Renormalization and a conjecture of Quillen on determinant lines.

Workshop *Mathematics of interacting QFT models*, York

Nguyen Viet Dang [1]

[1]Université Lyon 1

Motivation of the talk : copyright to Alex Schenkel.

1. QFT is a tool to learn something about geometry, e.g. invariants of manifolds via **differential geometry, analysis** ? *What QFT has to say about the geometry of space times*.
2. QFT leads to interesting **geometric structures** that are *parametrized* by geometric and functional objects ?

Motivation of the talk : copyright to Alex Schenkel.

1. QFT is a tool to learn something about geometry, e.g. invariants of manifolds via **differential geometry, analysis** ? *What QFT has to say about the geometry of space times*.
2. QFT leads to interesting **geometric structures** that are *parametrized* by geometric and functional objects ?

How ?

Motivation of the talk : copyright to Alex Schenkel.

1. QFT is a tool to learn something about geometry, e.g. invariants of manifolds via **differential geometry, analysis** ? *What QFT has to say about the geometry of space times*.
2. QFT leads to interesting **geometric structures** that are *parametrized* by geometric and functional objects ?

How ? Understand simple things from free quantum fields *interacting* with *external classical fields*, how they **depend** on the external field.

Finally, I am a **bad Euclidean guy**.

(M,g) closed, compact, connected Riemannian manifold dimension d, volume form dv, Δ Laplace–Beltrami.
Sequence $\sigma = \{0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \to +\infty\}$ and $(e_\lambda)_{\lambda\in\sigma}$ **eigenfunctions**

$$\Delta e_\lambda = \lambda e_\lambda.$$

Green kernel :

$$\boxed{\mathbf{G}(x,y) = \sum_{\lambda\in\sigma,\lambda>0} \lambda^{-1} e_\lambda(x) e_\lambda(y).}$$

Finally, I am a **bad Euclidean guy**.

(M,g) closed, compact, connected Riemannian manifold dimension d, volume form dv, Δ Laplace–Beltrami.
Sequence $\sigma = \{0=\lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \to +\infty\}$ and $(e_\lambda)_{\lambda\in\sigma}$ **eigenfunctions**

$$\Delta e_\lambda = \lambda e_\lambda.$$

Green kernel :

$$\boxed{\mathbf{G}(x,y) = \sum_{\lambda\in\sigma,\lambda>0} \lambda^{-1} e_\lambda(x) e_\lambda(y).}$$

Example (Circle)

$\mathbb{S}^1$ constant metric, volume form $d\theta$, $\Delta = -\partial_\theta^2$. Spectrum $\sigma = \mathbb{N}^2$ and eigenfunctions are usual $(\cos(nx), \sin(nx))_{n\in\mathbb{N}}$.

Euclidean QFT.

1. Constructive fields : Albeverio, Fröhlich, Gallavotti, Gawedzki, Glimm, Guerra, Jaffe, Kupiainen, Magnen, Nelson, Rivasseau, Seiler, Sénéor, Spencer, Simon, Sokal, Symanzik, Wightman just to name a few and many others.
2. Euclidean QFT + geometry : Dappiaggi–Drago–Rinaldi, Dimock, Kandel, Pickrell, Segal, Stolz–Teichner...

Example.

Free quantum fields interacting with external fields :

	Free Bosons
Fast	ϕ scalar
Slow	(potential V, metric g)
Operator	$\Delta_g + V$
Action	$S(\phi, V, g) = \frac{1}{2}\int_M \phi(\Delta_g + V)\phi dv$
Partition f. integrate fast field	$Z_g(V) = \int[D\phi]e^{-S(\phi,V,g)}$

Example.

Free quantum fields interacting with external fields :

	Free Bosons
Fast	ϕ scalar
Slow	(potential V, metric g)
Operator	$\Delta_g + V$
Action	$S(\phi, V, g) = \frac{1}{2}\int_M \phi(\Delta_g + V)\phi dv$
Partition f. integrate fast field	$Z_g(V) = \int [D\phi] e^{-S(\phi,V,g)}$

Goal : give a meaning to the functional integrals with emphasis on dependence in *slow field*.

Cartoon oversimplified picture of TQFT's

Topological Field Theories

$$\boxed{\text{closed manifold } M \longmapsto \text{ partition function } Z(M) \sim \sum_{n=0}^{\infty} \lambda^n \underbrace{F_n(M)}.}$$

$F_n(M)$ topological invariant independent of C^∞ structure of M.

Invariants.

Flat space : Belkale–Brosnan, Bogner–Weinzierl show Feynman amplitudes are **special numbers** i.e. **periods**. But on **curved space**, QFT numbers should depend on the metric g and thus could be anything.
More structure on M, complex, Riemannian, bundles $E \mapsto M$, how to get invariants of corresponding structures ?

Hierarchy of information.

$$\boxed{\underbrace{C^\infty \text{ Riemannian}}_{C^\infty\text{metric}} \subset \underbrace{\text{smooth manifold}}_{C^\infty\text{structure}} \subset \underbrace{\text{topological manifold}}_{\text{topology}}.}$$

Fixed M, metrics, diffeomorphisms and an isometry invariant.

Fixed M smooth compact. Metrics $\mathbf{Met}(M)$: open convex cone in symmetric 2–tensors $C^\infty\left(S^2T^*M\right)$. $\mathbf{Diff}(M)$ acts on metrics by pull–back $g \mapsto \varphi^* g$.

Fixed M, metrics, diffeomorphisms and an isometry invariant.

