1^+ \to \mathscr{I}_2^+$$

such that for $(\mathscr{I}_1^+, q_1, n_1) \in C_1$ and $(\mathscr{I}_2^+, q_2, n_2) \in C_2$

$$\varphi(\mathscr{I}_1^+) = \mathscr{I}_2^+, \qquad \varphi^* q_1 = q_2, \qquad \varphi^* n_1 = n_2.$$

This guarantees that for every asymptotically flat spacetime $(\mathcal{M}, g)$ with an initial choice of Ω_0 one can fix $\Omega \doteq \omega\Omega_0$ in order that the metric

$$\widetilde{g}|_{\mathscr{I}^+} = -2d\ell d\Omega + dS^2(\vartheta, \varphi).$$

This descends from the possibility to define in a neighbourhood of $\mathscr{I}^+$ a coordinate system $(\ell, \Omega, \vartheta, \varphi)$. $dS^2(\vartheta, \varphi)$ is the standard metric of a unit 2-sphere and $\ell \in \mathbb{R}$ is an affine parameter along the complete null geodesic forming $\mathscr{I}^+$ itself with tangent vector $n = \partial/\partial\ell$. In these coordinates one has finally the triple $(\mathscr{I}^+, q_B, n_B) \doteq (\mathbb{R} \times S^2, dS^2, \partial/\partial\ell)$.
We conclude this section with the definition of Bondi-Metzner-Sachs group.

Definition 1.3.8. The *Bondi-Metzner-Sachs group* $G_{\mathscr{I}}$ is the group of diffeomorphisms of $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$ which preserves the universal structure of $\mathscr{I}^+$, *i.e.*

$(\varphi(\mathscr{I}^+), \varphi^* q, \varphi^* n)$ differs from $(\mathscr{I}^+, q, n)$ at most by gauge transformation

$$\mathscr{I}^+ \mapsto \mathscr{I}^+, \qquad q \mapsto \omega^2 q, \qquad n \mapsto \omega^{-1} n$$

To give an explicit representation of $G_{\mathscr{I}}$ we need a suitable coordinate frame on $\mathscr{I}^+$. Having fixed the triple $(\mathscr{I}^+, q_B, n_B)$ one is still free to select an arbitrary coordinate frame on the sphere and, using the parameter u of the integral curves of n_B to complete the coordinate system, one is free to fix the origin of u depending on $\varsigma, \overline{\varsigma}$ generally. Taking advantage of the stereographic projection one may adopt complex coordinates $(\varsigma, \overline{\varsigma})$ on the (Riemann) sphere, $\varsigma = e^{i\varphi}\cot(\vartheta/2)$, ϑ, φ being the usual spherical coordinates.

Definition 1.3.9. Coordinates $(\ell, \varsigma, \overline{\varsigma})$ on $\mathscr{I}^+$ define a *Bondi frame* when $\varsigma, \overline{\varsigma} \in \mathbb{C} \times \mathbb{C}$ are complex stereographic coordinates on S^2 , $\ell \in \mathbb{R}$ (with the origin fixed arbitrarily) is the affine parameter of the integral curves of n and $(\mathscr{I}^+, q, n) = (\mathscr{I}^+, q_B, n_B)$.

In this frame the set $G_{\mathscr{I}}$ is nothing but $SO_{\uparrow}(3,1) \ltimes C^\infty(S^2)$, and $(\Lambda, f) \in SO_{\uparrow}(3,1) \ltimes C^\infty(S^2)$ acts on $\mathscr{I}^+$ as

$$\ell \to \ell' \doteq K_\Lambda(\varsigma, \overline{\varsigma})(\ell + f(\varsigma, \overline{\varsigma}))$$

$$\varsigma \to \varsigma' \doteq \Lambda\varsigma \doteq \frac{a_\Lambda \varsigma + b_\Lambda}{c_\Lambda \varsigma + d_\Lambda}$$

$$\overline{\varsigma} \to \overline{\varsigma}' \doteq \Lambda\overline{\varsigma} \doteq \frac{\overline{a}_\Lambda \overline{\varsigma} + \overline{b}_\Lambda}{\overline{c}_\Lambda \overline{\varsigma} + \overline{d}_\Lambda}$$

$$K_\Lambda(\varsigma, \overline{\varsigma}) \doteq \frac{(1 + \varsigma\overline{\varsigma})}{(a_\Lambda \varsigma + b_\Lambda)(\overline{a}_\Lambda \overline{\varsigma} + \overline{b}_\Lambda) + (c_\Lambda \varsigma + d_\Lambda)(\overline{c}_\Lambda \overline{\varsigma} + \overline{d}_\Lambda)} \quad \text{and} \quad \Pi^1(\Lambda) = \begin{bmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{bmatrix}$$

$\Pi : SL(2, \mathbb{C}) \to SO_{\uparrow}(3,1)$ is a surjective covering homomorphism. The matrix of coefficients a_Λ, b_Λ c_Λ, d_Λ is thus an arbitrary element of $SL(2, \mathbb{C})$ determined by Λ up on overall sign. $G_{\mathscr{I}}$ can be viewed as the semidirect product of $SO_{\uparrow}(3,1)$ and the Abelian group $C^\infty(S^2)$. The elements of this subgroup are called *supertranslations.*

To conclude this section, we introduce some terms which will be often used in the following in order to specify the support properties of the most relevant operators in the study of hyperbolic equations on globally hyperbolic spacetimes.

Definition 1.3.10. Let $\mathcal{M}$ be a globally hyperbolic spacetime and consider a region $\Omega \subseteq \mathcal{M}$. We say that Ω is:

- *spacelike-compact* if there exists a compact subset $K \subseteq \mathcal{M}$ such that $\Omega \subseteq J_{\mathcal{M}}(K)$;
- *future-(past-) compact* if its intersection with the causal past (future) of any point is compact, namely if $\Omega \cap J^+_{\mathcal{M}}(p)$ $\left(\Omega \cap J^+_{\mathcal{M}}(p)\right)$ is compact for each $p \in \mathcal{M}$;
- *timelike-compact* if it is both future- and past-compact.

1.4 Distribution on Manifold

The history of the theory of distributions is closely connected with the theory of differential partial equations. In this section we first recall the notion of distributions on $\mathbb{R}^n$ and then we remark on their generalization on arbitrary manifolds. Our treatment is based on [39] for more information see [40] and [41].

1.4.1 Distribution on $\mathbb{R}^n$

Let Ω be an open set in the real $n-$dimensional space $\mathbb{R}^n$ and let u be a continuous function defined in Ω. By *support* of u, denoted by *supp*(u), we mean *supp*$(u) \doteq \{x; x \in \Omega, u(x) \neq 0\}$. The support is thus the smallest closed subset of Ω outside where u vanishes.

Definition 1.4.1. By $\mathcal{E}(\Omega) \doteq C^k(\Omega)$, $0 \leq k \leq \infty$ we denote the set of all $k-$differentiable functions u defined in Ω and by $\mathcal{D}(\Omega) \doteq C_0^k(\Omega)$ we denote the set of all functions in $\mathcal{E}(\Omega)$ with compact support. The elements of $\mathcal{D}(\Omega)$ are called *test functions*.

$\mathcal{D}(\Omega)$ is a real vector space. It can be given a Fréchet topology by defining the limit of a sequence of its elements. A sequence $f_k \in \mathcal{D}(\Omega)$ is said to converge to $f \in \mathcal{D}(\Omega)$ if the following two conditions are satisfied:

(i) there exists a compact set $K \subset \Omega$ containing the supports of all f_k

$$\bigcup_k \mathrm{supp}(f_k) \subset K$$

(ii) for each multiindex α, the sequence of partial derivatives $\partial^\alpha f_k$ tends uniformly to $\partial^\alpha f$

With this definition, $\mathcal{D}(\Omega)$ becomes a complete locally convex topological vector space satisfying the Heine–Borel property.

Definition 1.4.2. A linear map u on $\mathcal{D}(\Omega)$ is called a *distribution* if there exists a positive constant C and an integer k such that for all $f \in \mathcal{D}(\Omega)$ it holds

$$|u(f)| \leq C \max_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha f(x)|$$

The set of all distributions in Ω is denoted by $\mathcal{D}(\Omega)'$ and it is dual to $\mathcal{D}(\Omega)$.

Example 1.9. Let $n = 1$ and $\Omega = (0, 1)$. The linear map

$$u(f) = \sum_1^\infty f^k \frac{1}{k}$$

lies in $\mathcal{D}(\Omega)'$.

An equivalent form of Definition 1.4.2 is given by the following theorem, which we shall not prove.

Theorem 1.4.3. *A linear map u on $\mathcal{D}(\Omega)$ is a distribution if and only if $u(f_k) \to 0$ when $k \to \infty$ for every sequence $f_k \in \mathcal{D}(\Omega)$ such that*

(i) $\partial^\alpha f_k \to 0$ uniformly when $j \to 0$, for every multiindex α

(ii) there exist a fixed compact subset of Ω containing the supports of all f_k

A sequence satisfying the above properties is said to converge to 0 in $\mathcal{D}(\Omega)$.

Any locally integrable function $f \in L^1_{loc}(\Omega)$ defines a distribution in $\mathcal{D}(\Omega)'$ as:

$$\mathcal{D}(\Omega) \ni f \mapsto \int_\Omega |f(x)| dx.$$

Theorem 1.4.4. *Let $u \in \mathcal{D}(\Omega)$. If $supp(u)$ is a compact subset of Ω, then u has a finite order $N < \infty$. Let $K \subset V \subset \Omega$, where V is an open set. Then there exist finitely many functions $f_\beta \in \Omega$ with supports in V such that:*

$$u(f) = \sum_\beta (-1)^{|\alpha|} \int_\Omega f_\alpha(x)(\partial^\alpha u)(x) dx \qquad f \in \mathcal{D}(\Omega)$$

This theorem, proven in [42], justifies the notation used commonly in physics, where the evaluation of a distribution on a test function is written as an integral. However this characterization must be taken with care, since it is in general not possible to write a distribution U as a sum of measures with supports contained in the support of u.

A number of operations on functions can be extended to distributions. The most important of which is differentiation. If $u(x)$ is a C^k function, then both u and its derivatives $\partial_\alpha u$ can be identified with distributions.
For any $f \in \mathcal{D}$

$$(\partial_\alpha u, f) \doteq \int f(x)\partial_\alpha u(x) dx = \int (\partial_\alpha(u(x)f(x)) - u(x)\partial_\alpha f(x)) dx$$

As $u(x)f(x)$ has compact support, $\int \partial_\alpha(u(x)f(x))dx$ vanishes. Hence

$$(\partial_\alpha u, f) = -\int u(x)\partial_\alpha f(x) dx = -(u, \partial_\alpha f)$$

The derivative of a distribution is defined by generalizing this identity

Theorem 1.4.5. *If $u \in \mathcal{D}'$, then the linear forms*

$$(\partial_\alpha u, f) = -(u, \partial_\alpha f)$$

are distributions and are the derivatives of u. If $u \in C^1$, then its distributional derivatives are identical with its classical derivatives. The map $u \mapsto \partial_\alpha u$ are continuous maps $\mathcal{D}' \to \mathcal{D}'$

By iteration, one obtains the derivatives of higher order as

$$(\partial^\alpha u, f) = (-1)^{|\alpha|}(u, \partial^\alpha f).$$

If $u = f$, a locally integrable function, then

$$(\partial^\alpha f, f) = (-1)^{|\alpha|} \int f(x)\partial^\alpha f(x)dx.$$

Conversely, any distribution can be written locally in this form

Theorem 1.4.6 (Structure theorem)**.** *Let $u \in \mathcal{D}'$ and let Ω be an open set with compact closure. Then there exists a continuous function $f(x)$ and a multiindex α such that $u = \partial^\alpha$ in Ω.*

Theorems 1.4.5 and 1.4.6, whose proof can be found in [39], are fundamental to define a linear differential operator of order m.

Definition 1.4.7. Let $\{a^\alpha(x)\}_{|\alpha|\leq m}$ be a set of C^∞ functions and u a distribution. A continuous linear map $P : \mathcal{D}' \to D'$

$$\mathcal{D}' \ni u \to Pu = \sum_{|\alpha|\leq m} a^\alpha \partial^\alpha u$$

is called a *linear differential operator of order m.*

Explicitly, for all $f \in \mathcal{E}$ the distribution Pu is

$$(Pu, f) = (u, P^* f)$$

where

$$P^* f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha(a^\alpha f)$$

is another mth order differential operator, called the *formal adjoint* of P. The formal adjoint of P^* is P itself.

Now we discuss the singularity structure of distributions.

Definition 1.4.8. The *singular support* of $u \in \mathcal{D}(\Omega)'$, *sing supp*(u), is the smallest closed subset $\mathcal{O}$ such that $u|_{\Omega\setminus\mathcal{O}} \in \mathcal{E}(\Omega \setminus \mathcal{O})$.

If a distribution has a nonempty singular support we can give a further characterization of its singularity structure by specifying the direction in which it is singular. This is exactly the purpose of the definition of a wave front set.

Definition 1.4.9. For a distribution $u \in \mathcal{D}(\Omega)'$ the *wavefront set* $\mathrm{WF}(u)$ is the complement in $\Omega \times \mathbb{R}^n \setminus \{0\}$ of the set of points $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ such that there exist

(i) a function $f \in \mathcal{D}(\Omega)$ with $f(x) = 1$

(ii) an open conic neighbourhood $\mathcal{C}$ of ξ, with

$$\sup_{\xi\in\mathcal{C}}(1+|\xi|)^N|\widehat{f}\cdot(\xi)|<\infty \qquad \forall N\in\mathbb{N}_0$$

Introducing the concept of regular direction, it is possible to recast the Definition 1.4.9 so to give a more "physical" interpretation.

Definition 1.4.10. A neighborhood $\mathcal{C}$ of $x_0 \in \mathbb{R}^n$ is called *conic* if $x \in \mathcal{C}$ implies $\lambda x \in \mathcal{C}$ for all $\lambda \in (0,\infty)$. Let be $u \in \mathcal{D}'(\Omega)$ a distribution, a pair $(x,\xi) \in \Omega\times\mathbb{R}^n\setminus\{0\}$ is a *regular direction* for u if there exists $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) \neq 0$, a conic neighbourhood $\mathcal{C}$ of ξ and constants $C_N, N \in \mathbb{N}$ such that

$$|\widehat{\varphi u}(\xi)| < \frac{C_N}{1+|\xi|^N} \qquad \forall\, k\in\mathcal{C}, N\in\mathbb{N}$$

$\widehat{\varphi u}$ is said to be rapidly decreasing as $k\to\infty$

Definition 1.4.11. Let be $u \in \mathcal{D}'(\Omega)$ a distribution. The *wavefront set* of u is defined to be

$$WF(u) \doteq \big\{(x,\xi)\in\Omega\times\mathbb{R}^n\setminus\{0\} \mid (x,\xi) \text{ is not a regular direction for } u\,\big\}$$

With Definition 1.4.11, we have given the idea of the wavefront set as the set of points and directions along which a distribution takes singular values.

Example 1.10. Let $u \in C_0^\infty(\mathbb{R}^n)$ be an everywhere smooth function. Every pair $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$ must be regular, so that

$$WF(u)=\emptyset.$$

The great advantages one gains in using wavefront set are due to its properties.

Theorem 1.4.12. *Let be $u\in\mathcal{D}'(\mathbb{R}^n)$ a distribution, then follow:*

(i) if u is smooth, it holds $WF(u)=\emptyset$,

(ii) for every $u,v\in\mathcal{D}'(\mathbb{R}^n)$ and $\forall\alpha,\beta\in\mathbb{C}$ it holds

$$WF(\alpha u+\beta v)\subseteq WF(u)\cup WF(v)$$

,

(iii) if P is an arbitrary partial differential operator it holds $WF(Pu)\subset WF(u)$,

(iv) Let $U,V\subset\mathbb{R}^m$, let $u\in\mathcal{D}'(V)$ and let $\chi: U\to V$ be a diffeomorphism. The pull-back $\chi^*(u)$ of u defined by $\chi^*u(f)=u(\chi_*f)$ for all $f\in\mathcal{D}(U)$ fulfils

$$WF(\chi^*u)=\chi^*WF(u)\doteq\{(\chi^{-1}(x),\chi^*\xi)|(x,\xi)\in WF(u)\}$$

where χ^*k denotes the push-forward of χ in the sense of cotangent vectors, *(v) let be $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$ two distributions and let*

$$WF(u_1) \oplus WF(u_2) \doteq \{(x, \xi_1 + \xi_2) \mid (x, \xi_1) \in WF(u_1) \text{ and } (x, \xi_2) \in WF(u_2)\}$$

if $\xi_1 + \xi_2 \neq 0$, the product $u_1 u_2$ is a well-defined distribution in $\mathcal{D}'(\mathbb{R}^n)$ and it holds that

$$WF(u_1 u_2) \subseteq WF(u_1) \cup WF(u_2) \cup \big(WF(u_1) \oplus WF(u_2)\big).$$

Moreover, if $u_1, u_2 \in C^\infty(\mathbb{R}^n)$ it reduces to the standard point-wise product of smooth functions.

Condition (iv) establishes that the wave front set transforms covariantly under diffeomorphisms as an element of $T^*\mathbb{R}^m$, and we can extend its definition to distributions on general curved manifolds $\mathcal{M}$ by patching together wave front sets in different coordinate gluing of $\mathcal{M}$.

1.4.2 Distribution on $\mathcal{M}$

Before introducing a theory of distributions on manifolds, it is useful to define the concept of bundle and, in particularly, of vector bundle. For more information see [43]

Definition 1.4.13. A *bundle* is a triple $(E, \pi, \mathcal{M}) \doteq \xi$, where $\pi : E \to \mathcal{M}$ is a map. The space $\mathcal{M}$ is called the *base space*, the space E is called the *total space*, and the map π is called *projection* of the bundle. For each $q \in \mathcal{M}$ the space $F_q = \pi^{-1}(q)$ is called the *fibre* of the bundle over $q \in \mathcal{M}$.

Intuitively, one thinks of a bundle as a union of fibres $\pi^{-1}(q)$ for $q \in \mathcal{M}$ parametrized by $\mathcal{M}$ and "glued together" by the topology of the space E.
A vector bundle is a bundle with an additional vector space structure on each fibre.

Definition 1.4.14. A k-dimensional *vector bundle* ξ over F is a bundle $(E, \pi, \mathcal{M})$ together with the structure of a k-dimensional vector space over F on each fibre $\pi^{-1}(q)$ such that the following local triviality condition is satisfied:

(i) each point $q \in \mathcal{M}$ has an open neighbourhood U and a $U-$isomorphism $h : U \times F^k \to \pi^{-1}(U)$ such that the restriction $q \times F^k \to \pi^{-1}(q)$ is a vector space isomorphism for each $q \in \mathcal{M}$.

An F-vector bundle is called *real vector bundle* if $F = \mathbb{R}$ and a *complex vector bundle* if $F = \mathbb{C}$.

Now let $\mathcal{M}$ be a manifold equipped with a smooth volume density dV, a Riemannian metric g and consider a real or complex vector bundle ξ. The space of compactly supported smooth sections valued in E will be denoted by $\mathcal{D}(\mathcal{M}, E)$.

We equip E and $T^*\mathcal{M}$ with connections, both denoted by ∇. They induce counterparts on the tensor bundles $\otimes_k T^*\mathcal{M} \otimes E$. For a continuously differentiable section $f \in \mathcal{E}(\mathcal{M}, E)$ the covariant derivative is a continuous section in $T^*\mathcal{M} \otimes E$, $\nabla f \in \mathcal{E}(\mathcal{M}, \otimes_k T^*\mathcal{M} \otimes E)$.
For a subset $U \subset \mathcal{M}$ and $f \in C^k(\mathcal{M}, E)$, we define the C^k-*norm* as

$$\| f \|_{C^k(U)} \doteq \max_{\mu=0,\dots,k} \sup_{q \in U} |\nabla^\mu f(q)|.$$

If U is compact, then different choices of the metrics and of the connections yield equivalent norms $\| \cdot \|_{C^k(U)}$.
The elements of $\mathcal{D}(\mathcal{M}, E)$ are referred to as test sections valued in E. We define a notion of convergence of test sections.

Definition 1.4.15. Let $f, f_n \in \mathcal{D}(\mathcal{M}, E)$. We say that the sequence $(f_n)_n$ converges to f in $\mathcal{D}(\mathcal{M}, E)$ if the following two conditions are satisfied:

(i) there exists a compact set $K \subset \mathcal{M}$ such that the supports of all f_n are contained in K, *i.e.*, $supp(f_n) \subset K$ for all n;

(ii) the sequence $(f_n)_n$ converges to f in all C^k -norms over K, *i.e.*, for each $k \in \mathbb{N}$

$$\| f - f_n \|_{C^k(K)} \xrightarrow[n\to\infty]{} 0.$$

We fix a finite-dimensional $\mathbb{K}$-vector space W . Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex.

Definition 1.4.16. A $\mathbb{K}$-linear map $F : \mathcal{D}(\mathcal{M}, E^*) \to W$ is called a distribution in E with values in W if it is continuous in the sense that for all convergent sequences $f_n \to f$ in $\mathcal{D}(\mathcal{M}, E^*)$ one has $F[f_n] \to F[f]$. We write $\mathcal{D}'(\mathcal{M}, E, W)$ for the space of all W -valued distributions in E.

We provide two important examples of distributions.

Example 1.11. Pick a bundle $E \to \mathcal{M}$ and a point $q \in \mathcal{M}$. The delta-distribution δ_q is an E^*-valued distribution in E. For $f \in \mathcal{D}(\mathcal{M}, E^*)$ it is defined by

$$\delta_q[f] = f(q).$$

Example 1.12. Every locally integrable section $\varphi \in L^1_{loc}(\mathcal{M}, E)$ can be interpreted as a $\mathbb{K}$-valued distribution in E by setting for any $f \in \mathcal{D}(\mathcal{M}, E^*)$

$$\varphi[f] \doteq \int_{\mathcal{M}} \varphi(f) dV.$$

All definitions and properties mentioned in the previous section extend easily to manifolds, since locally they are isomorphic to $\mathbb{R}^n$.

1.5 Wave Equation

We start with our study of wave operators, based on [44], that is consider an equation of the form $Pu = f$ where P is a normally hyperbolic (see below) operator acting on sections in a vector bundle. Solving wave equations on all of the Lorentzian manifold $\mathcal{M}$ is, in general, possible only if $\mathcal{M}$ is globally hyperbolic.

Definition 1.5.1. Let $\mathcal{M}$ be a Lorentzian manifold and let $E \to \mathcal{M}$ be a $\mathbb{K}$ vector bundle. A linear differential operator $P : \mathcal{E}(\mathcal{M}, E) \to \mathcal{E}(\mathcal{M}, E)$ of second order will be called *normally hyperbolic* if its principal symbol is given by the metric:

$$\sigma_P(X) = -\langle X, X\rangle \cdot id_{E_X} \qquad \forall x \in \mathcal{M}, \forall X \in T^*\mathcal{M}.$$

In other words if we choose local coordinates on $\mathcal{M}$ and a local trivialization of E, then

$$P = -\sum_{\mu\nu}^{n} g^{\mu\nu}\partial_\mu\partial_\nu + \sum_{\mu}^{n} A_\mu\partial_\mu + B$$

where A_μ and B are matrix-valued coefficients depending smoothly on local coordinates.

We want to find operators which are inverses of P when restricted to suitable spaces of sections. We will see that existence of such operators is basically equivalent to the existence of fundamental solutions.

Proposition 1.5.2. *Let $\mathcal{M}$ be a time oriented connected Lorentzian manifold and P be a normally hyperbolic operator acting on sections in a vector bundle E over $\mathcal{M}$. Then there exist two linear maps $G_\pm : \mathcal{D}(\mathcal{M}, E) \to C^\infty(\mathcal{M}, E)$ satisfying*

(i) $P \circ G_\pm = G_\pm \circ P = id_{\mathcal{D}(\mathcal{M},E)}$

(ii) $supp(G_\pm f) \subset J_\pm^{\mathcal{M}}(supp(f))$ *for all* $f \in \mathcal{D}(\mathcal{M}, E)$

called advanced (retarded) Green operator for P.

Not all linear differential operators admit Green operators, but, indeed, those who do, are of primary physical relevance, since, as we shall show in the next section, one can associate to them a distinguished algebra of observables, built out of $G_\pm$. For this reason we can encompass these special operators in a specific class.

Definition 1.5.3. Let $\mathcal{M}$ be a time oriented connected Lorentzian manifold and P be a normally hyperbolic operator acting on sections of a vector bundle E over $\mathcal{M}$. P is called *Green-hyperbolic* if it admits advanced and retarded Green operators.

Starting from G_+ and G_- it is also possible to define an new operator

Definition 1.5.4. Let $\mathcal{M}$ be a timeoriented connected Lorentzian manifold, P be a Green-hyperbolic operator acting on sections in a vector bundle E over $\mathcal{M}$ and let $G_\pm$ be the advanced and retarded Green operators for P. Then $G = G_+ - G_-$ is the *causal propagator* for P.

The causal propagator entails the full characterization of the space of solutions of the wave equation $Pu = 0$. G satisfies properties similar to (i) and (ii) of Proposition 1.5.2

(i) $P \circ G = 0$

(ii) $supp(Gf) \subset J_{\mathcal{M}}(supp(f))$ for all $f \in \mathcal{D}(\mathcal{M}, E)$

We outline the last result of this section which has an important physical interpretation.

Proposition 1.5.5. *Let $\mathcal{M}$ be a timeo riented connected Lorentzian manifold, P be a Green-hyperbolic operator acting on smooth sections of a vector bundle E over $\mathcal{M}$ and let G be the causal propagator. The space $\mathcal{S}(\mathcal{M})$ of solutions with spacelike-compact support of the equation $Pu = 0$ on $\mathcal{M}$ is isomorphic to the quotient space $\mathcal{D}(\mathcal{M}, E)/P(\mathcal{D}(\mathcal{M}, E))$ via the map $I : \mathcal{D}(\mathcal{M}, E)/P(\mathcal{D}(\mathcal{M}, E)) \to \mathcal{S}(\mathcal{M})$ that $[f] \mapsto Gf$.*

This proposition guarantees the possibility to endow $\mathcal{S}(\mathcal{M})$ with a symplectic form, which allows to use an holographic principle to construct Hadamard states for the linearized Einstein equations.

Chapter 2

Quantum Field Theory in curved spacetimes: an algebraic approach

Contents

We apply the analytical theory of wave equations and develop a few mathematical basic tools necessary to introduce the quantization of fields. These are henceforth meant as sections in vector bundles, which dynamics is ruled by a suitable wave equation. There are two different approaches.
In the more traditional one a quantum field is constructed as a distribution satisfying the wave equation in a weak sense, via a mode expansion. These distributions take value in the selfadjoint operators on the Fock space realized using the creation and annihilation operators associated to the modes, up to the selection of a cyclic vector which lies in the kernel of all annihilation operators. The construction of the Fock space is however crucially dependent upon the choice of modes. For the case of Minkowski spacetime, the resulting Hilbert space is the unique Fock space whose vacuum state vector is covariant under the action of the Poincaré group: Heuristically, the physical interpretation is that all inertial observers will agree on what is to be regarded as the vacuum state. The fact that we have covariance under the action of the Poincaré group entails that it is meaningful to talk about particles, but for curved spacetimes this is no longer the case.

A general curved spacetime will possess no symmetries that can be used to pick out a preferred representation, through invariance of a vacuum state, and thus the particle interpretation becomes difficult. It seems that for a quantum field theory on curved spacetimes the approach of local quantum physics is more appropriate. The idea is to associate to a physical system a suitable algebra of observables which encodes the relevant physical properties of isotony, locality and covariance. Once the algebra has been constructed and once a continuous, linear, positive functional on the algebra has been assigned, it is possible to identify via GNS theorem the observables as linear operators acting on a suitable Hilbert space. The choice of the algebra is not random but it must fulfil a set of axioms, first formulated by Haag and Kastler in [45]. Although they assumed to work in Minkowski spacetime, it is possible to generalize these axioms to a generic spacetime [46].
As said by mathematical physicist Edward Nelson: *"quantization is a mystery, but second quantization is a functor"*. Using the formalism of category theory, it is possible to construct a local covariant field as a functor from the category of globally hyperbolic Lorentzian manifolds equipped with a formally selfadjoint normally hyperbolic operator to the category of C^* -algebras. This functorial interpretation of locally covariant quantum field theory on curved spacetimes has been introduced in [12], [47] and [48].

Before sketching the algebraic approach, it is useful to introduce the $*$-algebras. We follow [44] and for more information see [49] and [50].

2.1 C^*-algebras

Definition 2.1.1. Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra, let $\| \cdot \|$ be a norm on the $\mathbb{C}$-vector space $\mathcal{A}$, and let $* : \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$, be a $\mathbb{C}$-antilinear map. Then $(\mathcal{A}, \| \cdot \|, *)$ is called a *unital C^*-algebra*, if $(\mathcal{A}, \| \cdot \|)$ is complete with respect to the norm for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the following conditions hold true:

(i) $\exists\, \mathbb{1} \in \mathcal{A}$ s.t. $a\mathbb{1} = \mathbb{1}a = a$

(ii) $a^{**} = a$

(iii) $(ab)^* = b^* a^*$

(iv) $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$

(v) $\| ab \| \leq \| a \| \| b \|$

(vi) $\| a^* \| = \| a \|$

(vii) $\| a^* a \| = \| a \|^2$

The conditions (i) to (iv) define a *unital $*$-algebra*.

Example 2.1. Let $\mathcal{M}$ be a differentiable manifold. We call

$$\mathcal{A} \doteq \mathcal{D}(\mathcal{M}) \doteq C_0^\infty(\mathcal{M})$$

$\mathcal{D}(\mathcal{M})$ the *algebra of smooth functions of compact support.* All $f \in \mathcal{D}(\mathcal{M})$ are bounded and we may define a norm and a $*$-operation as follows:

$$\| f \| \doteq \sup_{x \in \mathcal{M}} |f(x)|$$

$$f^*(x) \doteq \overline{f(x)}$$

$(\mathcal{D}(\mathcal{M}), \| \cdot \|, *)$ satisfies all axioms of a commutative C^*-algebra except for $(\mathcal{A}, \| \cdot \|)$ not being complete.

Definition 2.1.2. An element a of a C^*-algebra is called *selfadjoint* if $a = a^*$.

Like any algebra a C^*-algebra $\mathcal{A}$ has at most one unit $\mathbb{1}$. Let b be another unit, then

$$\mathbb{1} = \mathbb{1} \cdot b = b$$

For all $a \in \mathcal{A}$ we have

$$\mathbb{1}^* a = (\mathbb{1}^* a)^{**} = (a^* \mathbb{1}^{**})^* = (a^* \mathbb{1})^* = a^{**} = a$$

and similarly it descends $a\mathbb{1}^* = a$. Thus $\mathbb{1}^*$ is also a unit. By uniqueness $\mathbb{1} = \mathbb{1}^*$,*i.e.*, the unit is selfadjoint. Furthermore,

$$\| \mathbb{1} \| = \| \mathbb{1}^* \mathbb{1} \| = \| \mathbb{1} \|^2$$

hence $\| \mathbb{1} \| = \mathbb{1}$ or $\| \mathbb{1} \| = 0$. In the second case $\mathbb{1} = 0$ and therefore $\mathcal{A} = 0$. Hence we may (and will) from now on assume that $\| \mathbb{1} \| = \mathbb{1}$.

Definition 2.1.3. Let $\mathcal{A}$ be a C^*-algebra with unit. An element $a \in \mathcal{A}$ is called

(i) *normal* if $aa^* = a^*a$

(ii) *an isometry* if $a^*a = \mathbb{1}$

(iii) *unitary* if $a^*a = aa^* = \mathbb{1}$

In particular, sefladjoint elements are normal. In a commutative algebra, all elements are normal.

Definition 2.1.4. Let $\mathcal{A}$ and $\mathcal{B}$ a C^*-algebras. An algebra homomorphism

$$\pi : \mathcal{A} \to \mathcal{B}$$

is called a $*$-*morphism* if for all $a \in A$ we have

$$\pi(a^*) = \pi(a)^*$$

A map $\pi : \mathcal{A} \to \mathcal{A}$ is called $*$-*automorphism* if it is an invertible $*$-morphism.

2.2 The algebra of observables

Since our ultimate goal is the quantization of a free field theory, our starting point will still be the field $\Phi : \mathcal{M} \to \mathbb{R}$ which solves a linear field equation $P\Phi = 0$, where P is an hyperbolic operator (see Definition 1.5.1).
Hence we call $\mathcal{S}(\mathcal{M})$ the space of solutions built out of smooth, compactly supported initial data:

$$\mathcal{S}(\mathcal{M}) = \{\Phi_f \in \mathcal{E}(\mathcal{M}) \,|\, \exists\, f \in \mathcal{D}(\mathcal{M}) \text{ and } \Phi_f = G(f)\}$$

where G is the causal propagator (see Definition 1.5.4). $\mathcal{S}(\mathcal{M})$ can be equipped with a bilinear, skew-symmetric non degenerate form $\sigma : \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M}) \to \mathbb{R}$. In this way $\mathcal{S}(\mathcal{M})$ becomes a symplectic space and it can be interpreted as the phase space of the theory. Classical observables are maps $F_f : \mathcal{S}(\mathcal{M}) \to \mathbb{R}$, or equivalently from $\mathcal{D}(\mathcal{M})$ into $\mathbb{R}$ if we recall the properties of G, the causal propagator. In between the classical observables of especial relevance are those of the form

$$F_f(\Phi) \doteq \Phi(f) = \int_{\mathcal{M}} d^4x \sqrt{|g|}\Phi(x) f(x)$$

whose Poisson brackets for $f \neq g \in \mathcal{D}(\mathcal{M})$ yield

$$\{F_f, F_g\} = G(f, g) \tag{2.2.1}$$

As a matter of fact our quest will be to find a way to collect all our observables into an algebra $\mathcal{A}$, or in other words we look for a map $\varphi : \mathcal{D}(\mathcal{M}) \to \mathcal{A}$ where the symbol φ is here used without any reference a priori to the field, which was previously indicated as Φ. The choice of $\mathcal{A}$ is not random, but, actually, it must fulfil the requirements of Definition 2.1.1: An example is represented by the Borchers-Uhlmann algebra. For a more detailed treatment see [20].

Definition 2.2.1. The *Borchers-Uhlmann algebra* $\mathscr{A}(\mathcal{M})$ is defined as

$$\mathscr{A}(\mathcal{M}) \doteq \bigoplus_{n=1}^{\infty} \mathcal{D}(\mathcal{M})^{\otimes n} \qquad \text{and} \qquad \mathcal{D}(\mathcal{M})^0 \doteq \mathbb{C}$$

The algebra $\mathscr{A}(\mathcal{M})$ satisfies the following properties:

(i) there exists a product defined by linear extension of the product of $\mathcal{D}(\mathcal{M})^n$;

(ii) there exists a $*-$operation defined by the antilinear extension of $|f^*|(x_1, \ldots, x_n) = \overline{f}(x_n, \ldots, x_1)$;

(iii) there exists a sequence $\{f_k\}_k = \{\otimes_n f_k\}_k$ of elements in $\mathscr{A}(\mathcal{M})$ that converges to $f = \otimes_n f^{(n)}$ if $f_k^{(n)}$ tends to $f^{(n)}$ for all n in the locally convex topology of $\mathcal{D}(\mathcal{M})^n$. There exists moreover N such that $f_k = 0$ for all $n > N$ and all k

(iv) all elements of $\mathscr{A}(\mathcal{M})$ are finite linear combinations of multi-component test functions

The Borchers-Uhlmann algebra has no information on the dynamics of the underlying field. To solve this problem, one can quotient $\mathscr{A}(\mathcal{M})$ by an ideal $\mathscr{I}$ generated by elements of the form

$$-\imath G(f,g) \oplus (f \otimes g - g \otimes f) \quad \text{and} \quad Pf$$

Definition 2.2.2. The *field algebra* $\mathscr{F}(\mathcal{M})$ defined as

$$\mathscr{F}(\mathcal{M}) \doteq \mathscr{A}(\mathcal{M})/\mathscr{I}$$

is equipped with the product, $*$-operation and topology descending from $\mathscr{A}(\mathcal{M})$. If $\mathcal{O}$ is an open subset of $\mathcal{M}$, $\mathscr{F}(\mathcal{O})$ denotes the algebra obtained by allowing only test functions with support in $\mathcal{O}$

Lemma 2.2.3. *The field algebra $\mathscr{F}(\mathcal{M})$ fulfils the time-slice axiom: Let Σ be a Cauchy surface of $(\mathcal{M}, g)$ and let $\mathcal{O}$ be an arbitrary causally convex neighbourhood of Σ. Then $\mathscr{F}(\mathcal{O}) = \mathscr{F}(\mathcal{M})$.*

From the above discussion, one can infer that φ plays indeed the role of a field in the algebra and thus its symbol can be correctly exchanged with Φ.

The field algebra is not the only possible choice which fulfils the requirements of the Definition 2.1.1: There exists a second possibility, the Weyl algebra.

Definition 2.2.4. We call *Weyl algebra* $\mathfrak{W}(\mathcal{M})$ the unital C^*-algebra generated by the abstract symbols $W(\varphi)$, for all$\varphi \in \mathcal{S}(\mathcal{M})$ such that for all $\varphi, \psi \in \mathcal{S}(\mathcal{M})$ the following conditions are satisfied:

(i) $W(0) = 1$

(ii) $W(-\varphi) = W(\varphi)^*$

(iii) $W(\varphi) \cdot W(\psi) = e^{-\imath\sigma(\varphi,\psi)/2} W(\varphi + \psi)$.

The symplectic vector space $(\mathcal{S}(\mathcal{M}), \sigma)$ in which is defined a Weyl algebra $\mathfrak{W}(\mathcal{M})$ is commonly referred as *Weyl system*. $\mathfrak{W}(\mathcal{M})$ is unique up to $*$-isometries (see [49]).

As defined, the Weyl algebra cannot be read as map from $C_0^\infty(\mathcal{M})$ into suitable algebra elements. In order to solve this potential problem, recall that every $\Phi \in \mathcal{S}(\mathcal{M})$ can be written as $\Phi = G(f)$ for a suitable $f \in C_0^\infty(\mathcal{M})$ (see Proposition 1.5.2). Furthermore on account of (2.2.1) and of the subsequent identity, we also know that $\sigma(\Phi, \Psi) = G(f,g)$ where $\Phi = G(f)$ and $\Psi = G(g)$. In other words we have identified a $*$-isomomorphism which associates to each generator $W(\Phi)$ another generator $V([f])$ such that

$$V([f]^* = V(-[f]) \qquad \text{and} \qquad V([f])V([g]) = e^{\frac{\imath}{2}G([f],[g])} V([f] + [g])$$

for all $[f], [g] \in C_0^\infty(\mathcal{M})/J$ where $J \doteq \{f \sim g \text{ if } \exists h \in C_0^\infty(\mathcal{M}) \text{ fulfilling } f - g = Gh\}$

2.3 Haag-Kastler axioms

The algebraic approach was initially developed by Haag and Kastler in [45] as a way to lay down a collection of axioms which should be obeyed by an algebra, that describes the observables of a quantum field theory in Minkowski spacetime. One begins by considering the open relatively compact subsets of Minkowski spacetime to ensure that one is only taking into account those observables that can be measured within a finite region, such as a laboratory. This rules out global observables such as total energy and charge. For each such subset $\mathcal{O}$, a quantum field theory assigns an algebra $\mathcal{A}(\mathcal{O})$, which will contain all local observables that can be measured within the region $\mathcal{O}$. Each $\mathcal{A}(\mathcal{O})$ is assumed to be a C^*-algebra. The smallest C^*-algebra containing the union of all the algebras over all the regions with compact closure will be denoted by $\mathcal{A}$ and is known as the algebra of observables for the spacetime. To ensure that such a net describes the observables of a quantum field theory, certain extra conditions need to be imposed on it. These conditions are the Haag-Kastler axioms and we now state them in their modern form:

(i) *Isotony*: for all open sets $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{M}$, then $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')$ and the full $*$-algebra of local observables is defined as the union of all $\mathcal{A}(\mathcal{O})$, with $\mathcal{O} \subset \mathcal{M}$ contractible open bounded,

(ii) *Causality*: for all open sets $\mathcal{O}, \mathcal{O}' \subset \mathcal{M}$ spacelike separated $[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0$,

(iii) *Covariance*: for any isometry $h : \mathcal{O} \to \mathcal{O}'$ we can associate $\alpha_h : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\mathcal{O}')$ which maps $(\alpha_h V)(f) \doteq V(f \circ h^{-1})$.

The first condition ensures that if a measurement can be made in a region $\mathcal{O}$, which satisfies $O \subset \mathcal{O}'$, then that measurement can also be performed in the second region $\mathcal{O}'$. The second condition is where relativity enters: Measurements made in causally disconnected regions cannot influence each other. If one can extend all relations to include unbounded regions, which contain a Cauchy surface for the ambient spacetime, then a time-slice condition might hold true. The existence of a time-slice condition is important for making physical predictions: One can determine the state of system by examining its expectation values on all elements of the algebra. However, if one had to do this for elements of the algebra that were localised anywhere within the spacetime, then it would be totally impractical to obtain the state. Even if the time-slice condition holds, it is still impractical to know all expectation values in a whole slice. The final condition express the idea that a theory on a globally spacetime should be covariant under its isometry group. The preceding formulation has since been implemented by the methods of locally covariant quantum field theory due to Brunetti, Fredenhagen and Verch [48], which permit to generalized on all physically admissible spacetimes, using methods from category theory.