Fixed M smooth compact. Metrics $\mathbf{Met}(M)$: open convex cone in symmetric 2–tensors $C^\infty\left(S^2T^*M\right)$. $\mathbf{Diff}(M)$ acts on metrics by pull–back $g \mapsto \varphi^* g$.

Definition (polygon Feynman amplitude)

For $V \in C^\infty(M, \mathbb{C})$,

$$c_n(g, V) = \int_{M^n} \mathbf{G}(x_1, x_n)V(x_1)\dots\mathbf{G}(x_n, x_1)V(x_n)dv_n(g_n) \tag{1}$$

where $dv_n(g_n)$ *volume form on* (M^n, g_n) *for* g_n *natural metric on* M^n.

Fixed M, metrics, diffeomorphisms and an isometry invariant.

Fixed M smooth compact. Metrics $\mathbf{Met}(M)$: open convex cone in symmetric 2–tensors $C^\infty\left(S^2T^*M\right)$. $\mathbf{Diff}(M)$ acts on metrics by pull–back $g\mapsto\varphi^*g$.

Definition (polygon Feynman amplitude)

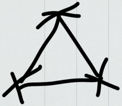
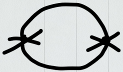
For $V\in C^\infty(M,\mathbb{C})$,

$$c_n(g,V)=\int_{M^n}\mathbf{G}(x_1,x_n)V(x_1)\dots\mathbf{G}(x_n,x_1)V(x_n)dv_n(g_n)\tag{1}$$

where $dv_n(g_n)$ *volume form on* (M^n,g_n) *for* g_n *natural metric on* M^n.

Claim : if $V=1$ and $c_n(g,1)$ converges then isometry invariant :

$$\int_{M^n}(\varphi^*\mathbf{G})(x_1,x_n)\dots(\varphi^*\mathbf{G})(x_n,x_1)dv_n(\varphi^*g_n)=\int_{M^n}\mathbf{G}(x_1,x_n)\dots\mathbf{G}(x_n,x_1)dv_n.$$

Feynman Graph	Amplitude, V potential, $G\in\mathcal{D}'(M\times M)$
	$\int_{M\times M\times M} G(x,y)V(y)G(y,z)V(z)G(z,x)V(x)$
	$\int_{M\times M} G(x,y)V(y)G(y,x)V(x)$
	$\int_{M} G(x,x)V(x)$

Another cartoon.

$$\boxed{\text{Riemannian mfd } (M, g), \text{potential } V \overset{\text{QFT}}{\longmapsto} Z_g(\lambda, V) = \left\langle \exp\left(-\frac{\lambda}{2}\int_M V : \phi^2 : dv\right)\right\rangle}$$

where

$$Z_g(\lambda, 1) = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^n \lambda^n}{2} c_n(g, 1)\right)$$

isometry invariant of metric g i.e. depends only on **Riemannian structure**.

Another cartoon.

$$\boxed{\text{Riemannian mfd } (M,g), \text{potential } V \xmapsto{\text{QFT}} Z_g(\lambda, V) = \left\langle \exp\left(-\frac{\lambda}{2}\int_M V : \phi^2 : dv\right)\right\rangle}$$

where

$$Z_g(\lambda,1) = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^n\lambda^n}{2} c_n(g,1)\right)$$

isometry invariant of metric g i.e. depends only on **Riemannian structure**.

When $\dim(M) = 4$, is $Z_g(\lambda, V)$ well–defined ?

Another cartoon.

$$\boxed{\text{Riemannian mfd } (M,g), \text{potential } V \xmapsto{\text{QFT}} Z_g(\lambda, V) = \left\langle \exp\left(-\frac{\lambda}{2}\int_M V : \phi^2 : dv\right)\right\rangle}$$

where

$$Z_g(\lambda, 1) = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^n\lambda^n}{2} c_n(g,1)\right)$$

isometry invariant of metric g i.e. depends only on **Riemannian structure**.

When $\dim(M) = 4$, is $Z_g(\lambda, V)$ well–defined ?
What information on (M,g) contained in $Z_g(\lambda, 1)$?

Another cartoon.

$$\boxed{\text{Riemannian mfd } (M,g), \text{potential } V \overset{\text{QFT}}{\longmapsto} Z_g(\lambda,V) = \left\langle \exp\left(-\frac{\lambda}{2}\int_M V : \phi^2 : dv\right)\right\rangle}$$

where

$$Z_g(\lambda,1) = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^n\lambda^n}{2} c_n(g,1)\right)$$

isometry invariant of metric g i.e. depends only on **Riemannian structure**.

When $\dim(M) = 4$, is $Z_g(\lambda,V)$ well–defined ?
What information on (M,g) contained in $Z_g(\lambda,1)$?
Two flavors :

- fixed smooth mfd M, question about metric g,
- M has **no fixed diffeo** type (C^∞ structure), question about **pair** (M,g).

Moduli space of metrics.

Fixed M.

Definition (Moduli space of metrics)

$\mathcal{R}(M) = C^\infty$ *Metrics* $\mathbf{Met}(M)/C^\infty$ *Diffeos* $\mathbf{Diff}(M)$.
$[g] \in \mathcal{R}(M)$ *called* **isometry class** *of metric.*

Moduli space of metrics.

Fixed M.

Definition (Moduli space of metrics)

$\mathcal{R}(M) = C^\infty$ *Metrics* $\mathbf{Met}(M)/C^\infty$ *Diffeos* $\mathbf{Diff}(M)$.
$[g] \in \mathcal{R}(M)$ *called* **isometry class** *of metric.*

Group $\mathbf{Diff}(M)$ acts on $\mathbf{Met}(M)$ **not free** since some metrics have **non trivial isometry groups** :

$$\varphi^* g = g, \varphi \neq Id \in \mathbf{Diff}(M). \tag{2}$$

Moduli space of metrics.