2.4 Category theory

Definition 2.4.1. A *category* $\mathfrak{C}$ consists of the following data:

(i) a collection $obj\mathfrak{C}$, whose elements will be called *objects of the category*,

(ii) a collection of morphisms between objects, *i.e.* $f : A \to B$, where $A, B \in obj\mathfrak{C}$. Furthermore it is required that for all $f : A \to B$ and for all $g : B \to C$ there exists $h : A \to C$ such that $h = g \circ f$, where $A, B, C \in obj\mathfrak{C}$

(iii) for every object A, there exist a morphism $\mathbb{1}_A \in \mathfrak{C}(A; A)$ called the *identity* on A.

These data are subject to the following requirements:

(i) *Associativity*: given any morphism $f \in \mathfrak{C}(A; B)$, $g \in \mathfrak{C}(B; C)$, $h \in \mathfrak{C}(C; D)$ it holds that $h \circ (g \circ f) = (h \circ g) \circ f$

(ii) *Identity*: given any morphism $f \in \mathfrak{C}(A; B)$, $g \in \mathfrak{C}(B; C)$, it holds $\mathbb{1}_B \circ f = f$ and $g \circ \mathbb{1}_B = g$

In the theory of categories, it is possible to define, also, a map between categories.

Definition 2.4.2. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ two categories. A *functor* $\mathcal{F}$ is a "map"

$$\mathcal{F} : \quad \mathfrak{C}_1 \to \mathfrak{C}_2$$

such that for all $A \in obj\mathfrak{C}_1$, there exists $B \in obj\mathfrak{C}_2$ such that $B = F(A)$ and for each morphism $f : A_1 \to A_2$ there exists $g : F(A_1) \to F(A_2)$, where $A_1, A_2 \in obj\mathfrak{C}_1$. In shorthand we write $g = F(f)$.

This paragraph was only a very brief introduction to category theory and for more information see [51], [52] [53].
The formalism of category theory is useful to construct a locally covariant quantum field theory according to [48] as a functor from a category of spacetimes to a category of suitable algebras. The first step in this construction is the definition of an appropriate category of spacetimes. We have already explained in section 1.3 that four-dimensional, oriented and time-oriented, globally-hyperbolic spacetimes are the physically sensible Lorentzian manifolds. It is therefore natural to take them as the objects of a category of spacetimes. Regarding the morphisms, one could think of various possibilities to select them among all possible maps between the spacetimes under consideration. However, to be able to emphasise the local nature of a quantum field theory, we shall consider isometric embeddings between spacetimes. Heuristically, the underling ideas is to require locality by asking that a quantum field theory on a "small" spacetime can be embedded into a larger spacetime without having information about the remainder of the larger spacetime. A sensible quantum field theory will depend moreover on the orientation and time-orientation and the causal structure of the underlying manifold, we should therefore only consider embeddings that preserve these structures. To this avail, we follow [48]. Let us now resume the above considerations in a definition.

Definition 2.4.3. The *category of spacetimes* $\mathfrak{Man}$ consists of the class of objects $obj(\mathfrak{Man})$ constituted by globally hyperbolic, four-dimensional, oriented and time-oriented spacetimes $(\mathcal{M}, g)$. Given two spacetimes $(\mathcal{M}_1, g_{\mathcal{M}_1})$ and $(\mathcal{M}_2, g_{\mathcal{M}_2})$ in $obj(\mathfrak{Man})$, the morphisms $\chi \in hom_{\mathfrak{Man}}((\mathcal{M}_1, g_{\mathcal{M}_1}), (\mathcal{M}_2, g_{\mathcal{M}_2}))$ are orientation and time orientation preserving isometric embeddings

$$\chi : (\mathcal{M}_1, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$$

such that:

(i) if $\gamma : [a, b] \to \mathcal{M}_2$ is any causal curve and $\gamma(a), \gamma(b) \in \chi(\mathcal{M}_1)$ then the whole curve must in the image $\chi(\mathcal{M}_1)$, *i.e.* $\gamma(t) \in \chi(\mathcal{M}_1)$ for all $t \in (a, b)$;

For any $\chi \in hom_{\mathfrak{Man}}((\mathcal{M}_1, g_{\mathcal{M}_1}), (\mathcal{M}_2, g_{\mathcal{M}_2}))$ and $\zeta \in hom_{\mathfrak{Man}}((\mathcal{M}_2, g_{\mathcal{M}_2}), (\mathcal{M}_3, g_{\mathcal{M}_3}))$ the composition rule $\zeta \circ \chi$ is defined as the composition of maps. Hence $\zeta \circ \chi : (\mathcal{M}_1, g_{\mathcal{M}_1}) \to (\mathcal{M}_3, g_{\mathcal{M}_3})$ is a well-defined map which is a diffeomorphism onto its range $\zeta(\chi(\mathcal{M}_1))$ and isometric; also property (i) is fulfilled, and hence $\zeta \circ \chi \in hom_{\mathfrak{Man}}((\mathcal{M}_1, g_{\mathcal{M}_1}), (\mathcal{M}_3, g_{\mathcal{M}_3}))$. The associativity of the composition rule follows that of the composition of maps. Each $hom_{\mathfrak{Man}}((\mathcal{M}, g_{\mathcal{M}}), (\mathcal{M}, g_{\mathcal{M}}))$ possesses a unit element, given by the identity map $id_{\mathcal{M}} : x \to x$, $x \in \mathcal{M}$.

Before concluding this section we give a last definition, which will be used in the following section.

Definition 2.4.4. The *category of C^*-algebra* $\mathfrak{Alg}$ consists of objects $obj(\mathfrak{Alg})$, all unital C^*-algebras, whose morphisms are faithful (injective) unit-preserving $*$-homomorphisms. Given $\alpha \in hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_2)$ and $\alpha' \in hom_{\mathfrak{Alg}}(\mathcal{A}_2, \mathcal{A}_3)$, the composition $\alpha' \circ \alpha$ is again defined as the composition of maps and it is seen to be an element in $hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_3)$. The unit element in $hom_{\mathfrak{Alg}}(\mathcal{A}, \mathcal{A}))$ is for any $\mathcal{A} \in Obj(\mathfrak{Alg})$ given by the identity map $id_A : \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \in \mathcal{A}$.

2.5 Axioms of Locally Covariant Theory

As anticipated at the end of section 2.3, a locally covariant theory incorporates in a local sense the principle of general local covariance of general relativity. So locally covariant quantum field theories will be described mathematically in terms of covariant functors between the categories of globally hyperbolic spacetimes and that of C^*-algebras. Moreover, locally covariant quantum fields can be described in this framework as natural transformations between certain functors. The usual Haag-Kastler framework can be regained from this approach as a special case. It is possible to resume in a three point the *axioms of locally covariant theory*:

(i) to each globally hyperbolic time-oriented spacetime $(\mathcal{M}, g)$ we associate a unital C^*-algebra $\mathcal{A}(\mathcal{M})$;

(ii) let $\chi : \mathcal{M} \to \mathcal{N}$ be an isometric embedding which preserves causal relations, then there exists an injective homomorphism $\alpha_\chi : \mathcal{A}(\mathcal{M}) \to \mathcal{A}(\mathcal{N})$;

(iii) let $\chi : \mathcal{M} \to \mathcal{N}$ and $\zeta : \mathcal{N} \to \mathcal{L}$ be isometric embeddings. Then $\alpha_{\zeta\circ\chi} = \alpha_\zeta \circ \alpha_\chi$;

The first three conditions characterize a quantum field theory as a covariant functor $\mathcal{F}$ from the category $\mathfrak{Man}$ to the category $\mathfrak{Alg}$. It is possible to add additional two conditions which correspond to causality and to the time-slice axiom.

(iv) let $\chi_i : \mathcal{M}_i \to \mathcal{N}$, $i = 1, 2$, be morphisms with causally disjoint images. Then the images of $\mathcal{A}(\mathcal{M}_1)$ and $\mathcal{A}(\mathcal{M}_2)$ represent independent subsystems of $\mathcal{A}(\mathcal{N})$ in the sense that the algebras $\mathcal{A}(\mathcal{M}_1)$ and $\mathcal{A}(\mathcal{M}_2)$ commute and $\mathcal{A}(\mathcal{M}_1)\otimes\mathcal{A}(\mathcal{M}_2) \mapsto \mathcal{A}(\mathcal{M}_1)\vee\mathcal{A}(\mathcal{M}_2)$ defines an isomorphism from the tensor product $\mathcal{A}(\mathcal{M}_1) \otimes \mathcal{A}(\mathcal{M}_2)$ to the algebra generated by $\mathcal{A}(\mathcal{M}_1)$ and $\mathcal{A}(\mathcal{M}_2)$;

(v) let $\chi : \mathcal{M} \to \mathcal{N}$ be a morphism such that its image contains a Cauchy surface of $\mathcal{N}$. Then α_χ is an isomorphism.

In quantum field theory fields are defined as distributions with values in the algebra of observables. They are required to transform covariantly under isometries of spacetime. At first sight, it seems that the latter requirement becomes moot on generic spacetimes. Furthermore, it seems to be difficult to compare fields which are defined on different spacetimes. Yet it turns out that the locally covariant framework offers the possibility for a new interpretation of the concept of fields.

Definition 2.5.1. A *locally covariant quantum field* is a natural transformation Φ between the functors $\mathcal{D}$ and $\mathcal{A}$, *i.e.* for any object $(\mathcal{M}, g)$ in $\mathfrak{Man}$ there exists a morphism $\Phi_{(\mathcal{M},g)} : \mathcal{D}(\mathcal{M}, g) \to \mathcal{A}(\mathcal{M})$ in $\mathfrak{Alg}$ such that for each given morphism $\chi \in hom_{\mathfrak{Man}}((\mathcal{M}, g_\mathcal{M}), (\mathcal{N}, g_\mathcal{N}))$ holds the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}(\mathcal{M}) & \xrightarrow{\Phi(\mathcal{M})} & \mathcal{A}(\mathcal{M}) \\ \chi_*\big\downarrow & & \big\downarrow\alpha_\chi \\ \mathcal{D}(\mathcal{N}) & \xrightarrow{\Phi(\mathcal{N})} & \mathcal{A}(\mathcal{N}) \end{array}$$

The commutativity of the diagram means that $\alpha_\chi \circ \Phi_\mathcal{M} = \Phi_\mathcal{N} \circ \chi_*$, *i.e.* the requirement of covariance for fields.

2.6 Algebraic states

The last step in the algebraic approach consist of representing the algebra of observable on a suitable Hilbert space: The quest to find a representation either of the Weyl or of the field algebra is the most difficult aspect of the algebraic approach to quantum field theory, even Minkowskian spacetime. A method to tackle to this problem was given by Gelfand, Naimark and Segal and their theorem represents one more important result of the whole algebraic approach. As a starting point, let us to define the concept of algebraic state.

Definition 2.6.1. We define an *algebraic state* of a $*$-algebra $\mathcal{A}$ a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ which satisfies the following properties:

(i) $\omega(\mathbb{1}) = 1$

(ii) $\omega(a^*a) \geq 0 \ \forall a \in \mathcal{A}$

Let us show that, whenever we represent a $*$-algebra on a Hilbert space via linear operators, we can automatically construct several states.

Lemma 2.6.2. *Let $\mathcal{A}$ be any topological $*$-algebra with an identity element and $\mathscr{H}$ a Hilbert space with scalar product $(\cdot,\cdot)$, such that there exists a strongly continuous representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$, where $\mathcal{D}$ is a dense subspace of $\mathscr{H}$, $\mathcal{L}(\mathcal{D})$ is the space of continuous linear operators on $\mathcal{D}$ and where $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. Then, for any $\psi \in \mathcal{D}$ of unit norm, the functional $\omega_\psi : \mathcal{A} \to \mathbb{C}$ defined as $\omega_\psi(a) \doteq (\psi, \pi(a)\psi)$ is a state on $\mathcal{A}$.*

Proof. Let $\psi \in \mathcal{D}$ be any element such that $\|\psi\|_{\mathscr{H}} = 1$. Let $\omega_\psi(a) \doteq (\psi, \pi(a)\psi)$. Per construction ω_ψ is linear and continuous since π is linear and strongly continuous. $\omega_\psi(\mathbb{1}) = 1$ follows from $\|\psi\|_{\mathscr{H}} = 1$ and $\pi(\mathbb{1}) = \mathrm{id}_{\mathcal{D}}$, π being a representation. To conclude we notice that

$$\omega_\psi(a^*a) \doteq (\psi, \pi(a^*a)\psi) = (\psi, \pi(a)^*\pi(a)\psi) = \|\pi(a)\psi\|^2_{\mathscr{H}} \geq 0,$$

where we exploited $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$. □

As we can directly infer from the proof, the assignment of $(\psi, \pi, \mathscr{H})$ allows to construct a state in the algebraic sense and the overall result depends strictly on the choice of ψ. Notice that one is not forced to choose a single element of norm 1, but it is possible to consider a linear combination of vectors in $\mathscr{H}$, say $\sum_i \psi_i$ such that $\| \sum_i \psi_i \| = 1$. The whole discussion is motivated by following theorem[1]

Theorem 2.6.3 (GNS)**.** *Let ω be a state on a topological $*$-algebra $\mathcal{A}$ with a unit element. There exists a dense subspace $\mathcal{D}$ of a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$, as well as a representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ and a unit vector $\Omega \in \mathcal{D}$, such that $\omega = (\Omega, \pi(\cdot)\Omega)$ and $\mathcal{D} = \pi(\mathcal{A})\Omega$. The* GNS triple $(\mathcal{D}, \pi, \Omega)$ *is determined up to unitary equivalence.*

Proof. The first step consists of endowing $\mathcal{A}$ with an inner product defined as $(a,b)_\bullet \doteq \omega(a^*b)$ for each $a, b \in \mathcal{A}$. This is per construction sesquilinear and positive semidefinite, since $(a,a)_\bullet = \omega(a^*a) \geq 0$. We need to check Hermiticity, namely that $\overline{(a,b)_\bullet} = (b,a)_\bullet$ holds for all $a, b \in \mathcal{A}$. To this avail one needs to take into account the following two identities:

$$\begin{aligned} 4a^*b &= (a+b)^*(a+b) - (a-b)^*(a-b) - i(a+ib)^*(a+ib) + i(a-ib)^*(a-ib); \\ 4b^*a &= (a+b)^*(a+b) - (a-b)^*(a-b) + i(a+ib)^*(a+ib) - i(a-ib)^*(a-ib). \end{aligned}$$

The positivity requirement on ω yields the Cauchy-Schwarz inequality for $(\cdot,\cdot)_\bullet$, namely $|(a,b)_\bullet|^2 \leq (a,a)_\bullet (b,b)_\bullet$, but it does not ensure that non-degeneracy holds for this sesquilinear form. Hence, we have to single out the vanishing elements introducing the subset $\mathcal{I} = \{a \in \mathcal{A} \mid \omega(a^*a) = 0\}$. This is a closed left ideal of $\mathcal{A}$ (but $\mathcal{I}^* \nsubseteq \mathcal{I}$ in general):

[1]The proof of the Theorem 2.6.3 is slavishly copied from [54]

- It is a closed subset of $\mathcal{A}$ being the preimage of 0 under the continuous map $a \in \mathcal{A} \mapsto \omega(a^*a) \in \mathbb{R}$;

- Using the Cauchy-Schwarz inequality, we note that $\omega(ba) = 0 = \omega(a^*b)$ for each $a \in \mathcal{I}$ and for each $b \in \mathcal{A}$. In particular, this means that $\mathcal{I} = \{a \in \mathcal{A} \mid \omega(ba) = 0\ , \forall b \in \mathcal{A}\}$, showing that $\mathcal{I}$ is a vector subspace of $\mathcal{A}$;

- $\omega((ba)^*ba) = \omega((a^*b^*b)a) = 0$ for each $a \in \mathcal{I}$ and each $b \in \mathcal{A}$, hence $\mathcal{I}$ is a left ideal.

We can thus define the vector space $\mathcal{D} \doteq \mathcal{A}/\mathcal{I}$, where the latter is the set of equivalence classes $[a]$ induced by the following equivalence relation: $a \sim a'$ if and only if there exists $b \in \mathcal{I}$ such that $a' = a + b$. We can endow $\mathcal{D}$ with the positive definite Hermitian non-degenerate sesquilinear form $(\cdot, \cdot)$ defined as $([a], [b]) \doteq (a, b)'$ for all $[a], [b] \in \mathcal{D}$, where a and b are any representative of the equivalence classes $[a]$ and $[b]$ respectively. This is well defined as a consequence of the remarks made above and it endows $\mathcal{D}$ with a pre-Hilbert structure. Taking the completion of $(\mathcal{D}, (,))$ produces a Hilbert space $\mathscr{H}$. The representation π can be induced via left multiplication exploiting the fact that $\mathcal{I}$ is a left ideal, namely we introduce $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ defined via $\pi(b)[a] = [ba]$ for all $b \in \mathcal{A}$ and for all $[a] \in \mathcal{D}$. For each $a, b \in \mathcal{A}$ it is easy to check that $\pi(ab) = \pi(a)\pi(b)$, while $\pi(a^*) = \pi(a)^*$ follows from the identity $([a^*b], [c]) = ([b], [ac])$, so that π turns out to be a representation. Furthermore, setting $\Omega \doteq [\mathbb{1}]$, one has $(\Omega, \pi(a)\Omega) = \omega(\mathbb{1}^*a\mathbb{1}) = \omega(a)$ for each $a \in \mathcal{A}$ and $\pi(\mathcal{A})\Omega = \mathcal{D}$. This concludes the identification of the GNS triple. Let us now tackle uniqueness. Suppose that one can find another realization of ω as $(\mathcal{D}', \pi', \Omega')$ and let us introduce the operator $U : \mathcal{D} \to \mathcal{D}'$ such that $U(\pi(a)\Omega) \doteq \pi'(a)\Omega'$ for each $a \in \mathcal{A}$. This is well-defined since $\pi(a)\Omega = 0$ means $\omega(a^*a) = 0$, which yields $\|\pi'(a)\Omega'\|'^2 = 0$. Furthermore U preserves the scalar product, namely $(U[a], U[b])' = (\pi'(a)\Omega', \pi'(b)\Omega')' = (\Omega', \pi'(a^*b)\Omega')' = \omega(a^*b) = ([a], [b])$, and has an inverse $U^{-1} : \mathcal{D}' \to \mathcal{D}$, defined as $U^{-1}(\pi'(a)\Omega') \doteq \pi(a)\Omega$ for each $a \in \mathcal{A}$, preserving the scalar products as well. Thus it can be extended to a unitary operator from $\mathscr{H}$ to $\mathscr{H}'$, the Hilbert space obtained via completion of $\mathcal{D}'$. In other words this means that $\Omega' = U\Omega$ and that the defining relation for U can also be read as $\pi'([a]) = U\pi(a)U^{-1}$. This is nothing but the statement that the two representations π and π' are unitary equivalent. □

Let us conclude this section with a definition.

Definition 2.6.4. Let $\mathcal{A}$ be a $*$-algebra and $\mathscr{F}$ the field algebra.

(i) A state ω on $\mathcal{A}(\mathcal{M})$ is said to be *mixed*, if it is a convex linear combination of states, *i.e.* $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, where $\lambda < 1$ and $\omega_i = \omega$ are states on $\mathcal{A}$. Otherwise a state is said to be *pure*.

(ii) A state ω on $\mathscr{F}(\mathcal{M})$ is said *even*, if it is invariant under $\Phi(f) \to -\Phi(f)$, *i.e.* it has vanishing n-point functions for all odd n.

(iii) An even state on $\mathscr{F}(\mathcal{M})$ is called *quasifree* or *Gaussian* if, for all odd n $\omega_n(f_1,\ldots,f_n)=0$ and for all even n

$$\omega_n(f_1,\ldots,f_n)=\sum_{\pi_n\in S'_n}\prod_{i=1}^{n/2}\omega_2(f_{\pi_n(2i-1)},f_{\pi_n(2i)})$$

Here, S'_n denotes the set of ordered permutations of n elements which satisfies the following condition

$$\pi_n(2i-1)<\pi_n(2i)\quad\text{for}\quad 1\le i\le n/2,\quad \pi_n(2i-1)<\pi_n(2i+1)\quad\text{for}\quad 1\le i\le n/2$$

(iv) Let α_t denote a one-parameter group of $*$-automorphisms on $\mathcal{A}$, *i.e.* for arbitrary elements A, B of $\mathcal{A}$,

$$\alpha_t(A*B)=(\alpha_t(A))^*\alpha_t(B)\quad \alpha_t(\alpha_s(A))=\alpha_{t+s}(A)\quad \alpha_0(A)=A$$

A state ω on $\mathcal{A}$ is called α_t-*invariant* if $\omega(\alpha_t(A))=\omega(A)$ for all $A\in\mathcal{A}$.