Fixed M.

Definition (Moduli space of metrics)

$\mathcal{R}(M) = C^\infty$ *Metrics* $\mathbf{Met}(M)/C^\infty$ *Diffeos* $\mathbf{Diff}(M)$.
$[g] \in \mathcal{R}(M)$ *called* **isometry class** *of metric.*

Group $\mathbf{Diff}(M)$ acts on $\mathbf{Met}(M)$ **not free** since some metrics have **non trivial isometry groups** :

$$\varphi^* g = g, \varphi \neq Id \in \mathbf{Diff}(M). \tag{2}$$

In Alex Schenkel's talk these are the self–arrows.

Moduli space of metrics.

Moduli space of metrics.

Theorem (Ebin 1967, Fischer 1970, Bourguignon 1975)

1. $\mathcal{R}(M)$ *endowed with quotient topology is Hausdorff and distance* **d**.
2. $\mathcal{R}(M)$ *not manifold but rather an orbifold whose* **regular points** $\mathcal{G}(M)$ *are isometry classes of metrics* $[g]$ *having* **no isometries**.
3. $\mathcal{G}(M)$ *is open dense in* $\mathcal{R}(M)$ *and has* C^∞*-manifold structure.*

Moduli space of metrics.

Theorem (Ebin 1967, Fischer 1970, Bourguignon 1975)

1. *$\mathcal{R}(M)$ endowed with quotient topology is Hausdorff and distance* **d**.
2. *$\mathcal{R}(M)$ not manifold but rather an orbifold whose* **regular points** *$\mathcal{G}(M)$ are isometry classes of metrics $[g]$ having* **no isometries**.
3. *$\mathcal{G}(M)$ is open dense in $\mathcal{R}(M)$ and has C^∞-manifold structure.*

Concretely : $[g_i] \to [g] \in \mathcal{R}(M)$ if there is a sequence of **representative** $g_i \to g$ in C^∞.

Moduli space of metrics.

Theorem (Ebin 1967, Fischer 1970, Bourguignon 1975)

1. *$\mathcal{R}(M)$ endowed with quotient topology is Hausdorff and distance* **d**.
2. *$\mathcal{R}(M)$ not manifold but rather an orbifold whose* **regular points** *$\mathcal{G}(M)$ are isometry classes of metrics $[g]$ having* **no isometries**.
3. *$\mathcal{G}(M)$ is open dense in $\mathcal{R}(M)$ and has C^∞-manifold structure.*

Concretely : $[g_i] \rightarrow [g] \in \mathcal{R}(M)$ if there is a sequence of **representative** $g_i \rightarrow g$ in C^∞.
Moduli space $\mathcal{M}_g$ from string theory is orbifold because mapping class group does not act freely.

Subsets of $\mathcal{R}(M)$.

For $0 < \varepsilon < 1$, $\mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]}$ moduli space of metrics whose sectional curvatures are contained in $[-\varepsilon^{-1},-\varepsilon]$.

Subsets of $\mathcal{R}(M)$.

For $0 < \varepsilon < 1$, $\mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]}$ moduli space of metrics whose sectional curvatures are contained in $[-\varepsilon^{-1}, -\varepsilon]$.

Example

$d = 2$ Riemann surface $g \geqslant 2$. $d = 3$ Many hyperbolic 3–manifolds...

Subsets of $\mathcal{R}(M)$.

For $0 < \varepsilon < 1$, $\mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]}$ moduli space of metrics whose sectional curvatures are contained in $[-\varepsilon^{-1}, -\varepsilon]$.

Example

$d = 2$ *Riemann surface* $g \geqslant 2$. $d = 3$ *Many hyperbolic* 3*–manifolds...*

Singular stratum $\partial\mathcal{G} \subset \mathcal{R}(M)$ of metric classes $[g]$ admitting **isometries**. Then

$$\boxed{\mathcal{G}_{\geqslant\varepsilon} = \{[g] \in \mathcal{G} \text{ s.t. } \mathbf{d}\left([g], \partial\mathcal{G}\right) \geqslant \varepsilon\}.}$$

Renormalized partition function

Proposition

$2 \leqslant d \leqslant 4$, $\varepsilon > 0$, $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ heat regularized GFF. $: \phi_\varepsilon^2(x) := \phi_\varepsilon^2(x) - \langle \phi_\varepsilon^2(x) \rangle$ and the renormalized partition functions :

$$Z_g(\lambda) = \lim_{\varepsilon \to 0^+} \left\langle \exp\left(-\frac{\lambda}{2} \int_M V(x) : \phi_\varepsilon^2(x) : dv \right) \right\rangle, d = (2,3),$$

$$Z_g(\lambda) = \lim_{\varepsilon \to 0^+} \left\langle \exp\left(-\frac{\lambda}{2} \int_M V(x) : \phi_\varepsilon^2(x) : dv - \frac{\lambda^2 \int_M V^2(x) dv}{64\pi^2} |\log(\varepsilon)| \right) \right\rangle, d = 4.$$

Renormalized partition function

Proposition

$2\leqslant d\leqslant 4$, $\varepsilon>0$, $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ *heat regularized GFF.* : $\phi_\varepsilon^2(x) := \phi_\varepsilon^2(x)-\langle\phi_\varepsilon^2(x)\rangle$ *and the renormalized partition functions :*

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+}\left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x) : \phi_\varepsilon^2(x) : dv\right)\right\rangle, d=(2,3),$$

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+}\left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x) : \phi_\varepsilon^2(x) : dv - \frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\right\rangle, d=4.$$

For small $|\lambda|, \|V\|_\infty$ *:*

$$Z_g(\lambda,V)=\exp\left(P(\lambda,V)+\sum_{n>\frac{d}{2}}\frac{(-1)^n c_n(g,V)\lambda^n}{2n}\right) \tag{3}$$

Renormalized partition function

Proposition

$2\leqslant d\leqslant 4$, $\varepsilon>0$, $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ heat regularized GFF. $:\phi_\varepsilon^2(x):=\phi_\varepsilon^2(x)-\langle\phi_\varepsilon^2(x)\rangle$ and the renormalized partition functions :

$$\begin{aligned} Z_g(\lambda) &= \lim_{\varepsilon\to0^+}\left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv\right)\right\rangle, d=(2,3),\\ Z_g(\lambda) &= \lim_{\varepsilon\to0^+}\left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv-\frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\right\rangle, d=4. \end{aligned}$$

For small $|\lambda|,\|V\|_\infty$:

$$Z_g(\lambda,V)=\exp\left(P(\lambda,V)+\sum_{n>\frac{d}{2}}\frac{(-1)^n c_n(g,V)\lambda^n}{2n}\right) \tag{3}$$

and Z_g^{-2} extends as **entire function** *on $\mathbb{C}$ with zeroes in $-\sigma(V\Delta^{-1})$.*

Renormalized partition function

Proposition

$2 \leqslant d \leqslant 4$, $\varepsilon > 0$, $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ heat regularized GFF. $:\phi_\varepsilon^2(x) := \phi_\varepsilon^2(x) - \langle \phi_\varepsilon^2(x) \rangle$ and the renormalized partition functions :

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+} \left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x) :\phi_\varepsilon^2(x): dv\right)\right\rangle, d = (2,3),$$

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+} \left\langle \exp\left(-\frac{\lambda}{2}\int_M V(x) :\phi_\varepsilon^2(x): dv - \frac{\lambda^2 \int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\right\rangle, d = 4.$$

For small $|\lambda|, \|V\|_\infty$:

$$Z_g(\lambda, V) = \exp\left(P(\lambda, V) + \sum_{n>\frac{d}{2}} \frac{(-1)^n c_n(g,V)\lambda^n}{2n}\right) \tag{3}$$

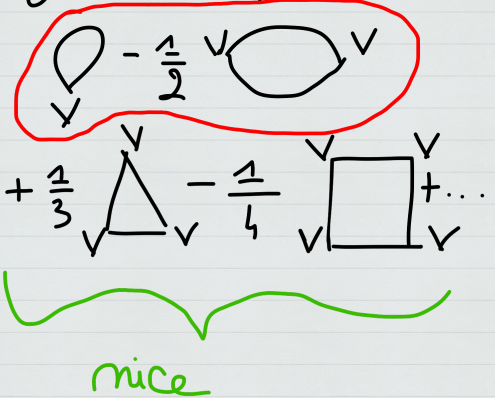
and Z_g^{-2} extends as **entire function** *on $\mathbb{C}$ with zeroes in $-\sigma(V\Delta^{-1})$.*

See also Dappiaggi–Drago–Rinaldi.

$$d = 4$$

$$\log\det(I + \Delta^{-1}V) =$$

diverge



Main result heuristics.

Random variable X given by probability distribution μ or Fourier–Laplace transform of μ.

Main result heuristics.

Random variable X given by probability distribution μ or Fourier–Laplace transform of μ.

When $d = (2,3)$, Riemannian rigidity in **negative curvature** where fluctuations of $\int_M : \phi^2(x) : dv$ are encoded by probability distribution of random variable $\int_M : \phi^2(x) : dv$ or partition function Z_g.

Wick square also in Fewster–Hollands, Sanders, Dappiaggi–Drago–Rinaldi, Moretti

Main result heuristics.

Random variable X given by probability distribution μ or Fourier–Laplace transform of μ.

When $d = (2, 3)$, Riemannian rigidity in **negative curvature** where fluctuations of $\int_M : \phi^2(x) : dv$ are encoded by probability distribution of random variable $\int_M : \phi^2(x) : dv$ or partition function Z_g.

Wick square also in Fewster–Hollands, Sanders, Dappiaggi–Drago–Rinaldi, Moretti

When $d = 4$, $\int_M : \phi^2(x) : dv$ no longer random variable ! Only renormalized Z_g well–defined and similar rigidity result fixing Z_g.

Corollary of main results.

$d = 4$, N **finite dimensional** submfd of $\mathcal{G} \subset \mathcal{R}(M)$. Choose $\varepsilon \in (0, 1)$,

Corollary of main results.

$d = 4$, N **finite dimensional** submfd of $\mathcal{G} \subset \mathcal{R}(M)$. Choose $\varepsilon \in (0,1)$,

$$\boxed{[g] \in N \cap \mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]} \longmapsto \underbrace{(c_3(g), c_4(g), \dots, c_n(g), \dots) \in \mathbb{R}^{\mathbb{N}}}_{\textbf{sequence Feynman amplitudes are local coordinates}}.}$$

Corollary of main results.