2.6.1 Hadamard states

Definition 2.6.1 is too general encompassing also states which do not have physically reasonable properties. To restrict this freedom, it seems reasonable to look at the situation in Minkowski spacetime. We require that all ground states possess the same ultraviolet properties, *i.e.* they have the same UV energy behaviour, namely they satisfy the so-called *Hadamard condition* which we would like to explain in a more formally manner later. We will see that this condition ensures us that the possibility to regularize quantum observables, building a generalization of Wick polynomials which are used in perturbative quantum field theory. As a by product this yields also finite quantum fluctuation for observables.
In this subsection, we present only the analysis of Hadamard states for real scalar fields following [54]. The counterpart for highter spin fields can be found in [20] and [21].

Let us consider the field algebra: It contains only linear combinations of products of free fields at different points. However, if one wants to treat interacting fields in perturbation theory, one needs a notion of normal ordering, *i.e.* a way to define field monomials like $\Phi^2(x)$. So, let us consider a massless scalar field in Minkowski spacetime. Its two-point function is

$$\omega_2(x,y)\doteq\omega_2(\Phi(x)\Phi(y))=\lim_{\varepsilon\downarrow 0}\frac{1}{4\pi^2}\frac{1}{(x-y)^2+i\varepsilon(x_0-y_0)+\varepsilon^2}\tag{2.6.1}$$

where $\varepsilon\in\mathbb{R}$. For $\varepsilon=0$ this is a smooth function if x and y are spacelike or timelike separated. It is singular at $(x-y)^2=0$, but the singularity is 'good enough' to give a finite result when smearing $\omega_2(x,y)$ with two test functions. Hence, ω_2 is a well-defined (tempered) distribution. However, if we were to define $\Phi^2(x)$ by some "limit" like

$$\Phi^2(x) \doteq \lim_{x\to y} \Phi(x)\Phi(y)$$

the expectation value of the resulting object would "blow up". A solution to this problem is to define field monomials by appropriate regularising subtractions. For a squared field, this is achieved by setting

$$:\!\Phi^2(x)\!: \doteq \lim_{x\to y} (\Phi(x)\Phi(y) - \omega_2(x,y)\mathbb{1})$$

The field $:\!\Phi^2(x)\!:$ has a finite expectation value. In the standard Fock space picture, one writes the field in terms of creation and annihilation operators in momentum space *i.e.*

$$\Phi(x) = \frac{1}{\sqrt{2\pi^3}} \int \frac{d\mathbf{k}}{\sqrt{2k_0}} a^\dagger_{\mathbf{k}} e^{ikx} + a_{\mathbf{k}} e^{-ikx}.$$

This procedure is equivalent to the above defined subtraction of the vacuum expectation value. However, having defined the Wick polynomials is not enough. We would include them in the Borchers-Ulhmann algebra. Using the mode-expansion picture, one can compute

$$:\!\Phi^2(x)\!: :\!\Phi^2(y)\!: = :\!\Phi^2(x)\Phi^2(y)\!: + 4 :\!\Phi(x)\Phi(y)\!: \omega_2(x,y) + 2\left(\omega_2(x,y)\right)^2$$

Obviously, $\omega_2(x,y)$ is singular, and one could wonder whether the singularities are mild enough to yield finite integrals when evaluated with test functions: In terms of a mode decomposition, one could equivalently wonder whether the momentum space integrals, appearing in the definition of $:\!\Phi(x)\Phi(y)\!:$ via normal ordering converge in a suitable topology. In Minkowski spacetime, on account of the energy property of the vacuum state, this is indeed the case. In more detail, the Fourier decomposition of the two-point function ω_2 is written as

$$\omega_2(x,y) = \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dk\, \Theta(k_0)\delta(k^2) e^{ik(x-y)} e^{-\varepsilon k_0} \tag{2.6.2}$$

where $\Theta(k_0)$ denotes the Heaviside step function. We see that the Fourier transform of w_2 has only support on the forward lightcone. This corresponds to the fact that we have associated the positive frequency modes to the creation operators in the above mode expansion of the quantum field. This insight allows to determine the square of $\omega_2(x,y)$ by a convolution in Fourier space

$$\begin{aligned}(\omega_2(x,y))^2 &= \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp\, \Theta(q_0)\delta(q^2)\Theta(p_0)\delta(p^2) e^{i(q+p)(x-y)} e^{-\varepsilon(p_0+q_0)} \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq\, \Theta(q_0)\delta(q^2)\Theta(k_0-q_0)\delta((k-q)^2) e^{ik(x-y)} e^{-\varepsilon k_0}.\end{aligned}$$

Since ω_2 has only support in one "energy direction", namely the positive one, the intersection of its Fourier support and the same support evaluated with negative momentum is compact, and the convolution therefore well-defined. Moreover, as this statement only relies on the large momentum behaviour of the Fourier transforms, it holds equally in the case of massive fields, as the mass shell approaches the light cone for large momenta. It looks promising to promote the Hadamard condition to a good selection criterion for physical states in Minkowski spacetime. It is indeed also used in quantum field theory in curved spacetimes in order both to select physical states among all possible ones and to discuss perturbatively interacting theories.

We shall outline the definition of the Hadamard condition. It can be formulated in two ways, one being a generalisation of the position space form (2.6.1) of the vacuum two-point function and the other being a generalisation of its momentum space form (2.6.2). We shall start our review of the Hadamard condition by considering the microlocal aspects of Hadamard states. Heuristically, the language of *microlocal analysis* is well-suited because generic curved spacetimes are not translationally invariant and therefore do not allow to define a global Fourier transform and a related global energy positivity condition. Moreover, recalling by Definition 1.4.11 that the wavefrontset is the collection of all points and directions along a distribution taking singular values, it is useful to characterize the wavefront set with test functions. From equations (2.6.1) and (2.6.2) and from Theorem 1.4.12 one can infer that the wavefront set of the two-point function (for $m \geq 0$) in Minkowski spacetime is

$$\begin{aligned} WF(\omega_2) &= \left\{(x,y,k,-k) \in T^*\mathbb{R}^8 \setminus \{0\} \mid x \neq y,\ (x-y)^2 = 0,\ k||(x-y),\ k_0 > 0\right\} \\ &\cup \ \left\{(x,x,k,-k) \in T^*\mathbb{R}^8 \setminus \{0\} \mid k^2 = 0,\ k_0 > 0\right\}, \end{aligned} \tag{2.6.3}$$

where $k||(x-y)$ entails that k is parallel to the vector connecting the points x and y. It is the condition $k_0 > 0$ in particular, which encodes the energy positivity of the vacuum state. We can now rephrase our observation that the pointwise square of $\omega_2(x,y)$ is a well-defined distribution by noting that $WF(\omega_2) \oplus WF(\omega_2)$ does not contain the zero section. In contrast, we know that the δ-distribution $\delta(x)$ is singular at $x = 0$ and that its Fourier transform is a constant. Hence, its wave front set reads

$$WF(\delta) = \{(0,k) \in T^*\mathbb{R} \setminus \{0\} \mid k \in \mathbb{R} \setminus \{0\}\},$$

. We see that the δ-distribution does not have a "one-sided" wave front set and, hence, cannot be squared. The same holds if we view δ as a distribution $\delta(x,y)$ on $C_0^\infty(\mathbb{R}^2)$. Then

$$WF(\delta(x,y)) = \{(x,x,k,-k) \in T^*\mathbb{R}^2 \setminus \{0\} \mid k \in \mathbb{R} \setminus \{0\}\}.$$

The previous discussion suggests that a generalisation of equation (2.6.3) to curved spacetimes is the sensible requirement to select physical states. We shall now define such a generalisation.

Definition 2.6.5. Let ω be a state on the field algebra $\mathcal{F}(M)$ (see Definition 2.2.2): We say that ω fulfils the *Hadamard condition* and is therefore a *Hadamard state* if its two-point correlation function ω_2 fulfils

$$WF(\omega_2) = \left\{(x, y, k_x, -k_y) \in T^*M^2 \setminus \{0\} \mid (x, k_x) \sim (y, k_y),\ k_x \triangleright 0\right\} .$$

Here, $(x, k_x) \sim (y, k_y)$ implies that there exists a null geodesic c connecting x to y such that k_x is coparallel and cotangent to c at x and k_y is the parallel transport of k_x from x to y along c. Finally, $k_x \triangleright 0$ means that the covector k_x is future-directed.

Example 2.2. All vacuum states and thermal equilibrium states on ultrastatic spacetimes (*i.e.* spacetimes with a metric $ds^2 = -dt^2 + h_{\mu\nu}dx_\mu dx_\nu$, with $h_{\mu\nu}$ time independent) are of Hadamard form.

Example 2.3. Given a Hadamard state ω on the field algebra $\mathcal{F}(M)$ and a smooth solution Ψ of the field equation $P\Psi = 0$, one can construct a coherent state by redefining the quantum field $\widehat{\Phi}(x)$ as $\widehat{\Phi}(x) \mapsto \widehat{\Phi}(x) + \Psi(x)\mathbb{1}$. The induced coherent state has two-point function $\omega_{\Psi,2}(x, y) = \omega_2(x, y) + \Psi(x)\Psi(y)$, which is Hadamard since $\Psi(x)$ is smooth.

Example 2.4. The *Bunch-Davies state* on de Sitter spacetime fulfils the Hadamard condition. It has been shown in [17] and [55] that this result can be generalised to asymptotically de Sitter spacetimes, where distinguished Hadamard states can be constructed by means of a holographic argument. These states are generalisations of the Bunch-Davies state in the sense that the aforementioned holographic construction yields the Bunch-Davies state in de Sitter spacetime.

2.7 Holographic principle

One of the main obstacles in combining in a unique theory general relativity and quantum mechanics consists in the understanding of the role and of the number of quantum degrees of freedom of gravity. A new direction in this investigation has been proposed by G. 't Hooft in [56], who suggested the famous *holographic principle*: The physical information in a spacetime is fully encoded on the boundary of the region under consideration. 't Hooft paper represented a starting point for countless research papers which led to an extension of the Bekenstein-Hawking result on black hole entropy to a wider class of spacetime regions. A weaker version of the holographic principle states that any quantum field theory living in a D-dimensional spacetime can be fully described by means of a second theory living on a suitable submanifold, with codimension 1, which is not necessary boundary of the former. However, this principle does not give any prescription on how to construct an holographic counterpart of a given quantum field theory. A few example are known, the most notable being the so-called anti deSitter - conformal field theory (AdS/CFT) correspondence [57] (or Maldacena conjecture or gauge/gravity duality). We outline and we use in the construction of Hadamard states for linearized gravity another implementation of the the holographic principle known also as *bulk to boundary correspondence* (BBC). Our treatment is based on [17], in which, in order to implement the holographic principle in four-dimensional asymptotically flat spacetimes $(\mathcal{M}, g)$, it is proposed to construct a correspondence between a theory living on $\mathcal{M}$ and a quantum field theory living on the conformal boundary $\mathscr{I}$ of $\mathcal{M}$.

2.7.1 Bulk to Boundary Corrispondece

The paradigm of the Bulk to Boundary Corrispondence is to encode the information of a QFT defined in the bulk of a manifold into a counterpart living on the boundary. Our treatment is a briefly introduction. For more information refer to [17].
This correspondence is motivated by the following Proposition.

Proposition 2.7.1. *Assume that $(\mathcal{M}, g)$ is asymptotically flat (see Definition 1.3.7) with associated unphysical space $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\widetilde{g}|_{\mathcal{M}} = \Omega^2 g$. Suppose that there exists an open set $\widetilde{V} \subset \widetilde{\mathcal{M}}$ with $\overline{\mathcal{M} \cap J^-(\mathscr{I}^-)} \subset \widetilde{V}$ (the closure being referred to $\widetilde{\mathcal{M}}$) such that $(\widetilde{V}, \widetilde{g})$ is globally hyperbolic so that $(\mathcal{M} \cap V, g)$ is globally hyperbolic, too. If $\Phi : \mathcal{M} \cap \widetilde{V} \to \mathbb{C}$ has compactly supported Cauchy data on some Cauchy surface of $\mathcal{M} \cap \widetilde{V}$ and satisfies the massless conformal Klein-Gordon equation,*

$$\Box\Phi - \frac{1}{6}R\Phi = 0 \tag{2.7.1}$$

where $\Box$ is d'Alembert operator and R is the scalar curvature, the following conditions are satisfy:

(i) the field $\widetilde{\Phi} \doteq \Omega^{-1}\Phi$ can be extended uniquely into a smooth solution in $(\widetilde{V}, \widetilde{g})$ of

$$\Box\widetilde{\Phi} - \frac{1}{6}\widetilde{R}\widetilde{\Phi} = 0$$

(ii) for every smooth positive factor ω defined in a neighbourhood of $\mathscr{I}^-$ used to rescale $\Omega \to \omega\Omega$ in such a neighbourhood, $(\omega\Omega)^{-1}\Phi$ extends to smooth field Ψ on $\mathscr{I}^-$ uniquely.

Proof. Let $\mathcal{M}_{\widetilde{V}} := \mathcal{M} \cap \tilde{V}$ and the symbol "tilde" written on a causal set indicates that the metric $\widetilde{g}$ is employed, otherwise the used metric is g. Notice that $\tilde{J}^-(\mathcal{M}) \cap \mathscr{I} = \emptyset$ so that $J^-(p; \mathcal{M}_{\widetilde{V}}) = \tilde{J}^-(p; \tilde{V})$ if $p \in \mathcal{M}_{\widetilde{V}}$. $(\mathcal{M}_{\widetilde{V}}, g)$ is globally hyperbolic because it is strongly causal and the sets $J^-(p; \mathcal{M}_{\widetilde{V}}) \cap J^+(q; \mathcal{M}_{\widetilde{V}})$ are compact for $p, q \in M_{\widetilde{V}}$ (see sec. 8 in [34]). Indeed, $(\tilde{V}, \widetilde{g})$ is strongly causal and thus $(M_{\widetilde{V}}, g)$ is strongly causal. If $p, q \in \mathcal{M}_{\widetilde{V}}$, $J^-(p; \mathcal{M}_{\widetilde{V}}) \cap J^+(q; M_{\widetilde{V}})$ is compact because $J^-(p; \mathcal{M}_{\widetilde{V}}) \cap J^+(q; \mathcal{M}_{\widetilde{V}}) = \tilde{J}^-(p; \tilde{V}) \cap \tilde{J}^+(q; \tilde{V})$ and $\tilde{J}^-(p; \tilde{V}) \cap \tilde{J}^+(q; \tilde{V})$ is compact since $(\tilde{V}, \widetilde{g})$ is globally hyperbolic. As a consequence, we can use in $M_{\widetilde{V}}$ (but also in $\tilde{V}$) standard results of solutions of Klein-Gordon equation with compactly supported Cauchy data in globally hyperbolic spacetimes [34]. (a) Let S be a spacelike Cauchy surface for $(M_{\widetilde{V}}, g)$. It is known [34] that, in any open subset of M and under the only hypothesis $\widetilde{g} = \Omega^2 g$, is valid for Φ if and only if is valid for $\tilde{\Phi} := \Omega^{-1}\Phi$. The main idea of the proof is to associate Φ with Cauchy data for

$\tilde{\Phi}$ on a suitable Cauchy surface of the larger spacetime $(\tilde{V}, \widetilde{g})$, so that the unique maximal solution $\tilde{\Phi}$ of uniquely determined in $(\tilde{V}, \widetilde{g})$ by those Cauchy data, on a hand is well defined on $\mathscr{I} \subset \widetilde{V}$, on the other hand it is a smooth extension of $\Omega^{-1}\Phi$ initially defined in $M_{\widetilde{V}}$ only. Let K_S be the compact support of Cauchy data of Φ on S. As $\widetilde{V}$ is homeomorphic to the product manifold $\mathbb{R} \times \Sigma$, $\mathbb{R}$ denoting a global time coordinate on $\widetilde{V}$ and Σ being a spacelike Cauchy surface of $\widetilde{V}$, one can fix Σ in the past of the compact set K_S. Since K_S is compact and the class of the open sets $\tilde{I^-}(p; \tilde{V}) \cap \tilde{I^+}(q; \tilde{V})$ with $p, q \in \mathcal{M}_{\widetilde{V}}$ is a basis of the topology of $\mathcal{M}_{\widetilde{V}}$, it is possible to determine a *finite* number of points $p_1, \dots p_n \in \mathcal{M}_{\widetilde{V}}$ in the future of K_S in order that $\cup_i I^-(p_i; \mathcal{M}_{\widetilde{V}}) \supset K_S$. In this way one also has $\cup_i J^-(p_i; \mathcal{M}_{\widetilde{V}}) = \cup_i \tilde{J^-}(p_i; \tilde{V}) \supset K_S$. On the other hand, as is well known $\cup_i \tilde{J^-}(p_i; \tilde{V}) \cap D^+(\Sigma)$ is compact and, in particular, $K_\Sigma := \cup_i \tilde{J^-}(p_i; \tilde{V}) \cap \Sigma = \cup_i J^-(p_i; M_{\widetilde{V}}) \cap \Sigma$ is compact too, it being a closed subset of a compact set. Notice that, outside $J^-(K_S; \mathcal{M}_{\widetilde{V}}) \cup J^+(K_S; \mathcal{M}_{\widetilde{V}})$ the field Φ vanishes in $\mathcal{M}_{\widetilde{V}}$. Thus we are naturally lead to consider compactly supported (in K_Σ) Cauchy data on Σ for the equation , obtained by restriction of $\Omega^{-1}\Phi$ and its derivatives to Σ. Let $\tilde{\Phi}$ be the unique solution of in the whole globally hyperbolic spacetime $(\tilde{V}, \widetilde{g})$, associated with those Cauchy data on Σ. By construction $\tilde{\Phi}$ must be an extension to $(\tilde{V}, \widetilde{g})$ of $\tilde{\Phi}$ defined in $\mathcal{M}_{\widetilde{V}}$ (more precisely in $\tilde{D^+}(\Sigma; \widetilde{V}) \cap \mathcal{M}_{\widetilde{V}} = D^+(\Sigma \cap \mathcal{M}_{\widetilde{V}}; \mathcal{M}_{\widetilde{V}})$), since they satisfy the same equation and have the same Cauchy data on Σ. The proof concludes by noticing that $\mathscr{I} \subset \tilde{V}$ and thus $\psi \doteq \tilde{\Phi}|_{\mathscr{I}^-}$ is, in fact, a smooth extension to $\mathscr{I}$ of $\tilde{\Phi}$.
(b) The case with $\omega \neq 1$ is now a trivial consequence of what proved above replacing Ω with $\omega\Omega$ in the considered neighbourhood of $\mathscr{I}^-$ where $\omega > 0$. □

Proposition[2] 2.7.1 entails that, at level of classical fields, there exist a correspondence between solutions of the field equation (2.7.1) and an associated field Ψ defined on $\mathscr{I}^-$. In the next paragraph we show how to implement at a level of algebras of observables such correspondence. If it can be implemented in terms of an injective $*$-homomorphism, the algebra of the bulk can be realized as subalgebra of the observables of the boundary counterpart.

Bosonic QFT in the bulk

Consider a real linear bosonic QFT in $(\mathcal{M}, g)$ based on the symplectic space $(\mathcal{S}(\mathcal{M}), \sigma_\mathcal{M})$, where $\mathcal{S}(\mathcal{M})$ is the space of solutions Φ of equation

$$P\Phi = 0$$

where P is the Klein-Gordon operator $P = \Box + \xi R + m^2$ with $\Box = -\nabla_a \nabla^a$, $m > 0$ and $\xi \in \mathbb{R}$ constants. The symplectic form σ_M is:

$$\sigma_\mathcal{M}(\Phi_1, \Phi_2) \doteq \int_\Sigma (\Phi_2 \nabla_N \Phi_1 - \Phi_1 \nabla_N \Phi_2) \, d\mu_g^{(\mathcal{S})} \tag{2.7.2}$$

[2]The proof of the Proposition 2.7.1 is slavishly copied from [17]

for all $\Phi_1, \Phi_2 \in \mathcal{S}(\mathcal{M})$. Σ is an arbitrary Cauchy surface of M with normal unit future-directed vector N and 3-volume measure $d\mu_g^{(\Sigma)}$ induced by g. For any symplectic space $(\mathcal{S}(M), \sigma_M)$, it is possible to construct an algebra of observables, a Weyl algebra $\mathfrak{W}(\mathcal{M})$ in these cases. This algebra is, up to (isometric) $*$-isomorphisms, unique and its generators $W_{\mathcal{M}}(\Phi) \neq 0$, $\Phi \in \mathcal{S}(\mathcal{M})$, satisfy *Weyl commutation relations*

$$W_{\mathcal{M}}(-\Phi) = W_{\mathcal{M}}(\Phi)^* \qquad W_{\mathcal{M}}(\Phi)W_{\mathcal{M}}(\Psi) = e^{i\sigma_M(\Phi,\Psi)/2}W(\Phi+\Psi) \tag{2.7.3}$$

where $\mathfrak{W}(\mathcal{M})$ represents the basic set of quantum observables associated with the bosonic field Φ propagating in the bulk spacetime $(\mathcal{M}, g)$.