$d = 4$, N **finite dimensional** submfd of $\mathcal{G} \subset \mathcal{R}(M)$. Choose $\varepsilon \in (0,1)$,

$$\boxed{[g] \in N \cap \mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]} \longmapsto \underbrace{(c_3(g), c_4(g), \dots, c_n(g), \dots) \in \mathbb{R}^{\mathbb{N}}}_{\textbf{sequence Feynman amplitudes are local coordinates}} .}$$

Sequence $(M_i, g_i)_i$ with given Z_g plus condition on curvature. $\exists$ **finite number of manifolds** $(M'_1, \dots, M'_k)$ and on each M'_j a **compact family of metrics** $\mathcal{M}'_j$ such that each of the manifolds M_j is diffeomorphic to one of the M'_i and isometric to an element of $\mathcal{M}'_i$.

Main result, dim $(2,3)$.

Theorem (D 2019)

1. *N **finite dimensional** submfd of $\mathcal{G}\subset\mathcal{R}(M)$ s.t. $\partial N\subset\partial\mathcal{G}$. $\forall\varepsilon>0$, the set of classes of metrics $[g]\in N\cap\mathcal{R}(M)_{\leqslant-\varepsilon}\cap\mathcal{G}_{\geqslant\varepsilon}$ s.t. $\int_M:\phi^2(x):dv$ has* **given probability distribution is finite**.
2. *When $d=2$, the genus and diffeo type determined.*
3. *When $d=3$, for a sequence $(M_i,g_i)_{i\in\mathbb{N}}$ of Riemannian 3–mfds of negative curvature s.t. $\int_M:\phi^2(x):dv$ has* **given probability distribution**, *$(M_i)_i$ contains* **finitely many diffeo types** *and one can extract a subsequence s.t. M_i has* **fixed diffeo type** *and $g_i\to g$.*

Main result, dim 4

Theorem (D 2019)

1. N **finite dimensional** *submfd of* $\mathcal{G} \subset \mathcal{R}(M)$ *s.t.* $\partial N \subset \partial \mathcal{G}$. $\forall \varepsilon \in (0,1)$, *the set of classes of metrics* $[g] \in N \cap \mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$ **with given partition function** Z_g **is finite**.
2. *For a sequence* $(M_i, g_i)_{i\in\mathbb{N}}$ *of Riemannian* 4*–mfds of negative curvature bounded in some compact interval s.t.* Z_g **is given**, $(M_i)_i$ *contains* **finitely many diffeo types** *and one can extract a subsequence s.t.* M_i *has* **fixed diffeo type** *and* $(M_i, g_i) \to (M, g)$.

Chiral fermions.

Example (Chiral fermions coupled to external YM)

Chiral fermions on $\mathbb{S}^4$ *coupled to external YM potential* $(A_\mu)_\mu$.

$$\boxed{D_A = i\sigma^\mu \left(\nabla_\mu + eA_\mu\right)} \tag{4}$$

where σ_μ *obey the Weyl algebra* $\sigma_\mu\sigma_\nu^\dagger + \sigma_\nu\sigma_\mu^\dagger = 2\delta_{\mu\nu}$.

Chiral fermions.

Example (Chiral fermions coupled to external YM)

Chiral fermions on $\mathbb{S}^4$ *coupled to external YM potential* $(A_\mu)_\mu$.

$$\boxed{D_A = i\sigma^\mu \left(\nabla_\mu + eA_\mu\right)} \tag{4}$$

where σ_μ *obey the Weyl algebra* $\sigma_\mu\sigma_\nu^\dagger + \sigma_\nu\sigma_\mu^\dagger = 2\delta_{\mu\nu}$.

Example (Quillen 1985)

Hermitian bundle $B \mapsto M$ *on Riemann surface. Affine space*

$$\mathcal{A} = \overline{\partial} + \omega, \;\; \omega \in \Omega^{0,1}(M, End(B)) \tag{5}$$

parametrizes complex structures on B *hence* $E_+ = B$, $E_- = \Lambda^{0,1}T^*M \otimes B$.

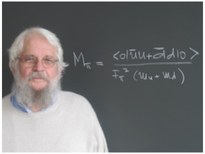
Example.

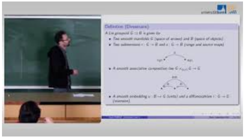
	Chiral Fermions
Fast	$(\Psi_\pm)$ spinor
Slow	(gauge potential A, metric g)
Operator	$\underbrace{D+A}_{\text{twisted Dirac}}$
Action	$S(\Psi, A, g) = \int_M \langle \Psi_-, (D+A)\,\Psi_+ \rangle$
Partition f. integrate fast field	$Z(A) = \int [\mathcal{D}\Psi_+ \mathcal{D}\Psi_-] e^{-S(\Psi, A, g)}$

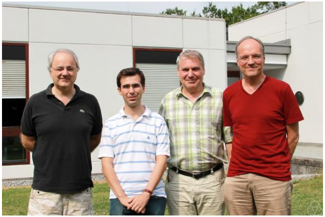
Example.

	Chiral Fermions
Fast	$(\Psi_\pm)$ spinor
Slow	(gauge potential A, metric g)
Operator	$\underbrace{D+A}_{\text{twisted Dirac}}$
Action	$S(\Psi, A, g) = \int_M \langle \Psi_-, (D+A)\Psi_+ \rangle$
Partition f. integrate fast field	$Z(A) = \int [\mathcal{D}\Psi_+ \mathcal{D}\Psi_-] e^{-S(\Psi, A, g)}$

Goal : give a meaning to the functional integrals with emphasis on dependence in *slow field*.