Bosonic QFT on $\mathscr{I}^-$ and $G^-_{\mathscr{I}}$-invariant states

Recalling Definition 1.3.7 on page 24 $\mathscr{I}^- \equiv \mathbb{R}\times S^2$, consider the symplectic space $(\mathcal{S}(\mathscr{I}^-), \sigma)$, where

$$\mathcal{S}(\mathscr{I}^-) \doteq \left\{\Psi \in C^\infty(\mathbb{R}\times S^2) \mid \Psi\,, \partial_\ell\Psi \in L^2(\mathbb{R}\times S^2, d\ell\wedge dS^2(\vartheta,\varphi)\right\}$$

dS^2 being the standard volume form of the unit 2-sphere and the symplectic form σ is given by

$$\sigma(\Psi_1,\Psi_2) \doteq \int_{\mathbb{R}\times S^2}\left(\Psi_2\frac{\partial\Psi_1}{\partial\ell} - \Psi_1\frac{\partial\Psi_2}{\partial\ell}\right) d\ell\wedge dS^2(\vartheta,\varphi)$$

for all $\Psi_1, \Psi_2 \in \mathcal{S}(\mathscr{I}^-)$. As in the previous paragraph, we associate to $(\mathcal{S}(\mathscr{I}^-), \sigma)$ the Weyl algebra $\mathfrak{W}(\mathscr{I}^-)$ whose generators $W(\Psi) \neq 0$ satisfy the Weyl commutation relations (2.7.3). It is possible to prove that $(\mathcal{S}(\mathscr{I}^-), \sigma)$ is invariant under the pull-back action of $G_{\mathscr{I}^-}$, where $G_{\mathscr{I}^-}$ is the BMS group (see Definition (1.3.8)). In other words,

(i) $\Psi\circ g \in \mathcal{S}(\mathscr{I}^-)$ if $\Psi \in \mathcal{S}(\mathscr{I}^-)$;

(ii) $\sigma(\Psi_1\circ g, \Psi_2\circ g) = \sigma(\Psi_1, \Psi_2)$ for all $g \in G^-_{\mathscr{I}}$ and $\Psi_1, \Psi_2 \in \mathcal{S}(\mathscr{I}^-)$.

As a consequence, $G_{\mathscr{I}^-}$ induces a $*$-automorphism $\alpha : \mathfrak{W}(\mathscr{I}^-) \to \mathfrak{W}(\mathscr{I}^-)$, uniquely individuated by the requirement

$$\alpha_g(W(\Psi)) \doteq W(\Psi\circ g^{-1})$$

with $\Psi \in \mathcal{S}(\mathscr{I}^-)$ and $g \in G_{\mathscr{I}^-}$.

$G^-_{\mathscr{I}}$-invariant algebraic states on $\mathfrak{W}(\mathscr{I}^-)$

Consider the quasifree state ω on $\mathfrak{W}(\mathcal{S}(\mathscr{I}^-))$ defined as follows: if $\Psi_1, \Psi_2 \in \mathcal{S}(\mathscr{I}^-)$, then

$$\begin{aligned} &\omega(W(\Psi)) = e^{-\mu(\Psi,\Psi)/2} \\ &\mu(\Psi_1,\Psi_2) \doteq Re \int_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\Psi_1}(k,\vartheta,\varphi)}\widehat{\Psi_2}(k,\vartheta,\varphi)dk\wedge dS^2(\vartheta,\varphi) \end{aligned} \tag{2.7.4}$$

the bar denoting complex conjugation. $\Theta(k)$ is the Heaviside step function and $\widehat{\Psi}$ stands for the l-Fourier-Plancherel transform of Ψ:

$$\widehat{\Psi}(k,\vartheta,\varphi) \doteq \int_{\mathbb{R}} \frac{e^{ikl}}{\sqrt{2\pi}}\Psi(l,\vartheta,\varphi)dl \quad (k,\vartheta,\varphi) \in \mathbb{R}\times S^2.$$

For every $\Psi_1, \Psi_2 \in \mathcal{S}(\mathscr{I}^-)$, the constraint

$$|\sigma(\Psi_1,\Psi_2)|^2 \leq 4\,\mu(\Psi_1,\Psi_1)\mu(\Psi_2,\Psi_2)$$

must hold for every quasifree state and it is fulfilled by the scalar product μ, viz:

$$\sigma(\Psi_1,\Psi_2) \doteq -2Im \int_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\Psi_1}(k,\vartheta,\varphi)}\widehat{\Psi_2}(k,\vartheta,\varphi)dk\wedge dS^2(\vartheta,\varphi).$$

Consider the GNS representation $(\mathfrak{H},\Pi,\Upsilon)$ of ω. Since ω is quasifree, $\mathfrak{H}$ is a bosonic Fock space $\mathcal{F}_+(\mathscr{H})$ with cyclic vector Υ given by the Fock vacuum. The 1-particle Hilbert $\mathscr{H}$ space is obtained as the Hilbert completion of the complex space generated by the "positive-frequency parts" $\Theta\widehat{\Psi} \doteq K_\mu\Psi$, of every wavefunction $\Psi \in \mathcal{S}(\mathscr{I}^-)$, with the scalar product $\langle\cdot,\cdot\rangle$ individuated by μ. In our case

$$\langle K_\mu\Psi_1, K_\mu\Psi_2\rangle \doteq \int_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\Psi_1}(k,\vartheta,\varphi)}\widehat{\Psi_2}(k,\vartheta,\varphi)dk\wedge dS^2(\vartheta,\varphi).$$

The map $K_\mu : \mathcal{S}(\mathscr{I}^-) \to \mathscr{H}$ is $\mathbb{R}$-linear. As ω is quasifree, it is *regular*, so that symplectically-smeared field operators $\sigma(\Psi_1,\Psi_2)$ are defined in $\mathcal{F}_+(\mathscr{H})$ via Stone theorem, namely:

$$\Pi(W(t\Psi_1)) = e^{-it\sigma(\Psi_2,\Psi_1)}$$

with $t \in \mathbb{R}$ and $\Psi \in \mathcal{S}(\mathscr{I}^-)$ and these operators have the usual form in terms of creator and annihilator operators.
The state ω satisfies two theorems, which are proved in [55]. The first concerns the invariance properties of ω.

Theorem 2.7.2. *The state ω defined in (2.7.4) whose GNS triple is $(\mathfrak{H},\Pi,\Upsilon)$, satisfies the following properties:*

(i) it is invariant under the $*$-automorphisms representation $G_{\mathscr{I}^-} \ni g \mapsto \alpha_g$, namely: $\omega(\alpha_g(a)) = \omega(a)$ for all $a \in \mathfrak{W}(I^-)$ and $g \in G_{\mathscr{I}^-}$,

(ii) *the unique unitary representation* $U : G_{\mathscr{I}^-} \ni g \mapsto Ug$ *that implements* α *in* $\mathfrak{H}$ *leaving* Υ *invariant, namely,*

$$U_g a U_g^* = \alpha_g(a) \quad \textit{and} \quad U_g \Upsilon = \Upsilon \quad \forall a \in \Omega(I^-), g \in G_{\mathscr{I}_-}$$

leaves $\mathscr{H}$ *invariant and it is determined by* $U|_{\mathscr{H}}$ *completely.* U *has the tensorialised form*

$$U = I \oplus U|_{\mathscr{H}} \oplus (U|_{\mathscr{H}} \oplus U|_{\mathscr{H}}) \oplus (U|_{\mathscr{H}} \oplus U|_{\mathscr{H}} \oplus U|_{\mathscr{H}}) \dots$$

(iii) *the unitary representation* $U|_{\mathscr{H}} : \mathscr{H} \to \mathscr{H}$ *is irreducible.*

A second important result concerns the positive-energy/uniqueness properties of ω. In Minkowski QFT positivity of energy guarantees that, under small (external) perturbations, the system does not collapse to lower and lower energy states. In general spacetimes the notion of energy is associated with a notion of Killing time. This interpretation can be extended to this case too: the positive-energy requirement is fulfilled for the "asymptotic" notion of time associated with limit values $\widetilde{Y}$ towards $\mathscr{I}^-$ of a timelike future-directed vector field Y in $\mathcal{M}$. The positive-energy property for ∂_ℓ, determine completely ω.

Theorem 2.7.3. *Consider the state* ω *defined in (2.7.4) and its GNS representation. The following holds.*

(i) *The state* ω *is the unique pure quasifree state on* $\mathfrak{W}(\mathscr{I}^-)$ *satisfying both:*

   (a) *it is invariant under* $\alpha^{(\partial_\ell)}$,

   (b) *the unitary group which implements* $\alpha^{(\partial_\ell)}$*, leaving fixed the cyclic GNS vector is strongly continuous with non negative self-adjoint generator (energy positivity condition).*

(ii) *Let* λ *be a pure (not necessarily quasifree) state on* $\mathfrak{W}(I^-)$ *which is* $G_{\mathscr{I}^-}$*-invariant.* w *is the unique state on* $\mathfrak{W}(I^-)$ *satisfying:*

(a) *it is invariant under* $\alpha^{(\partial_\ell)}$,

(b) *it belongs to the folium of* ω.

Interplay of QFT in $\mathcal{M}$ and QFT on $\mathscr{I}^-$

While in the previous section we have shown that there exists a preferred quasifree pure state λ invariant under the action of $SG^-_{\mathscr{I}}$, we induce a state $\omega_{\mathcal{M}}$ on the algebra of field observables in the bulk.
To this avail, we concentrate beforehand on the algebraic properties, establishing the existence of a nice interplay between $\mathfrak{W}(\mathscr{I}^-)$ and $\mathfrak{W}(\mathcal{M})$ under suitable hypotheses on the considered symplectic forms. Such interplay will be used to define ω_M in the next subsection.

Theorem 2.7.4. *Consider an asymptotically flat spacetime* $(\mathcal{M}, g)$ *and suppose that every* $\Phi \in \mathcal{S}(\mathcal{M})$ *extends smoothly to some* $\Gamma\Phi \in \mathcal{S}(\mathscr{I}^-)$ *in order that:*

$$\sigma_{\mathcal{M}}(\Phi_1, \Phi_2) = \sigma(\Gamma\Phi_1, \Gamma\Phi_2) \tag{2.7.5}$$

for every $\Phi_1, \Phi_2 \in \mathcal{S}(\mathcal{M})$.
Under these assumptions there exists an (isometric) $*$*-homomorphism* $\imath : \mathfrak{W}(\mathcal{M}) \to \mathfrak{W}(\mathscr{I}^-)$ *that identifies the Weyl* C^**-algebra of the bulk* $\mathcal{M}$ *with a sub* C^**-algebra of the boundary* $\mathscr{I}^-$*; it is completely determined by the requirement:*

$$\iota\left(W_M(\Phi)\right) \doteq W(\Gamma\Phi) \tag{2.7.6}$$

for all $\Phi \in \mathfrak{W}(M)$.

Proof. Notice that the linear map $\Gamma : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathscr{I}^-)$ has to be injective due to nondegenerateness of σ and of equation (2.7.5). Consider the sub Weyl-C^*-algebra $\mathcal{A}_{\mathcal{M}}$ of $\mathfrak{W}(\mathscr{I}^-)$ generated by the elements $W(\Gamma\Phi)$ with $\Phi \in \mathcal{S}(\mathcal{M})$. Since Weyl algebras are determined up to $*$-algebra isomorphisms, $\mathcal{A}_{\mathcal{M}}$ is nothing but the Weyl algebra associated with the symplectic space $(\Gamma(\mathcal{S}(\mathcal{M})), \sigma)$ and the map $\Gamma : \mathcal{S}(\mathcal{M}) \to \Gamma(\mathcal{S}(\mathcal{M}))$ is an isomorphism of symplectic spaces. Under these hypotheses, there exists a unique $*$-isomorphism $\iota : \mathfrak{W}(\mathcal{M}) \to \mathcal{A}_{\mathcal{M}} \subset \mathfrak{W}(\mathscr{I}^-)$ completely individuated by (2.7.6). □

The existence of $\Gamma : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathscr{I}^-)$ fulfilling (2.7.5) implies the existence of a isometric $*$-homomorphism $\iota : \mathfrak{W}(\mathcal{M}) \to \mathfrak{W}(\mathscr{I}^-)$. In this way the field observables of the bulk are mapped into a suitable counterpart on $\mathscr{I}^-$ as a bilinear product for every state $\omega : \mathfrak{W}(\mathscr{I}^-) \to \mathbb{C}$ which satisfies Definition 2.6.1 there exists $\omega_{\mathcal{M}} = \iota^*\omega : \mathfrak{W}(\mathcal{M}) \to \mathbb{C}$ defined on the generators of $W(\mathcal{M})$ as

$$\omega_{\mathcal{M}}(W(\varphi)) = \iota^*\omega(W(\varphi)) \doteq \omega(\iota W(\varphi)) = \omega(W(\Gamma\varphi))$$

namely, the state ω on $\mathfrak{W}(\mathscr{I}^-)$ induces a counterpart ω_M on $\mathfrak{W}(\mathcal{M})$ via pull-back.

Chapter 3

Linearized gravity

Contents

Around 1915, Einstein formulated his theory General Relativity, which provided explanations for phenomena, such as the precession of the perihelion of Mercury, and the deflection of light rays by massive objects. However, even before Einstein's general relativity, various experiments were revealing that an entirely new framework, far removed from the "classical" approaches to physics, was required to describe nature on a microscopic scale. This framework came to be known as quantum theory. From those years, a tremendous amount of research has been devoted to the attainment of a full quantum description of gravity. A number of candidate theories have been put forward, most notably canonical quantum gravity, string theory and loop quantum gravity, but up to now a quantum theory of gravity proves to be elusive. Let us begin with field theories in Minkowski spacetime, say Maxwell theory to be specific. Here, the basic dynamical field is represented by the tensor field $F_{\mu\nu}$ on Minkowski. The spacetime geometry provides the kinematical arena on which the field propagates. We can foliate this spacetime by a one-parameter family of three dimensional spacelike hypersurface, and analyse how the values of electric and magnetic fields on one of these hypersurfaces determine those on any other surface. In General Relativity there is no fixed background and the metric itself is the fundamental dynamical variable. Thus in the canonical approach quantum gravity the metric tensor is split in two parts: Ones which plays the role of background and the other one which represents the role of linear perturbations which propagate on the background, *i.e.* a dynamical field. Specifically, one quantizes these perturbations and treats them as another quantum field propagating on the chosen fixed classical background. This approach has found applications particularly in early universe cosmology, where one studies tensor fluctuations in the cosmic microwave background (CMB) (see [22, 58]). A particular class of spacetimes used in the study of cosmology are Friedmann-Robertson-Walker and de Sitter spacetimes. In these it is possible to quantize the perturbations using the framework of algebraic quantum field theory, see [55] and [23]. The goals of this thesis is to generalize the quantization of gravitational perturbations on asymptotically flat vacuum spacetimes (see 1.3.7) and to construct thereupon an Hadamard state (see 2.6.5).

Let us start with the linearization of Einstein equations. For more details refer to [34].

3.1 Linearized Einstein equations

Let $\mathcal{M}$ be a globally hyperbolic vacuum spacetime (see 1.3.6 and 1.3.7) and let g^0 be a solution of Einstein equations, namely

$$R_{\mu\nu}(g^0) = 0$$

and let g be an approximate solution of the form

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

We are interest to linearizing the Einstein equations and to this end we follow [59] We consider a one-parameter family of exact solutions $\lambda \mapsto g(\lambda)$ which solve

$$G_{\mu\nu}\{g(\lambda)\} = 0 \tag{3.1.1}$$

where λ measures the size of perturbation namely

$$g_{\mu\nu}(\lambda) = g^0_{\mu\nu} + \lambda h_{\mu\nu}$$

Equation (3.1.1) is difficult to solve. Nevertheless, we can obtain a much simpler equation expanding it in Taylor series

$$G_{\mu\nu}\{g(\lambda)\} = G_{\mu\nu}(g^0) + \lambda \frac{d}{d\lambda} G_{\mu\nu}(g) + o(\lambda^2)$$

By imposing that the right hand side vanishes order by order in λ. We obtain

$$\left. \frac{d}{d\lambda} G_{\mu\nu}\{g(\lambda)\} \right|_{\lambda=0} = 0 \tag{3.1.2}$$

Equations (3.1.2) are linear per construction and we can be express then in the following form

$$(\mathscr{L} h)_{\mu\nu} = 0 \tag{3.1.3}$$

where $\mathscr{L}$ is a linear operator. Equations (3.1.3) are referred to as the "linearization" of (3.1.1). As shown in [34]

$$(\mathscr{L}h)_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \nabla^{\alpha}\nabla_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu}h^{\alpha}{}_{\alpha} - \frac{1}{2}g_{\mu\nu}(\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta} - \Box h^{\alpha}{}_{\alpha}) \quad (3.1.4)$$

where $(\cdot,\cdot)$ represent the symmetric part.
The linearized equations (3.1.4) also follow from the Euler-Lagrange equations of

$$L = T^{\mu\nu\alpha\beta\delta\gamma}\nabla_{\mu}h_{\nu\alpha}\nabla_{\beta}h_{\delta\gamma} \quad (3.1.5)$$

where $T^{\mu\nu\alpha\beta\delta\gamma}$ is given by

$$\begin{aligned} T^{\mu\nu\alpha\beta\delta\gamma} = \frac{1}{4}(&g^{\mu\beta}g^{\nu\alpha}g^{\delta\gamma} + g^{\mu\gamma}g^{\beta(\nu}g^{\alpha)\delta} + g^{\beta(\nu}g^{\alpha)\gamma}g^{\mu\delta} \\ &- g^{\mu\beta}g^{\delta(\nu}g^{\alpha)\gamma} - g^{\mu(\delta}g^{\gamma)\beta}g^{\nu\alpha} - g^{\beta(\nu}g^{\alpha)\mu}g^{\delta\gamma}). \end{aligned} \quad (3.1.6)$$

The Lagrangian (3.1.5) comes from the second order expansion of the Einstein-Hilbert action

$$S = \int_{\mathcal{M}} R\, dV_g,$$

where dV_g denotes the metric volume element of $\mathcal{M}$.
The covariant conjugate momentum

$$\Pi^{\mu\nu\lambda} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\nabla_{\mu}h_{\nu\lambda}} = 2T^{\mu\nu\lambda\delta\alpha\beta}\nabla_{\delta}h_{\alpha\beta} \quad (3.1.7)$$

is given by

$$\begin{aligned} \Pi^{\mu\nu\lambda} = &-\frac{1}{2}\nabla^{\mu}h^{\nu\lambda} + \frac{1}{2}g^{\nu\lambda}\nabla^{\mu}h^{\alpha}{}_{\alpha} - \frac{1}{2}g^{\nu\lambda}\nabla_{\delta}h^{\mu\delta} + \\ &- \frac{1}{4}g^{\mu\lambda}\nabla^{\nu}h^{\alpha}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}\nabla^{\lambda}h^{\alpha}{}_{\alpha} + \frac{1}{2}\nabla^{\nu}h^{\mu\lambda} + \frac{1}{2}\nabla^{\lambda}h^{\mu\nu} \end{aligned}$$

and the Euler-Lagrange equations are

$$\nabla_{\lambda}\Pi^{\lambda\mu\nu} = (\mathscr{L}h)^{\mu\nu} = 0.$$

The principal symbol of (3.1.4) is

$$\begin{aligned} \sigma^{P}_{\alpha}(K)^{\alpha\beta}_{\mu\nu} = &-\frac{1}{2}K_{\varrho}K^{\varrho}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + K^{\lambda}K_{(\mu}\delta^{\beta}_{\nu)}\delta^{\alpha}_{\lambda} + \\ &- \frac{1}{2}K_{\mu}K_{\nu}g^{\lambda\varrho}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\varrho} - \frac{1}{2}g_{\mu\nu}(K^{\lambda}K^{\varrho}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\varrho} - K^{\varepsilon}K_{\varepsilon}g^{\lambda\varrho}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\varrho}). \end{aligned}$$

It is not hyperbolic. To recover hyperbolicity we notice that

$$\left(\mathscr{L}\left(\mathcal{L}_{\xi}g\right)\right)_{\mu\nu} = 0 \quad (3.1.8)$$

for any vector field ξ where $\mathcal{L}_\xi g$ stand for the Lie derivative. This entails that for any solution h of (3.1.4), $h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi g$ is again a solution. This corresponds to linear counterpart of the group of diffeomorphisms, which is the group of General Relativity.
Suppose we choose a solution h of (3.1.4) such that

$$\nabla^\mu h_{\mu\nu} \neq 0.$$

We can use the gauge freedom to select ones which verifies

$$\nabla^\mu h'_{\mu\nu} = 0. \tag{3.1.9}$$

Starting from $h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi g$, this tantamount to look for ξ_ν which solves the following differential equations

$$\Box \xi_\nu = -\nabla^\mu h_{\mu\nu}.$$

Existence of a solution for these equations is proved in [60]. It is possible to note that there exists a residual gauge symmetry, namely $h'_{\mu\nu} \mapsto \widetilde{h}_{\mu\nu} = h'_{\mu\nu} + \mathcal{L}_{\xi'} g$ such that $\Box \xi' = 0$. This implies $\nabla^\mu \widetilde{h}_{\mu\nu} = 0$. Such additional freedom can be used as follows. Suppose ${h'}^\mu{}_\mu \neq 0$. Let $\widetilde{h}$ be equal to $h + \mathcal{L}_{\widetilde{\xi}} g$ such that $\Box \widetilde{\xi}_\nu = 0$ then

$$\widetilde{h}^\mu{}_\mu = 0. \tag{3.1.10}$$

Can be required provided a ξ_ν solves the following differential equations

$$\Box \widetilde{\xi}_\nu = 0 \qquad \nabla^\mu \widetilde{\xi}_\mu = -{h'}^\mu{}_\mu.$$

Existence of a solution for these equations is guaranteed under the construction that the Cauchy surface is not compact [23]. This gauge is also known as *transverse traceless gauge* (TT-gauge) and using h in place of $\widetilde{h}$ we obtain

$$\begin{aligned} &\nabla^\mu h_{\mu\nu} = 0 \\ &h^\alpha{}_\alpha = 0. \end{aligned} \tag{3.1.11}$$

In the TT-gauge equations (3.1.4) descends to

$$-\frac{1}{2}\Box h_{\mu\nu} + \nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} = 0$$

From the defining properties of the Riemann tensor and using $R_{\mu\nu} = 0$ (see Definition 1.3.1)

$$\begin{aligned} \nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} &= \nabla_{(\mu} \nabla^\alpha h_{\nu)\alpha} + R^\alpha{}_{(\mu\nu)}{}^\gamma h_{\gamma\alpha} + R^\alpha{}_{(\mu|\alpha|}{}^\gamma h_{\nu)\gamma} = \\ &= \nabla_{(\mu} \nabla^\alpha h_{\nu)\alpha} + R^\alpha{}_{\mu\nu}{}^\gamma h_{\gamma\alpha}. \end{aligned}$$

the linearized Einstein equations becomes

$$\begin{aligned}(\mathscr{L}h)_{\mu\nu} &= \Box h_{\mu\nu} - 2R^{\alpha}{}_{\mu\nu}{}^{\gamma}h_{\gamma\alpha} = 0\\ \nabla^{\mu}h_{\mu\nu} &= 0\\ h^{\alpha}{}_{\alpha} &= 0.\end{aligned} \tag{3.1.12}$$

3.2 The structure of the space of solutions

In this section we present the construction of the phase space for the solutions of the linearized Einstein equations (3.1.12) . The existence and the uniqueness of these solutions is guaranteed by Proposition 1.5.2.