$M_\pi = \frac{\langle 0|\bar{u}u+\bar{d}d|0\rangle}{F_\pi^2 (m_u+m_d)}$





General framework

Chiral Dirac $D : C^\infty(E_+) \mapsto C^\infty(E_-)$ on pair $E_\pm$ of Hermitian bundles s.t. $\mathrm{Ind}(D) = 0, \ker(D) = \{0\}$ and $D^*D : C^\infty(E_+) \mapsto C^\infty(E_+)$ generalized Laplacian.
Affine space :

$$\mathcal{A} = D + C^\infty\left(Hom\left(E_+, E_-\right)\right) \tag{6}$$

of perturbations of D.

Fermions and Quillen **Det** line bundle.

Chiral Dirac operator acts on spinors of different chirality.

Fermions and Quillen **Det** line bundle.

Chiral Dirac operator acts on spinors of different chirality.
(E, F) dim $< +\infty$, $T : E \mapsto F$ invertible, meaning of $\det(T)$?
How to get Numbers ?

Fermions and Quillen **Det** line bundle.

Chiral Dirac operator acts on spinors of different chirality.
(E, F) dim $< +\infty$, $T : E \mapsto F$ invertible, meaning of $\det(T)$?
How to get Numbers ?
For every pair $vol(E), vol(F)$, ratio $\frac{T_* vol(E)}{vol(F)}$ hence **determinant line** $\simeq \frac{\Lambda^{top}(E)}{\Lambda^{top}(F)}$.

Fermions and Quillen **Det** line bundle.

Chiral Dirac operator acts on spinors of different chirality.
(E, F) dim $< +\infty$, $T : E \mapsto F$ invertible, meaning of $\det(T)$?
How to get Numbers ?
For every pair $vol(E), vol(F)$, ratio $\frac{T_* vol(E)}{vol(F)}$ hence **determinant line** $\simeq \frac{\Lambda^{top}(E)}{\Lambda^{top}(F)}$.
Recall affine space $\mathcal{A}$ of perturbations $D_A = D + A$ of some fixed Dirac D.
Quillen (1985) constructed **holomorphic line bundle** $\mathcal{L} \mapsto \mathcal{A}$ in Cauchy–Riemann case generalized by Bismut–Freed (1986) to families of Dirac operators.

Fermions and Quillen **Det** line bundle.

Chiral Dirac operator acts on spinors of different chirality.
(E, F) dim $< +\infty$, $T : E \mapsto F$ invertible, meaning of $\det(T)$?
How to get Numbers ?
For every pair $vol(E), vol(F)$, ratio $\frac{T_* vol(E)}{vol(F)}$ hence **determinant line** $\simeq \frac{\Lambda^{top}(E)}{\Lambda^{top}(F)}$.
Recall affine space $\mathcal{A}$ of perturbations $D_A = D + A$ of some fixed Dirac D.
Quillen (1985) constructed **holomorphic line bundle** $\mathcal{L} \mapsto \mathcal{A}$ in Cauchy–Riemann case generalized by Bismut–Freed (1986) to families of Dirac operators.

iii April 30, 1989 282

According to Stora the BRS analysis of anomalies can be understood in the case, where the gauge field is treated as an external field, and only the fermion field is a quantum field.

Consider then the quadratic Lagrangian $\tilde{\psi} \not{\partial}_A \psi$ where ψ is a "chiral" fermion. Here $\not{\partial}_A$ stands for half the Dirac operator from + spinors to − spinors. We know the functional integrals attached to this Lagrangian are equivalent to things associated to the determinant line of $\not{\partial}_A$. More precisely, one gives a meaning to the functional integrals

$$(*) \qquad \int D\tilde{\psi}\, D\psi\, e^{\int \tilde{\psi} \not{\partial}_A \psi}\, \psi \cdots \psi\, \tilde{\psi} \cdots \tilde{\psi}$$

by trivializing the determinant line.

If one expands around a gauge field A_0 with $\not{\partial}_{A_0}$ non-singular the diagrams have only vertices

and the propagator is given by a geometric series.

The infinities arise from loops. We have

$$\det(\not{\partial}_A) = \det(\not{\partial}_{A_0}) \exp\left\{-\sum_{n\geq 1} \frac{1}{n} \operatorname{tr}(K^n)\right\}$$

finite value for them.

In renormalized perturbation theory there is a technique of adding counter-terms to the Lagrangian to remove the infinities. Problem: Explain what these counter terms are in the present situation.

Here is an attempt to make some sense out of the above situation:

First of all we have the determinant line bundle $\mathcal{L}$ over the space $\mathcal{A}$ of gauge fields A. The functional integrals $(*)$ are canonical sections of $\mathcal{L}$, or better perhaps is to think of $(*)$ as linear maps on $\mathcal{L}^{-1}$, so that picking a nonzero elts of $\mathcal{L}_A^{-1}$ gives a meaning to these integrals as numbers.

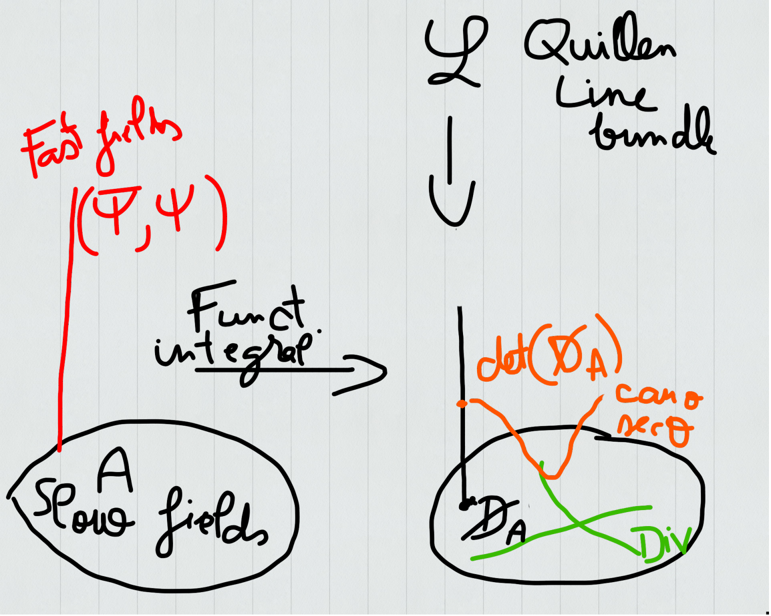
The issue in making sense of this QFT is just to construct a trivialization of the determinant line bundle.

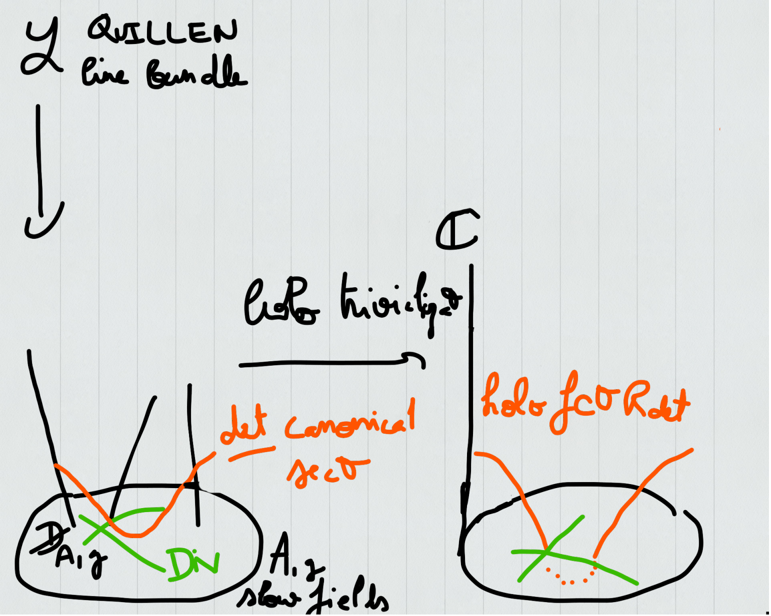
Secondly there are the ideas from Weinberg's paper on pseudo-particles, which are based in part on the Fredholm determinant theory. These ideas suggest that there are complex analytic trivializations of $\mathcal{L}$ unique up to a factor of the form $\exp\{\text{polynomial on } \mathcal{A}\}$, where the degree is bounded by the traces which have to be regularized.

We saw this is true over a Riemann surface. Here the ambiguity was $\exp\{\text{linear fn. on } \mathcal{A}\}$.

caused by the presence of zero models. However Fredholm really showed how to represent $\frac{1}{1-\lambda K}$ as the meromorphic operator function $\frac{Cof}{det}$, and I have the example of Reimann surfaces. 289

These considerations lead to the following conjectural picture. Over the space $\mathcal{A}$ of gauge fields there should be a principal bundle for the additive group of polynomial functions of degree $\leq d$, where d bounds the traces which have to be regularized. The idea is that near each A we should have a well-defined trivialization of $\mathcal{L}$ up to exp of such a polynomials. Moreover we should have a flat connection on this bundle.


Fast fields
$(\overline{\Psi},\Psi)$
Funct. integral
A
Slow fields
$\mathcal{L}$ Quillen Line bundle
$\det(D_A)$
cano sec
D_A
Div



QUILLEN
line bundle
$\mathbb{C}$
det canonical
sect
holo fct
$D_{A,g}$
Div
A, g
slow fields

Digression on determinants.

Complexity of entire function :

- **Order** $\rho(f) \geqslant 0$ of f, $\inf \rho$ s.t. $|f(z)| \leqslant Ae^{K|z|^{\rho}}$. Controls **growth** at infinity.
- **Divisor** of f, set of zeros $\{a_n | n \in \mathbb{N}\}$ of f counted with multiplicity. **Critical exponents** of $|a_n| \rightarrow +\infty$, $\inf \alpha > 0$ s.t. $\sum_n \frac{1}{|a_n|^{\alpha}} < +\infty$.

Digression on determinants.

Complexity of entire function :

- **Order** $\rho(f) \geqslant 0$ of f, inf ρ s.t. $|f(z)| \leqslant Ae^{K|z|^{\rho}}$. Controls **growth** at infinity.
- **Divisor** of f, set of zeros $\{a_n | n \in \mathbb{N}\}$ of f counted with multiplicity. **Critical exponents** of $|a_n| \to +\infty$, inf $\alpha > 0$ s.t. $\sum_n \frac{1}{|a_n|^{\alpha}} < +\infty$.

Theorem (Hadamard's factorization Theorem)

a_n sequence s.t. $\sum_n |a_n|^{-(p+1)} < +\infty$ but $\sum_n |a_n|^{-p} = \infty$. Then any entire function s.t. $Z(f) = \{a_n | n \in \mathbb{N}\}$ and $\rho(f) = p$ has **unique representation** *:*

$$\boxed{f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)} \tag{7}$$

where P polynomial of deg p, $E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^p}{p}}$ Weierstrass factor of order p and m vanishing order at 0.

Digression on determinants.

Complexity of entire function :

- **Order** $\rho(f) \geqslant 0$ of f, $\inf \rho$ s.t. $|f(z)| \leqslant Ae^{K|z|^{\rho}}$. Controls **growth** at infinity.
- **Divisor** of f, set of zeros $\{a_n | n \in \mathbb{N}\}$ of f counted with multiplicity. **Critical exponents** of $|a_n| \to +\infty$, $\inf \alpha > 0$ s.t. $\sum_n \frac{1}{|a_n|^{\alpha}} < +\infty$.