Theorem 3.2.1. *Let be $h \in \Gamma(\bigotimes_s^2 T^*\mathcal{M})$ and G the causal propagator. Then h solves the linearized Einstein equations namely*

$$\begin{aligned}L_{\mu\nu}(h) &= \Box h_{\mu\nu} - 2R^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = 0\\ \nabla^{\mu}h_{\mu\nu} &= 0\\ h^{\mu}{}_{\mu} &= 0\end{aligned}$$

if and only if $h_{\mu\nu} = G(\lambda_{\mu\nu})$ for every $\lambda_{\mu\nu} \in \Gamma(\bigotimes_s^2 T^*\mathcal{M})$ and if it holds the following condition:

$$\lambda_{\mu\nu} = \nabla_{\mu}\nabla_{\nu}\alpha + \varepsilon_{\mu\nu}$$

with $\alpha \in \mathcal{D}(\mathcal{M})$ and $\varepsilon \in \Gamma_0^{\infty}(\bigotimes_s^2 T^*\mathcal{M})$ which satisfies $\varepsilon^{\mu}{}_{\mu} = 0$ and $\nabla^{\mu}\varepsilon_{\mu\nu} = \Box v_{\nu}$, for an arbitrary vector field v.

Proof. Let be $P_{\mu\nu}{}^{\alpha\beta} \doteq \Box\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - 2R^{\alpha}{}_{\mu\nu}{}^{\beta}$ and let be $h_{\mu\nu}$ be such that $P_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta} = 0$. This implies that exists an operator

$$\begin{aligned}G : \Gamma(\otimes_s^2 T^*\mathcal{M}) &\to \Gamma(\otimes_s^2 T^*\mathcal{M})\\ \lambda_{\mu\nu} &\mapsto h_{\mu\nu} = G_{\mu\nu}{}^{\alpha\beta}(\lambda_{\alpha\beta}).\end{aligned}$$

There exists also an hyperbolic operator Q_0 which satisfies the following identities:

$$\begin{aligned}g^{\mu\nu}\left[Ph\right]_{\mu\nu} &= \Box g^{\mu\nu}h_{\mu\nu} - 2g^{\mu\nu}R^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\mu\nu} =\\ &= \Box h^{\mu}{}_{\mu} + 2R^{\alpha}{}_{\mu}{}^{\beta\mu}h_{\mu\nu} = \Box h^{\mu}{}_{\mu} = [Q_0 h]\end{aligned} \tag{3.2.1}$$

where in equation (3.2.1) we have used the symmetry properties of Riemannian tensor and $R_{\mu\nu} = 0$. In this way $g^{\mu\nu}P = Q_0 g^{\mu\nu}$. Using the properties of the advanced and retarded Green operators G (see 1.5.4), we have

$$G^{\pm}_{Q_0}g^{\mu\nu} = g^{\mu\nu}G^{\pm}$$

from which it descends per linearity that

$$G_{Q_0}g^{\mu\nu} = g^{\mu\nu}G$$

Hence $g^{\mu\nu}G^{\alpha\beta}_{\mu\nu}\lambda_{\alpha\beta}=0$ if and only if $\lambda\in\ker(G_{Q_0})$. It yields that

$$\begin{gathered}\lambda=\Box\alpha\ \Rightarrow\ g^{\mu\nu}\lambda_{\mu\nu}=g^{\mu\nu}(\nabla_\mu\nabla_\nu\alpha+\varepsilon_{\mu\nu})\mid\varepsilon^\mu{}_\mu=0\\ \Downarrow\\ \lambda_{\mu\nu}=\nabla_\mu\nabla_\nu\alpha+\varepsilon_{\mu\nu}.\end{gathered}\tag{3.2.2}$$

In a similar way we construct an other hyperbolic operator Q_1 which satisfies the following equation

$$\begin{aligned}\nabla^\mu\left[Ph\right]_{\mu\nu}&=\nabla^\mu\nabla^\alpha\nabla_\alpha h_{\mu\nu}-2(\nabla^\mu R^\alpha{}_{\mu\nu}{}^\beta)h_{\alpha\beta}-2R^\alpha{}_{\mu\nu}{}^\beta\nabla^\mu h_{\alpha\beta}=\\&=g^{\mu\lambda}g^{\chi\vartheta}\nabla_\lambda\nabla_\chi\nabla_\vartheta h_{\mu\nu}-2R^\alpha{}_{\mu\nu}{}^\beta\nabla^\mu h_{\alpha\beta}=\\&=g^{\mu\lambda}g^{\chi\vartheta}(\nabla_\chi\nabla_\lambda\nabla_\vartheta h_{\mu\nu})-R^\mu{}_\chi{}^{\varrho\chi}\nabla_\varrho h_{\mu\nu}+R^\mu{}_{\chi\mu}{}^\varrho\nabla^\chi h_{\varrho\nu}+\\&+R^\mu{}_{\chi\nu}{}^\varrho\nabla^\chi h_{\varrho\mu}-2R^\alpha{}_{\mu\nu}{}^\beta\nabla^\mu h_{\alpha\beta}=\\&=g^{\mu\lambda}g^{\chi\vartheta}(\nabla_\chi\nabla_\vartheta\nabla_\lambda h_{\mu\nu})+\nabla_\chi R^{\mu\chi}{}_\mu{}^\varrho h_{\varrho\nu}+\nabla_\chi R^{\mu\chi}{}_\nu{}^\varrho h_{\varrho\mu}+\\&+R^\mu{}_{\chi\nu}{}^\varrho\nabla^\chi h_{\varrho\mu}-2R^\alpha{}_{\mu\nu}{}^\beta\nabla^\mu h_{\alpha\beta}=\\&=\Box\nabla^\mu h_{\mu\nu}+2R^\mu{}_{\chi\nu}{}^\varrho\nabla^\chi h_{\mu\varrho}-2R^\alpha{}_{\mu\nu}{}^\beta\nabla^\mu h_{\alpha\beta}=\\&=\Box\nabla^\mu h_{\mu\nu}=Q_1\nabla^\mu h_{\mu\nu}\end{aligned}\tag{3.2.3}$$

where in equation (3.2.3) we have used the symmetry properties of the Riemann tensor, the Bianchi identity (see 1.1.5) and $R_{\mu\nu}=0$. Using the properties of advanced and retarded Green operators G (see 1.5.4), we have

$$G^\pm_{Q_1}\nabla^\mu=\nabla^\mu G^\pm$$

from which descends per linearity that

$$G_{Q_1}\nabla^\mu=\nabla^\mu G.$$

Hence $\nabla^\mu G^{\alpha\beta}_{\mu\nu}\lambda_{\alpha\beta}=0$ if and only if $\nabla\lambda\in\ker(G_{Q_2})$. It yields that

$$\nabla^\mu\lambda_{\mu\nu}=\Box v_\nu.\tag{3.2.4}$$

Replacing (3.2.2) in (3.2.4) we obtain

$$\nabla^\mu\varepsilon_{\mu\nu}=\Box v_\nu-\Box\nabla_\nu\alpha=\Box\widetilde{w}_\nu$$

where $\widetilde{w}_\nu\doteq\Box v_\nu-\Box\nabla_\nu\alpha$. This concludes the proof. $\Box$

As a straightforward consequence of Theorem 3.2.1 and defining the space of test functions $\Lambda(\mathcal{M})$

$$\begin{aligned}\Lambda(\mathcal{M})\doteq\{\lambda\in\Gamma(\otimes^2_sT^*\mathcal{M})\,|\,&\forall\alpha\in\mathcal{D}(\mathcal{M}),\,\forall v\in\Gamma^\infty_0(T^*\mathcal{M}),\,\forall\varepsilon\in\Gamma^\infty_0(\otimes^2_sT^*\mathcal{M})\\&\wedge\ \lambda_{\mu\nu}=\nabla_\mu\nabla_\nu\alpha+\varepsilon_{\mu\nu}\wedge\ \varepsilon_{\mu\nu}=0\wedge\ \nabla^\mu\varepsilon_{\mu\nu}=\Box v_\nu\}\end{aligned}\tag{3.2.5}$$

the space of solutions of linearized Einstein equations is defined as follow

$$\mathcal{S}(\mathcal{M}) \doteq \{h_\lambda \in \Gamma(\otimes_s^2 T^*\mathcal{M}) \,|\, \exists\, \lambda \in \Lambda(\mathcal{M}) \wedge h_{\mu\nu} = G(\lambda_{\mu\nu})\} \tag{3.2.6}$$

As shown in [23], (3.2.6) can be endowed with a pre-symplectic product whose action on perturbations $h^1, h^2 \in \mathcal{S}(\mathcal{M})$

$$\sigma_\Sigma(h^1, h^2) = \int_\Sigma (h^1_{\mu\nu}\pi_2^{\mu\nu} - h^2_{\mu\nu}\pi_1^{\mu\nu}) dV_g, \tag{3.2.7}$$

where Σ is a spacelike Cauchy surface with future-pointing unit normal vector n, dV_g denotes the volume element on Σ associated with the induced spatial metric g and π is defined in terms of the covariant conjugate momentum Π, given in (3.1.7), by

$$\pi^{\mu\nu} \doteq -n_\alpha \Pi^{\alpha\mu\nu}.$$

The product (3.2.7) is independent of the choice of Cauchy surface.

Lemma 3.2.2. *Given $h^1, h^2 \in \mathscr{S}(M)$ and two spacelike Cauchy surfaces Σ, Σ' then $\sigma_\Sigma(h^1, h^2) = \sigma_{\Sigma'}(h^1, h^2)$.*

Proof. Let the current of h^1 and h^2 be $j^\alpha(h^1, h^2) \doteq h^2_{\mu\nu}\Pi_1^{\alpha\mu\nu} - h^1_{\mu\nu}\Pi_2^{\alpha\mu\nu}$. The pre-symplectic product of these perturbations is

$$\sigma_\Sigma(h^1, h^2) = \int_\Sigma n_\alpha j^\alpha(h^1, h^2) dV_g. \tag{3.2.8}$$

Now, the divergence of the current is $\nabla_\alpha j^\alpha = h^2_{\mu\nu} L^{\mu\nu}(h^1) - h^1_{\mu\nu} L^{\mu\nu}(h^2) = 0$, where we used (3.1) and the symmetry properties of $T^{\mu\nu\alpha\beta\delta\gamma}$ (see (3.1.6)). Using the divergence theorem over the region bounded by the two Cauchy surfaces Σ, Σ' and the property according to which $Supp(h^i) \cap \Sigma^{(1)}$ is compact, with $i = 1, 2$, gives the desired result. □

To make the pre-symplectic product into a symplectic one, it is necessary to account for the degeneracies of (3.2.7), that is, non-trivial solutions whose pre-symplectic product with all solutions is zero. The subspace of degeneracies is also known as the *radical* of the pre-symplectic form $\sigma_\mathcal{M}$.
Let the subspace of pure gauge solutions be

$$\mathcal{G}(\mathcal{M}) \doteq \{\mathcal{L}_\xi g \,|\, \xi \in \Gamma_0^\infty(T^*\mathcal{M})\}$$

As shown in [23] $\mathcal{M}$ has compact Cauchy surfaces, the radical of σ is precisely the subspace of pure gauge solutions $\mathcal{G}(\mathcal{M})$, namely, given $h_1 \in \mathcal{S}(\mathcal{M})$ such that $\sigma_\mathcal{M}(h_1, h_2) = 0$ for all $h_2 \in \mathcal{S}(\mathcal{M})$, then $h_1 \in \mathcal{G}(\mathcal{M})$. In any spacetime for which $\mathcal{G}(\mathcal{M})$ is the radical of $\sigma_\mathcal{M}$, we obtain the complexified phase space as the quotient space

$$\mathcal{P}(\mathcal{M}) \doteq \mathcal{S}(\mathcal{M})/\mathcal{G}(\mathcal{M}) \tag{3.2.9}$$

with weakly non-degenerate symplectic product

$$\sigma_{\mathcal{M}}([h^1],[h^2]) = \int_\Sigma (h^1_{\mu\nu}\pi_2^{\mu\nu} - h^2_{\mu\nu}\pi_1^{\mu\nu})dV_g. \tag{3.2.10}$$

This is independent of the choice of gauge.

3.3 The algebra of observables

In section 2.2 we have defined classical observables for a scalar field as functions on the phase space $\mathcal{G}(\mathcal{M})$ and then we have constructed the field algebra and the Weyl algebra. In this section we generalize such construction to symmetric second-rank tensor fields the result obtained. Let be $\lambda \in \Lambda(\mathcal{M})$. We introduce the basic *observable* $H_\lambda : \mathcal{P}(\mathcal{M}) \to \mathbb{C}$:

$$H_\lambda([h]) \doteq h(\lambda) = \int h_{\mu\nu}\lambda^{\mu\nu}dV_g \tag{3.3.1}$$

The observables 3.3.4 do not depend from the choice of a representative; in fact if we chose another representation h', we can correlate them which

$$h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \xi_\nu \tag{3.3.2}$$

In this way equation(3.3.4) becomes

$$h'(\lambda) = \int h'_{\mu\nu}\lambda^{\mu\nu}dV_g = \int (h_{\mu\nu} + \nabla_\mu\xi_\nu)\lambda^{\mu\nu}dV_g \tag{3.3.3}$$

The vector field ξ is chosen to be compactly supported. Hence we obtain

$$h'(\lambda) = \int h_{\mu\nu}\lambda^{\mu\nu}dV_g \tag{3.3.4}$$

The classical observables satisfy three relations.

(i) *complex linearity*: $H_{\alpha\lambda+\beta\tilde{\lambda}}([h]) = \alpha H_\lambda([h]) + \beta H_{\tilde{\lambda}}([h])$ for all $\alpha, \beta \in \mathbb{C}$ and all $\lambda, \tilde{\lambda} \in \Lambda(M)$;

(ii) *Hermiticity*: $H_\lambda([h])^* = H_{\lambda^*}([h^*])$ for all $\lambda \in \Lambda(M)$;

(iii) *symmetry*: $H_\lambda([h]) = 0$ for all antisymmetric $\lambda \in \Lambda(M)$.

As a matter of fact we can collect all our observables into an algebra which satisfies Definition 2.1.1.

Definition 3.3.1. We call *Weyl algebra* $\mathfrak{W}(\mathcal{M})$ *for linearized Einstein equations* the unital C^*-algebra generated by the abstract symbols $W(h)$, for all $h \in \mathcal{P}(\mathcal{M})$ such that for all $h_1, h_2 \in \mathcal{P}(\mathcal{M})$ the following condition are satisfied:

(i) $W(0) = 1$

(ii) $W(-h) = W(h)^*$

(iii) $W(h_1) \cdot W(h_2) = e^{-\imath\sigma(h_1,h_2)/2} W(h_1 + h_2)$

where σ is the symplectic product defined in equation (3.2.10). The symplectic vector space $\mathcal{P}(\mathcal{M}) = (\mathcal{S}(\mathcal{M}), \sigma)$ for which it is defined a Weyl algebra $\mathfrak{W}(\mathcal{M})$ is referred as *Weyl system for linearized Einstein equations.*

Here $\sigma : \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M}) \to \mathbb{C}$ is defined so that $\sigma(G\lambda_{\mu\nu}, G\lambda'_{\mu\nu}) = G(\lambda_{\mu\nu}, \lambda'_{\mu\nu})$ for all $\lambda_{\mu\nu}$,$\lambda'_{\mu\nu} \in \Lambda(\mathcal{M})$ (3.2.5).

For completeness we construct also the field algebra for linearized Einstein equations. We start to construct a Borchers-Uhlmann algebra and then we single out an ideal to implements the dynamics of the underlying field.

Definition 3.3.2. The Borchers-Uhlmann algebra $\mathscr{A}(\mathcal{M})$ is defined as follow

$$\mathscr{A}(\mathcal{M}) = \bigoplus_{n=1}^{\infty} \Lambda(\mathcal{M})^{\otimes n} \qquad \text{and} \qquad \Lambda(\mathcal{M})^0 = \mathbb{C}$$

If we single out $\mathscr{A}(\mathcal{M})$ by an ideal $\mathscr{I}(\mathcal{M})$ generated by elements of the form

$$-\imath G(\lambda, \widetilde{\lambda}) \oplus (\lambda \otimes \widetilde{\lambda} - \widetilde{\lambda} \otimes \lambda) \quad \text{and} \quad P\lambda$$

The *field algebra* $\mathscr{F}(\mathcal{M})$ *for the linearized Einstein equations* is

$$\mathscr{F}(\mathcal{M}) \doteq \mathscr{A}(\mathcal{M})/\mathscr{I}$$

is equipped with the product, $*$-operation and topology descending from $\mathscr{A}(\mathcal{M})$.

The algebra $\mathscr{F}(\mathcal{M})$ satisfies the following properties:

(i) there exists a product defined by linear extension of the tensor product of $\Lambda(\mathcal{M})^n$;

(ii) there exists a $*-$operation defined by the antilinear extension of $|\lambda^*|(x_1, \ldots, x_n) = \overline{\lambda}(x_n, \ldots, x_1)$;

(iii) there exists a sequence $\{\lambda_k\}_k = \{\otimes_n \lambda_k\}_k$ of elements in $\mathscr{F}(\mathcal{M})$ that converges to $\lambda = \otimes_n \lambda^{(n)}$ if $\lambda_k^{(n)}$ converges to $\lambda^{(n)}$ for all n in the locally convex topology of $\Lambda(\mathcal{M})^n$. There exists moreover N such that $\lambda_k = 0$ for all $n > N$ and all k

(iv) all elements of $\mathscr{F}(\mathcal{M})$ are finite linear combinations of multi-component test functions.

The distinction between $\mathfrak{W}(\mathcal{M})$ and $\mathscr{F}(\mathcal{M})$ is primarily technical: Any sufficiently regular representation $\widetilde{\pi} : \mathfrak{W}(\mathcal{M}) \to B(\mathscr{H})$ in a Hilbert space $\mathscr{H}$ induces a representation π of $\mathscr{F}(\mathcal{M})$ as unbounded operators on $\mathscr{H}$, so that

$$\pi(W(G\lambda)) = e^{\imath\widetilde{\pi}(h(\lambda))}$$

for any test function λ. The main advantage of the Weyl algebra are the following:

(i) it is generally easier to work with bounded rather than unbounded operators;

(ii) expectation values of the $W(h)$ take a simple form in quasifree states (2.7.4)

The main advantages of the field algebra are:

(i) to define a quasifree state we need a presymplectic product on the contrary to Weyl algebra which requires a weak non degenerate symplectic product;

(ii) via Definition 2.6.4 is always well-defined the n-point correlation function.

Both Weyl algebra and field algebra for linearized Einstein equations satisfies the Haag-Kastler axioms (see section 2.3) per construction.

3.4 Hadamard states

The last step in the algebraic approach consists of representing the algebra of observables on a suitable Hilbert space. We have seen that using the GNS Theorem 2.6.3 it is possible to find a representation either of the Weyl or of the field algebra. Yet the construction of an algebraic state is fundamental.
In section 2.7.1 we have outlined the bulk to boundary correspondence and we applied it to a scalar field. A key ingredient has been the existence of a correspondence between solutions of the field equation in the bulk and a scalar field defined intrinsically in $\mathscr{I}^-$. With the next theorem, we proposed a correspondence between solutions of equations (3.1.12) between bulk and $\mathscr{I}^-$.

Theorem 3.4.1. *Assume that $(\mathcal{M}, g)$ is asymptotically flat (see Definition 1.3.7) with associated unphysical space $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\widetilde{g}|_{\mathcal{M}} = \Omega^2 g$. Suppose that there exists an open set $\widetilde{V} \subset \widetilde{\mathcal{M}}$ with $\overline{\mathcal{M} \cap J^-(\mathscr{I}^-)} \subset \widetilde{V}$ such that $(\widetilde{V}, \widetilde{g})$ is globally hyperbolic so that $(\mathcal{M} \cap V, g)$ is globally hyperbolic, too. If $h \in \mathcal{S}(\mathcal{M})$ the following condition is satisfied:*

(i) the field $\widetilde{h} \doteq \Omega h$ can be extended uniquely to of

$$\begin{aligned}\widetilde{\nabla}^\alpha \widetilde{\nabla}_\alpha \widetilde{h}_{\mu\nu} \; - \frac{4}{3}\, \widetilde{\nabla}_{(\mu} \widetilde{\nabla}^\alpha \widetilde{h}_{\nu)\alpha} + \frac{2}{3}\, \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \widetilde{h}^{\alpha\beta} \widetilde{g}_{\mu\nu} \\ - 2\, \widetilde{R}_{\mu\alpha\beta\nu} \widetilde{h}^{\alpha\beta} - \frac{1}{2}\, \widetilde{R}_{\alpha\beta} \widetilde{h}^{\alpha\beta} \widetilde{g}_{\mu\nu} - \frac{1}{6}\, \widetilde{R}\, \widetilde{h}_{\mu\nu} = 0\end{aligned} \tag{3.4.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection depending from $\widetilde{g}$ while $\widetilde{R}_{\mu\alpha\beta\nu}$, $\widetilde{R}_{\mu\nu}$ and $\widetilde{R}$ are respectively the Riemann, the Ricci tensor and scalar curvature depending from $\widetilde{g}$.

Proof. Let be $h \in \Gamma(\bigotimes_s^2 T^*\mathcal{M})$ which satisfies the TT gauge (3.1.11)

$$\begin{aligned}\nabla^\mu h_{\mu\nu} &= 0 \\ h &= 0\end{aligned}$$

and let be $\Phi : \mathcal{M} \to \mathcal{N}$ a conformal transformation, which map g in $\widetilde{g} = \Omega^2 g$. Then equation (3.1.11) becomes

$$\widetilde{\nabla}^\mu \widetilde{h}_{\mu\nu} = 3\Omega^{-3}\nabla^\mu \Omega h_{\mu\nu} \neq 0$$
$$\widetilde{h} = \Omega g^{\mu\nu} h_{\mu\nu} = 0.$$

This allows to construct conformally covariant equations for symmetric second-rank traceless tensor. For this aim, we use The Weyl-to-Riemann method, defined in subsection 1.2.1. In this construction we follow [33]: The author, in his calculation, employs the signature $(+,-,-,-)$ a which we shift for convenience. Only at the end of the calculation we can switch back to our signature arranging the signs.
Let us write the most general second order equation for symmetric second-rank traceless tensors in Weyl space

$$\widetilde{D}_{\mu\nu} + \widetilde{U}_{\mu\nu} = 0 \tag{3.4.2}$$

where the derivative part is given by

$$\begin{aligned}\widetilde{D}_{\mu\nu} = \widetilde{D}^2 h_{\mu\nu} + a_1\, \widetilde{D}_{(\mu}\widetilde{D}^\alpha h_{\nu)\alpha} + a_2\, \widetilde{D}_\alpha \widetilde{D}_\beta h^{\alpha\beta} g_{\mu\nu} +\\ + a_3\, \widetilde{D}_\mu \widetilde{D}_\nu h + a_4\, \widetilde{D}^2 h g_{\mu\nu}\end{aligned} \tag{3.4.3}$$

and the geometrical part reads

$$\begin{aligned}\widetilde{U}_{\mu\nu} = b_1\, \widetilde{R}_{\mu\alpha\beta\nu} h^{\alpha\beta} + b_2\, \widetilde{R}_{\alpha\beta} h^{\alpha\beta} g_{\mu\nu} + b_3\, \widetilde{R}_{\alpha(\mu} h_{\nu)}^{\ \alpha} +\\ + b_4\, \widetilde{R}\, h_{\mu\nu} + b_5\, \widetilde{R}_{\mu\nu} h + b_6\, \widetilde{R}\, g_{\mu\nu} h.\end{aligned} \tag{3.4.4}$$

The free parameters a_i, with $i = 1, \dots 4$ and b_j with $j = 1, \dots, 6$ are real constants. Both (3.4.3) and (3.4.4) are defined using the Weyl covariant derivative (1.2.2) and the Weyl connection (1.2.4), which are conformally covariant (see (1.2.1)). In this manner also equation (3.4.2) is conformally covariant. By direct inspection we have:

$$\begin{aligned}\widetilde{D}^2 h_{\mu\nu} = \Box\, h_{\mu\nu} - 2(\omega-1)\nabla_\alpha h_{\mu\nu}\varphi^\alpha + (\omega^2 - 2\omega - 2)\varphi^2 h_{\mu\nu} - (\omega-2)\nabla_\alpha \varphi^\alpha h_{\mu\nu} +\\ - 2\varphi_\alpha \nabla_\mu h^\alpha_{\ \nu} - 2\varphi_\alpha \nabla_\nu h^\alpha_{\ \mu} + 4\varphi_{(\mu}\nabla^\alpha h_{\nu)\alpha} - 8\varphi_{(\mu} h_{\nu)\alpha}\,\varphi^\alpha + g_{\mu\nu} h_{\alpha\beta}\varphi^\alpha\varphi^\beta,\end{aligned}$$

$$\begin{aligned}\widetilde{D}_{(\mu}\widetilde{D}_{|\alpha|} h_{\nu)}^\alpha = \nabla_{(\nu}\nabla_{|\alpha|} h^\alpha_{\mu)} - (\omega-4)\varphi_{(\mu}\nabla^\alpha h_{\nu)\alpha} + (2+\omega) g_{\mu\nu} h_{\alpha\beta}\varphi^\alpha\varphi^\beta +\\ - (\omega+2)\nabla_{(\nu} h^\alpha_{\ \mu)}\varphi_\alpha + +(\omega^2 - 2\omega - 8) h^\alpha_{\ (\mu}\varphi_{\nu)}\varphi_\alpha - (\omega+2)\nabla_\alpha \varphi_{(\mu} h_{\nu)}^\alpha,\end{aligned}$$

$$\widetilde{D}_\alpha \widetilde{D}_\beta h^{\alpha\beta} = \nabla_\alpha \nabla_\beta h^{\alpha\beta} - 2(1+\omega)\nabla_\alpha h^{\alpha\beta}\varphi_\beta + (2+\omega) h^{\alpha\beta}(\omega\varphi_\alpha\varphi_\beta - \nabla_\beta\varphi_\alpha),$$

$$\widetilde{D}_\mu \widetilde{D}_\nu h = 0,$$

$$\widetilde{R}_{\mu\alpha\beta\nu}h^{\alpha\beta} = R_{\mu\alpha\beta\nu}h^{\alpha\beta} + (\varphi_\alpha\varphi_\beta + \nabla_\alpha\varphi_\beta)h^{\alpha\beta}g_{\mu\nu} + \\ + \varphi^2 h_{\mu\nu} - 2h^\alpha{}_{(\mu}(\nabla_{|\alpha|}\varphi_{\nu)} + \varphi_{|\alpha|}\varphi_{\nu)}),$$

$$\widetilde{R}_{\alpha(\mu}h^\alpha_{\nu)} = R_{\alpha(\mu}h^\alpha_{\nu)} - 2h^\alpha{}_{(\mu}(\nabla_{|\alpha|}\varphi_{\nu)} + \varphi_{|\alpha|}\varphi_{\nu)}) - (\nabla_\alpha\varphi^\alpha - 2\varphi^2)h_{\mu\nu},$$

$$\widetilde{R}_{\alpha\beta}\, h^{\alpha\beta} = R_{\alpha\beta}\, h^{\alpha\beta} - 2(\varphi_\alpha\varphi_\beta + \nabla_\beta\varphi_\alpha)h^{\alpha\beta},$$

$$\widetilde{R} = R - 6(\nabla_\alpha\varphi^\alpha - \varphi^2),$$

where ω and φ are defined in equations (1.2.2). Focus on (3.4.3). We cancel all terms containing both φ and derivative of $h_{\mu\nu}$, to express (3.4.2) in terms of the Levi-Civita connection.

$$\begin{aligned}\widetilde{D}_{\mu\nu} = D_{\mu\nu} \; & - 2(\omega - 1)\varphi^\alpha\nabla_\alpha h_{\mu\nu} \\ & - (4 + a_1(\omega + 2))\varphi^\alpha\nabla_{(\mu}h_{\nu)\alpha} \\ & + (4 - a_1(\omega - 4))\varphi_{(\mu}\nabla^\alpha h_{\nu)\alpha} \\ & - 2(a_1 + a_2(\omega + 1))\varphi^\alpha\nabla^\beta h_{\alpha\beta}g_{\mu\nu} \\ & + \textit{non derivative terms},\end{aligned}$$

where

$$D_{\mu\nu} = \Box h_{\mu\nu} \; + a_1\, \nabla_{(\mu}\nabla^\alpha h_{\nu)\alpha} + a_2\, \nabla_\alpha\nabla_\beta h^{\alpha\beta}g_{\mu\nu}.$$

The resulting algebraic system reads

$$\begin{aligned}\omega - 1 &= 0, \\ 2 + a_1(\omega + 2) &= 0, \\ 2 - a_1(\omega - 4) &= 0, \\ a_1 + a_2(\omega + 1) &= 0.\end{aligned}$$

This system has the following set of solutions,

$$\omega = 1, \qquad a_1 = -\frac{4}{3}, \qquad a_2 = \frac{2}{3}.$$

The resulting expression for the derivative part reads

$$\widetilde{D}_{\mu\nu} = D_{\mu\nu} + 4\left(\varphi_\alpha h^\alpha_{(\mu}\varphi_{\nu)} + h^\alpha_{(\mu}\nabla_{\nu)}\varphi_\alpha\right) + \\ - (\nabla_\beta\varphi_\alpha + \varphi_\alpha\varphi_\beta)h^{\alpha\beta}g_{\mu\nu} + (\nabla_\alpha\varphi^\alpha - 3\varphi^2)h_{\mu\nu}. \tag{3.4.5}$$

Adding (3.4.4) to (3.4.5)

$$\begin{aligned}\widetilde{D}_{\mu\nu} + \widetilde{U}_{\mu\nu} = D_{\mu\nu} + U_{\mu\nu} &+ (1 - 2b_3 - 6b_4)\,\nabla_\alpha \varphi^\alpha h_{\mu\nu} \\ &+ (-3 + b_1 + 4b_3 + 6b_4)\,\varphi^2 h_{\mu\nu} \\ &+ 2(2 - b_1 - 2b_3)\,\left(\varphi_\alpha \varphi_{(\mu} + \nabla_{(\mu}\varphi_{|\alpha|}\right) h_{\nu)}^{\ \alpha} \\ &+ (-1 + b_1 - 2b_2)\,(\nabla_\beta \varphi_\alpha + \varphi_\alpha \varphi_\beta)\, h^{\alpha\beta} g_{\mu\nu} \\ &= 0\end{aligned} \tag{3.4.6}$$

where

$$\widetilde{U}_{\mu\nu} = b_1\, R_{\mu\alpha\beta\nu} h^{\alpha\beta} + b_2\, R_{\alpha\beta} h^{\alpha\beta} g_{\mu\nu} + b_3\, R_{\alpha(\mu} h_{\nu)}^{\ \alpha} + b_4\, R\, h_{\mu\nu}.$$

The resulting algebraic system reads

$$\begin{aligned} 1 - 2b_3 - 6b_4 &= 0, \\ -3 + b_1 + 4b_3 + 6b_4 &= 0, \\ 2 - b_1 - 2b_3 &= 0, \\ -1 + b_1 - 2b_2 &= 0. \end{aligned}$$

This system has an infinite set of solutions which can take the form

$$b_1 = 2 - 2\tau, \quad b_2 = \frac{1}{2} - 2\tau, \quad b_3 = \tau, \quad b_4 = \frac{1}{6} - \frac{1}{3}\tau.$$

Setting $\tau = 0$ and changing signature

$$g_{(+,-,-,-)} = -g_{(-,+,+,+)}$$

equation (3.4.6) reads

$$\begin{aligned}\Box h_{\mu\nu} \ &- \frac{4}{3}\nabla_{(\mu}\nabla^\alpha h_{\nu)\alpha} + \frac{2}{3}\nabla_\alpha \nabla_\beta h^{\alpha\beta} g_{\mu\nu} \\ &- 2\, R_{\mu\alpha\beta\nu} h^{\alpha\beta} - \frac{1}{2} R_{\alpha\beta} h^{\alpha\beta} g_{\mu\nu} - \frac{1}{6} R\, h_{\mu\nu} = 0.\end{aligned} \tag{3.4.7}$$

Bear in mind that (3.4.7) is written in terms of Levi-Civita connection and it is conformally covariant for construction. If we consider a vacuum spacetime, namely $R_{\mu\nu} = R = 0$, they are reduces to

$$\Box h_{\mu\nu} \ - \frac{4}{3}\nabla_{(\mu}\nabla^\alpha h_{\nu)\alpha} + \frac{2}{3}\nabla_\alpha \nabla_\beta h^{\alpha\beta} g_{\mu\nu} - 2\, R_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0. \tag{3.4.8}$$

These equations are not hyperbolic, but we are interested at solution which satisfies also TT gauge (3.1.11). With such additional constraint (3.4.8) reduces a

$$\Box h_{\mu\nu} - 2\, R_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0. \tag{3.4.9}$$

Since $(\mathcal{M}, g)$ is asymptotically flat it is possible to associate an unphysical spacetime

$(\widetilde{\mathcal{M}}, \widetilde{g})$, namely, there exists a conformal transformation defined as follows

$$\begin{aligned} \chi : (M, g) &\to (\widetilde{\mathcal{M}}, \widetilde{g}_{\widetilde{\mathcal{M}}}) \\ h &\mapsto \widetilde{h} = \Omega h. \end{aligned} \tag{3.4.10}$$

After transformation 3.4.10, (3.4.7) becomes

$$\begin{aligned} \widetilde{\nabla}^{\alpha}\widetilde{\nabla}_{\alpha}\widetilde{h}_{\mu\nu} \; - \frac{4}{3}\,\widetilde{\nabla}_{(\mu}\widetilde{\nabla}^{\alpha}\widetilde{h}_{\nu)\alpha} + \frac{2}{3}\,\widetilde{\nabla}_{\alpha}\widetilde{\nabla}_{\beta}\widetilde{h}^{\alpha\beta}\widetilde{g}_{\mu\nu} \\ - 2\,\widetilde{R}_{\mu\alpha\beta\nu}\widetilde{h}^{\alpha\beta} - \frac{1}{2}\,\widetilde{R}_{\alpha\beta}\widetilde{h}^{\alpha\beta}\widetilde{g}_{\mu\nu} - \frac{1}{6}\,\widetilde{R}\,\widetilde{h}_{\mu\nu} = 0 \end{aligned} \tag{3.4.11}$$

and the gauge TT (3.1.11)

$$\widetilde{\nabla}^{\mu}\widetilde{h}_{\mu\nu} = 3\Omega^{-3}\nabla^{\mu}\Omega h_{\mu\nu}\widetilde{h} = \Omega^{-1}g^{\mu\nu}h_{\mu\nu} = 0. \tag{3.4.12}$$

Equations (3.4.8) becames hyperbolic one we insert (3.4.12) since the second and the third terms became dependent only upon the first derivative of h hence they are subprincipal terms (see 1.5.1). This conclude our proof. □

We are interested to propagate our solution from the bulk to the conformal boundary $\mathscr{I}$ in which the conformal factor Ω take zero value. This might be hindered by which might that by second and third term in (3.4.8), blow up in the limit $\Omega \to 0$. This hurdle is bypassed in [61], in which is shown that a smooth propagation of a solution to $\mathscr{I}$ is ensured by switching to the Geroch-Xanthopoulous gauge. This entails that, at a level of classical fields, there exists a correspondence between solutions of the field equations (3.4.9) and a smooth field $\widetilde{h}$ on $\mathscr{I}^{-}$. This allows us to apply the bulk to boundary correspondence.
We are interested in formulating a quantum field theory on $\mathscr{I}$ and in constructing a well defined state. The first step consists of endowing the space of solutions on $\mathscr{I}$ with a symplectic product. Let us start considering $\mathscr{I}$ equipped with a collection of pairs of smooth fields (q, n) (see (1.3.4)) satisfying the following conditions:

(i) $q_{\mu\nu}n^{\nu} = 0$,

(ii) the Lie derivative $\mathcal{L}_n q_{\mu\nu} = 0$,

(iii) (q, n) and $(\overline{q}, \overline{n})$ are both in the collection C (see (1.3.4)) if and only if there exists a function ω on $\mathscr{I}$ such that

$$\overline{q}_{\mu\nu} = \omega^2 q_{\mu\nu}, \qquad \overline{n}^{\mu} = \omega^{-1}n^{\mu}, \qquad \mathcal{L}_n\omega = 0,$$

(iv) n^{μ} is a complete vector field and the space of its orbits is diffeomorphic to S^2.

Fix a conformal frame, *i.e.* a pair (q, n) from the collection C on $\mathscr{I}$ and denote by V the affine space of torsion-free connections D on $\mathscr{I}$ satisfying

$$D_{\alpha}q_{\mu\nu} = 0, \qquad D_{\mu}n^{\mu} = 0.$$

Finally, introduce the following equivalence relation on C:

$$D \sim D' \qquad \Longleftrightarrow \qquad (D_\mu - D'_\mu)K_\nu = f q_{\mu\nu} n^\alpha K_\alpha$$

for any function f and vector K on $\mathscr{I}$. Finally, denote with Γ the space of equivalence classes $\{D\}$. Γ has the structure of an affine space: By fixing any point $\{D^0\}$ as “origin”, the results vector space is the collection of symmetric tensor fields $\gamma_{\mu\nu}$ satisfying:

$$\gamma_{\mu\nu} n^\nu = 0, \qquad \gamma_{\mu\nu} q^{\mu\nu} = 0. \tag{3.4.13}$$

Let us introduce

$$\mathcal{S}(\mathscr{I}^-) \doteq \left\{ \gamma \in \Gamma(\otimes^2_S T^*\mathbb{R} \otimes^2_S T^*S^2) \,|\, \gamma\,, \partial_\ell \gamma \in L^2(\mathbb{R} \times S^2, d\ell \wedge dS^2(\vartheta,\varphi) \otimes M_4(\mathbb{C}) \right\}.$$

This is a symplectic space if endowed

$$\sigma_{\mathscr{I}} = \int_{\mathscr{I}} \left(\gamma_{\mu\nu} \mathcal{L}_n \overline{\gamma}_{\alpha\beta} - \overline{\gamma}_{\mu\nu} \mathcal{L}_n \gamma_{\alpha\beta} \right) q^{\mu\alpha} q^{\nu\beta}\, d\ell \wedge dS^2(\vartheta,\varphi).$$

We associate to such symplectic space the field algebra $\mathscr{F}(\mathscr{I}^-)$. It is constructed starting from the Borchers-Uhlmann algebra

$$\mathscr{A}(\mathscr{I}^-) = \mathbb{C} \oplus \mathcal{S}(\mathscr{I}^-) \oplus \left(\mathcal{S}(\mathscr{I}^-) \otimes \mathcal{S}(\mathscr{I}^-)\right) \oplus \left(\mathcal{S}(\mathscr{I}^-) \otimes \mathcal{S}(\mathscr{I}^-) \otimes \mathcal{S}(\mathscr{I}^-)\right) \oplus \ldots$$

and single out an ideal $\mathscr{I}$ which is generated by elements of the form

$$\gamma^1 \otimes \gamma^2 - \gamma^2 \otimes \gamma^1 - \imath \sigma_{\mathscr{I}}(\gamma^1, \gamma^2) \mathbb{1}$$

In this way the algebra of fields on $\mathscr{I}^-$ is

$$\mathscr{F}(\mathscr{I}^-) = \mathscr{A}(\mathscr{I}^-)/\mathscr{I}.$$

Having constructed the algebra of fields on $\mathscr{I}^-$ we are interested to identify an injective $*$-homomorphism ι

$$\iota : \mathscr{F}(\mathcal{M}) \to \mathscr{F}(\mathscr{I}^-)$$

where $\mathscr{F}(\mathcal{M})$ and $\mathscr{F}(\mathscr{I}^-)$ are the algebras of fields. This is tantamount to construct a symplectomorphim between $\mathcal{S}(\mathcal{M})$ and $\mathcal{S}(\mathscr{I}^-)$, the generated spaces for $\mathscr{F}(\mathcal{M})$ and $\mathscr{F}(\mathscr{I}^-)$ respectively. We look for a linear map $\Gamma : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathscr{I}^-)$ such that

$$\sigma_{\mathcal{M}}(h_1, h_2) = \sigma_{\mathscr{I}^-}(\Gamma\widetilde{h}_1, \Gamma\widetilde{h}_2).$$

Via Theorem 2.7.6, the corresponding injective $*$-homomorphism

$$\iota : \mathscr{F}(\mathcal{M}) \to \mathscr{F}(\mathscr{I}^-)$$

is defined via its action on the generators of $\mathscr{F}(\mathcal{M})$

$$\iota[h] \doteq \Gamma[h]$$

In this way the field observables of the bulk are mapped into counterparts for the theory on $\mathscr{I}^-$. If we define moreover an algebraic state $\omega_{\mathscr{I}} : \mathscr{F}(\mathcal{M}) \to C$ on $\mathscr{I}^-$, it induces a preferred state $\omega_{\mathcal{M}}$ on $\mathscr{F}(\mathcal{M})$ via pull-back:

$$\omega_{\mathcal{M}} \doteq \iota^* \omega_{\mathscr{I}^-}. \tag{3.4.14}$$

Theorem 3.4.2. *Let be h_1, h_2 solutions for linearized Einstein equations 3.1.12 and let χ a conformal transformation defined as follow*

$$\begin{aligned} \chi : (M, g) &\to (\widetilde{\mathcal{M}}, \widetilde{g}_{\widetilde{\mathcal{M}}}) \\ h &\mapsto \widetilde{h} = \Omega h. \end{aligned}$$

Then there exists a linear map $\Gamma : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathscr{I}^-)$ which satisfies

$$\sigma_{\mathcal{M}}(h_1, h_2) = \sigma_{\mathscr{I}^-}(\Gamma\widetilde{h}_1, \Gamma\widetilde{h}_2).$$

Proof. For a more exhaustive proof it is possible to see [61].

$$\sigma_{\mathcal{M}}(\left[h^1\right], \left[h^2\right]) = 3 \int_\Sigma \varepsilon^{\alpha\beta\mu}{}_\nu (h^1_{\alpha\delta} \nabla_\beta h^2_{\mu\gamma} - h^2_{\alpha\delta} \nabla_\beta h^1_{\mu\gamma}) dS^{\nu\delta\gamma}$$

where Σ is any Cauchy surface in $(\mathcal{M}, g)$ and $\left[h^1\right], \left[h^2\right] \in \mathcal{S}(\mathcal{M})$. This symplectic product is well-defined and using $\widetilde{\varepsilon}^{\alpha\beta\mu}{}_\nu = \Omega^2 \varepsilon^{\alpha\beta\mu}{}_\nu$ and $h_{\mu\nu} = \Omega^{-1}\widetilde{h}_{\mu\nu}$ one has

$$\begin{aligned} \sigma_{\mathcal{M}}(\left[h^1\right], \left[h^2\right]) = 3 \int_\Sigma \widetilde{\varepsilon}^{\alpha\beta\mu}{}_\nu \Big(&\widetilde{h}^1_{\alpha\delta} \nabla_\beta \widetilde{h}^2_{\mu\gamma} - \widetilde{h}^2_{\alpha\delta} \nabla_\beta \widetilde{h}^1_{\mu\gamma} + \\ &- \Omega^{-1}(\nabla_n \Omega) \widetilde{h}^1_{\alpha\delta} \widetilde{h}^2_{\mu\gamma} + \Omega^{-1}(\nabla_n \Omega) \widetilde{h}^2_{\alpha\delta} \widetilde{h}^1_{\mu\gamma} \Big) dS^{\nu\delta\gamma}. \end{aligned}$$

Since the 3-form appearing in the integrand is curl-free, we may choose for Σ a surface defined by Ω= const. Then, replacing ∇ with $\widetilde{\nabla}$ one obtains

$$\begin{aligned} \sigma_{\mathcal{M}}(\left[h^1\right], \left[h^2\right]) = 3 \int_\Sigma \widetilde{\varepsilon}^{\alpha\beta\mu}{}_\nu \Big(&\widetilde{h}^1_{\alpha\delta} \widetilde{\nabla}_\beta \widetilde{h}^2_{\mu\gamma} - \widetilde{h}^2_{\alpha\delta} \widetilde{\nabla}_\beta \widetilde{h}^1_{\mu\gamma} + \\ &- \Omega^{-1}(\widetilde{\nabla}_n \Omega) \widetilde{h}^1_{\alpha\delta} \widetilde{h}^2_{\mu\gamma} + \Omega^{-1}(\widetilde{\nabla}_n \Omega) \widetilde{h}^2_{\alpha\delta} \widetilde{h}^1_{\mu\gamma} \Big) dS^{\nu\delta\gamma}. \end{aligned}$$

Finally, replace Σ by $\mathscr{I}$. Using (3.4.13)

$$\widetilde{h}_{\mu\nu} n^\nu = 0 \qquad \mathcal{L}_n(\widetilde{h}_{\mu\nu} q^{\mu\nu}) = 0$$

one obtains

$$\sigma_{\mathscr{I}}(\left[h^1\right], \left[h^2\right]) = 3 \int_\Sigma \left(\gamma^1_{\alpha\beta} \mathcal{L}_n \gamma^2_{\mu\nu} - \gamma^2_{\alpha\beta} \mathcal{L}_n \gamma^1_{\mu\nu}\right) q^{\mu\alpha} q^{\nu\beta} dS$$

where $\gamma_{\mu\nu} = \widetilde{h}_{\mu\nu} - 1/2\,\widetilde{h}_{\alpha\beta}q^{\alpha\beta}q_{\mu\nu}$. □

In case the hypotheses of Theorem 3.4.2 are fulfilled. Another relevant consequence is the following: Any algebraic state $\omega_{\mathscr{I}} : \mathscr{F}(\mathscr{I}^-) \to \mathbb{C}$ can be pulled back on $\mathcal{S}(\mathcal{M})$ through ι to the state $\omega_\iota : \mathscr{F}(\mathcal{M}) \to \mathbb{C}$ defined as

$$\omega_{\iota,2}(h_1, h_2) = \iota^*\omega_{\mathscr{I},2}(h_1, h_2) \doteq \omega_{\mathscr{I},2}(\iota(h_1, h_2)) = \omega_{\mathscr{I},2}(\Gamma h_1, \Gamma h_2)$$

for all $h \in \mathscr{F}(\mathcal{M})$. In between all boundary states we would identify a distinguished choice by requiring invariance under the BMS group. According to Proposition 2.11 in [21], the following two point correlation function of the state $\omega_{\mathscr{I}}$ defined on the boundary algebra

$$\omega_{\mathscr{I},2}(h^1, h^2) = \lim_{\varepsilon\downarrow 0} \int_{\mathscr{I}} \frac{h^{1\ \mu\nu}(\ell_1, \vartheta, \varphi)\partial_{\ell_2} h^2_{\mu\nu}(\ell_2, \vartheta, \varphi)}{(\ell_1 - \ell_2 - \imath\varepsilon)}\, d\ell_1 d\ell_2 \wedge dS^2(\vartheta, \varphi)$$

is well defined and satisfies Hadamard condition 2.6.5. Furthermore in [18] is proved that this state is the unique quasifree pure state on $\mathfrak{W}(\mathscr{I}^-)$ which is invariant under the action on the BMS group. Under the action of the pull-back map, we obtains the state on the bulk algebra of fields:

$$\omega_{\iota,2}(h^1, h^2) = \lim_{\varepsilon\downarrow 0} \int_{\mathscr{I}} \frac{\Gamma h^{1\ \mu\nu}(\ell_1, \vartheta, \varphi)\partial_{\ell_2} \Gamma h^2_{\mu\nu}(\ell_2, \vartheta, \varphi)}{(\ell_1 - \ell_2 - \imath\varepsilon)}\, d\ell_1 d\ell_2 \wedge dS^2(\vartheta, \varphi).$$

This state turns out to be an Hadamard form and it is also invariant under the actions of all isometries of the bulk: In particular when the spacetime coincident with Minkowski, it is invariant also under the action of Poincaré group and it is possible to identify it with the vacuum state.

Chapter 4

Conclusions

In this thesis we provided a self-contained description of the quantization of linearized gravity. Hence, as a starting point, we laid down the necessary mathematical foundations. Although it does not include novel results, it offers a partly different perspective on some of the mentioned definitions and propositions.

Afterwards we discussed several aspects of quantum field theory on curved spacetimes in the algebraic approach. We introduced first the formalism first outlined by Haag and Kastler: The algebra of observables is seen as a $*$-algebra, which fulfils a suitable set of axioms motivated by physical requirements. Our treatment departs from the existing literature insofar that the field algebra and its properties are often only discussed for the scalar field. Since our goal is to focus on Hadamard states we employed the bulk to boundary correspondence in asymptotically flat spacetimes for an abstractly defined bosonic quantum field. The net advantage of such procedure is that it yields naturally an asymptotic vacuum state.

In Chap. 3 we tackled the main issue of this thesis, namely, the quantization of linearized gravity and the construction of an associated Hadamard state via the bulk to boundary correspondence. We introduced first the linearized Einstein equations and we investigated their classical space phase. Using the causal propagator, we construct the algebra of fields and the Weyl algebra. A full-fledged application of the bulk to boundary correspondence has been then used to select a Hadamard state for the field algebra which reduces on Minkowski spacetime to the Poincaré vacuum .

In other words we have constructed a quantum field theory of the free graviton within the rigorous mathematical framework of algebraic quantum field theory. In particular the existence of a concrete Hadamard state is useful and essential to develop a theory for which we can perform reliable calculations for the quantum corrections to classical gravity, of course under the assumption that these are small and thus a perturbative treatment is allowed. Such corrections are vital for instance when one considers cosmology: Via Hadamard state, one could calculate the expectation value of the renormalized stress-energy tensor and then the fluctuations in the cosmic microwave background [62]. Additionally, in [63] it is shown that an approach to quantum gravity within the algebraic framework is conceivable, but in their construction an algebraic state is found only for ultrastatic spacetimes. In particular there does not exist a deformation argument which could help us to prove existence of Hadamard state on an arbitrary background. In this respect this is another context where an application of our results might be of help. Nevertheless it is possible to give more and more motivations for the importance of studying the quantum aspects of gravity; yet, we prefer to conclude this thesis with a (very purposeful) quotation from Feynman:

"Physics is like sex. Sure you can get some interesting results,
but that's not why we do it. "

Bibliography

[1] Rudolf Haag. *Local quantum physics*, volume 2. Springer Berlin, 1996.

[2] Stephen W Hawking. Particle creation by black holes. *Communications in mathematical physics*, 43(3):199–220, 1975.

[3] William G Unruh. Notes on black-hole evaporation. *Physical Review D*, 14(4):870, 1976.

[4] James P Miller, Eduardo de Rafael, and B Lee Roberts. Muon (g- 2): experiment and theory. *Reports on Progress in Physics*, 70(5):795, 2007.

[5] Robert M Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. University of Chicago Press, 1994.

[6] Robert M Wald. The back reaction effect in particle creation in curved spacetime. *Communications in Mathematical Physics*, 54(1):1–19, 1977.

[7] Stephen A Fulling, Mark Sweeny, and Robert M Wald. Singularity structure of the two-point function in quantum field theory in curved spacetime. *Communications in Mathematical Physics*, 63(3):257–264, 1978.

[8] SA Fulling, FJ Narcowich, and Robert M Wald. Singularity structure of the two-point function in quantum field theory in curved spacetime, ii. *Annals of Physics*, 136(2):243–272, 1981.

[9] Marek J Radzikowski. Micro-local approach to the hadamard condition in quantum field theory on curved space-time. *Communications in mathematical physics*, 179(3):529–553, 1996.

[10] Hanno Sahlmann and Rainer Verch. Microlocal spectrum condition and hadamard form for vector-valued quantum fields in curved spacetime. *Reviews in Mathematical Physics*, 13(10):1203–1246, 2001.

[11] Romeo Brunetti and Klaus Fredenhagen. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Communications in Mathematical Physics*, 208(3):623–661, 2000.

[12] Stefan Hollands and Robert M Wald. Local wick polynomials and time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics*, 223(2):289–326, 2001.

[13] Romeo Brunetti, Michael Dütsch, and Klaus Fredenhagen. Perturbative algebraic quantum field theory and the renormalization groups. *Advances in Theoretical and Mathematical Physics*, 13(5):1541–1599, 2009.

[14] Stefan Hollands and Robert M Wald. Existence of local covariant time ordered products of quantum fields in curved spacetime. *Communications in mathematical physics*, 231(2):309–345, 2002.

[15] Christopher J Fewster and Rainer Verch. A quantum weak energy inequality for dirac fields in curved spacetime. *Communications in mathematical physics*, 225(2):331–359, 2002.

[16] Christopher J Fewster and Michael J Pfenning. A quantum weak energy inequality for spin-one fields in curved space–time. *Journal of Mathematical Physics*, 44:4480, 2003.

[17] Claudio Dappiaggi, Valter Moretti, and Nicola Pinamonti. Rigorous steps towards holography in asymptotically flat spacetimes. *Reviews in Mathematical Physics*, 18(04):349–415, 2006.

[18] Valter Moretti. Uniqueness theorem for bms-invariant states of scalar qft on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence. *Communications in mathematical physics*, 268(3):727–756, 2006.

[19] Claudio Dappiaggi, Thomas-Paul Hack, and Nicola Pinamonti. The extended algebra of observables for dirac fields and the trace anomaly of their stress-energy tensor. *Reviews in Mathematical Physics*, 21(10):1241–1312, 2009.

[20] Thomas-Paul Hack. On the backreaction of scalar and spinor quantum fields in curved spacetimes-from the basic foundations to cosmological applications. *arXiv preprint arXiv:1008.1776*, 2010.

[21] Claudio Dappiaggi and Daniel Siemssen. Hadamard states for the vector potential on asymptotically flat spacetimes. *Reviews in Mathematical Physics*, 25(01), 2013.

[22] Steven Weinberg. *Cosmology*, volume 1. 2008.

[23] Christopher J Fewster and David S Hunt. Quantization of linearized gravity in cosmological vacuum spacetimes. *Reviews in Mathematical Physics*, 25(02), 2013.

[24] LH Ford and Leonard Parker. Quantized gravitational wave perturbations in robertson-walker universes. *Physical Review D*, 16(6):1601, 1977.

[25] Manfredo P Do Carmo. *Riemannian geometry.* Springer, 1992.

[26] Mikio Nakahara. Geometry, topology and physics, (graduate student series in physics). 2003.

[27] Theodore Frankel. *The geometry of physics: an introduction.* Cambridge University Press, 2004.

[28] James R Munkres. *Topology*, volume 2. Prentice Hall Upper Saddle River, 2000.

[29] Yvonne Choquet-Bruhat. *Analysis, Manifolds and Physics, Part II-Revised and Enlarged Edition.* Access Online via Elsevier, 2000.

[30] John M Lee. *Introduction to smooth manifolds*, volume 218. Springer, 2012.

[31] Chris J Isham. *Modern Differential Geometry For Physists, 2/e.* Allied Publishers, 2002.

[32] Gerald B Folland. Weyl manifolds. *Journal of differential geometry*, 4(2):145–153, 1970.

[33] Sofiane Faci. Constructing conformally invariant equations using the weyl geometry. *Classical and Quantum Gravity*, 30(11):115005, 2013.

[34] Robert M Wald. *General relativity.* University of Chicago press, 2010.

[35] Stephen W Hawking and George Francis Rayner Ellis. *The large scale structure of space-time*, volume 1. Cambridge university press, 1973.

[36] Charles W Misner and John Archibald Wheeler. *Gravitation: Charles W. Misner, Kip S. Thorne, John Archibald Wheeler.* Macmillan, 1973.

[37] Roger Penrose. Asymptotic properties of fields and space-times. *Phys. Rev. Lett.*, 10:66–68, Jan 1963.

[38] VP Frolov. Null surface quantization and quantum field theory in asymptotically flat space-time. *Fortschritte der Physik*, 26(9):455–500, 1978.

[39] Friedrich Gerard Friedlander. *The wave equation on a curved space-time*, volume 2. Cambridge university press, 1975.

[40] Lars Hörmander. *The analysis of linear partial differential operators III: pseudo-differential operators*, volume 274. Springer, 2007.

[41] François Trèves. *Topological vector spaces, distributions and kernels*, volume 25. Dover Publications, 2006.

[42] Walter Rudin. *Functional analysis.* McGraw-Hill, Inc., New York, 1991.

[43] Dale Husemöller. *Fibre bundles*, volume 20. Springer, 1994.

[44] Christian Bär, Nicolas Ginoux, and Frank Pfäffle. *Wave equations on Lorentzian manifolds and quantization*. European Mathematical Society, 2007.

[45] Rudolf Haag and Daniel Kastler. An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5:848, 1964.

[46] J Dimock. Algebras of local observables on a manifold. *Communications in Mathematical Physics*, 77(3):219–228, 1980.

[47] Rainer Verch. A spin-statistics theorem for quantum fields on curved spacetime manifolds in a generally covariant framework. *Communications in Mathematical Physics*, 223(2):261–288, 2001.

[48] Romeo Brunetti, Klaus Fredenhagen, and Rainer Verch. The generally covariant locality principle–a new paradigm for local quantum field theory. *Communications in Mathematical Physics*, 237(1-2):31–68, 2003.

[49] Ola Bratteli and Derek W Robinson. *Operator algebras and quantum statistical mechanics 1: C^*-and W^*-algebras. Symmetry groups. Decomposition of states*, volume 1. Springer, 2003.

[50] Ola Bratteli and Derek W Robinson. *Operator Algebras and Quantum Statistical Mechanics: Vol.: 2: Equilibrium States, Models in Quantum Statistical Mechanics*. Springer-Verlag, 1981.

[51] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer verlag, 1998.

[52] Francis Borceux. Handbook of categorical algebra: Volume 1, basic category theory. 1994.

[53] Peter Szekeres. *A course in modern mathematical physics: Groups, Hilbert space and differential geometry*. Cambridge University Press, 2004.

[54] Marco Benini, Claudio Dappiaggi, and Thomas-Paul Hack. Quantum field theory on curved backgrounds – a primer. *Int.J.Mod.Phys.*, A28, 2013.

[55] Claudio Dappiaggi, Valter Moretti, and Nicola Pinamonti. Cosmological horizons and reconstruction of quantum field theories. *Communications in Mathematical Physics*, 285(3):1129–1163, 2009.

[56] Gerard 't Hooft. Dimensional reduction in quantum gravity. 1993.

[57] Ofer Aharony, Steven S Gubser, Juan Maldacena, Hirosi Ooguri, and Yaron Oz. Large n field theories, string theory and gravity. *Physics Reports*, 323(3):183–386, 2000.

[58] Martin Reuter and Frank Saueressig. Asymptotic safety, fractals, and cosmology. In *Quantum Gravity and Quantum Cosmology*, pages 185–226. Springer, 2013.

[59] JM Stewart and M Walker. Perturbations of space-times in general relativity. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 341(1624):49–74, 1974.

[60] Christian Bär and Klaus Fredenhagen. Quantum field theory on curved spacetimes. In *Quantum Field Theory on Curved Spacetimes*, volume 786, 2009.

[61] Abhay Ashtekar and Anne Magnon-Ashtekar. On the symplectic structure of general relativity. *Communications in Mathematical Physics*, 86(1):55–68, 1982.

[62] Nicola Pinamonti and Daniel Siemssen. Scale-invariant curvature fluctuations from an extended semiclassical gravity. *arXiv preprint arXiv:1303.3241*, 2013.

[63] Romeo Brunetti, Klaus Fredenhagen, and Katarzyna Rejzner. Quantum gravity from the point of view of locally covariant quantum field theory. *arXiv preprint arXiv:1306.1058*, 2013.

[64] Lorenzo Pantieri and Tommaso Gordini. L'arte di scrivere con latex. *PDF document. URL http://www. guitex. org/home*, 2009.