Theorem (Hadamard's factorization Theorem)

a_n sequence s.t. $\sum_n |a_n|^{-(p+1)} < +\infty$ but $\sum_n |a_n|^{-p} = \infty$. Then any entire function s.t. $Z(f) = \{a_n | n \in \mathbb{N}\}$ and $\rho(f) = p$ has **unique representation** *:*

$$\boxed{f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)} \tag{7}$$

where P polynomial of deg p, $E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}$ Weierstrass factor of order p and m vanishing order at 0.

Finding f with prescribed divisor **non unique** when critical exponent > 0, **polynomial ambiguity** P

Locality.

Definition (Local polynomial functionals)

$P : A \mapsto P(A)$, C^∞, local polynomial functional iff $P(A) = \int_M \Lambda(j^k A(x)) dv$, dv C^∞ density, Λ polynomial in jets of A. Denote $\mathcal{O}_{loc}$.

Example

$P(\varphi) = \int_{\mathbb{S}^1} \varphi^4(\theta) d\theta$.

Reformulation of defining functional integrals.

Find **analytic map**

$$A \in C^{\infty}(Hom(E_+, E_-)) \mapsto \mathcal{R}\det(D_A)$$

vanishing over $Z = \{A \text{ s.t. } \ker(D_A) \neq 0\}$, of minimal **order**, obtained by **local renormalization**.
Regularize Ψ by heat operator, set $\Psi_\varepsilon = e^{-\varepsilon D^* D}\Psi$:

$$\exists \textbf{ local counterterm } P_\varepsilon \in \mathcal{O}_{loc} \otimes \mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)], \text{ s.t.}$$

$$\mathcal{R}\det(D_A) = \lim_{\varepsilon \to 0^+} \int [D\overline{\Psi}][D\Psi] e^{\int_M \langle \overline{\Psi}, D\Psi \rangle + \langle \overline{\Psi}_\varepsilon, A\Psi_\varepsilon \rangle dv - P_\varepsilon(A)}.$$

Classify all solutions.

Local renormalization.

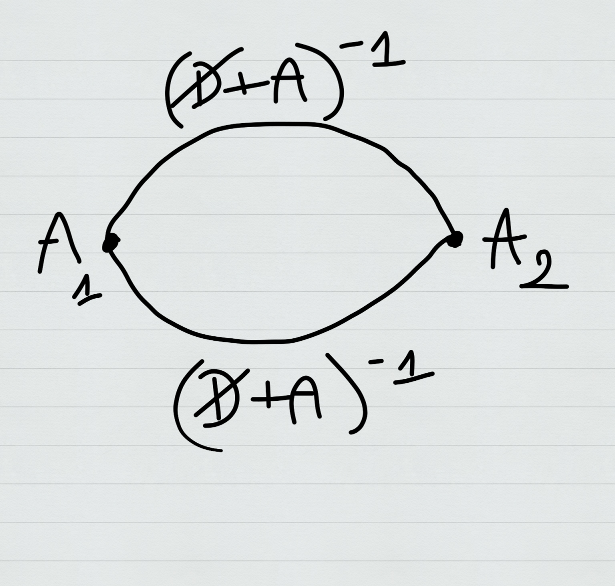
How to ensure that a renormalized determinant $\mathcal{R}\det$ comes from **local renormalization** ?

Definition (Axioms)

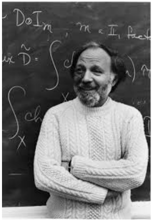
$\mathcal{R}\det$ *renormalized determinant on* $\mathcal{A} = D + C^\infty(Hom(E_+, E_-))$ *if*

- $\mathcal{R}\det$ **vanishes exactly on noninvertible elements**
- **Complexity** *order* $d+1$: $|\mathcal{R}\det(D_A)| \leqslant Ce^{K\|A\|^{d+1}_{C^m}}$ *for some norm* $\|.\|_{C^m}$.
- **Local renorm** $\mathcal{R}\det$ *satisfies equation (attributed to Witten by Kontsevich–Vishik) :* $\boxed{\delta_{A_1}\delta_{A_2}\log\mathcal{R}\det(D_A) = Tr_{L^2}\left(D_A^{-1}A_1 D_A^{-1}A_2\right)}$ *if* $supp(A_1)\cap supp(A_2)$ **disjoint**
- **Smoothness counterterms**, *wave front of the second derivative is contained in the conormal bundle :*
$$\boxed{\left(WF\left(\delta^2\log\mathcal{R}\det(D_A)\right)\cap T^\bullet_{d_2}M^2\right)\subset N^*\left(d_2\subset M^2\right).}$$

Identity from Kontsevich–Vishik



Suggestion from Singer (1985).



Observe $D^*D_A : E_+ \mapsto E_+$ elliptic with Laplace type principal symbol.

$$\boxed{A \mapsto \det_\zeta(D^*D_A) = \exp(-\frac{d}{ds}|_{s=0} \underbrace{\sum_{\lambda\in\sigma(D^*D_A)} \lambda^{-s}}_{\textbf{spectral zeta}})}$$

Renorm group

Theorem (D 2019)

- *Every* $\mathcal{R}\det$ *renormalized by subtraction of local counterterms.*
- *The zeta determinant* $\det_\zeta(D^*D_A)$ *is a renormalized determinant.*
- *Group* $(\mathcal{O}_{loc,\leqslant d},+)$ *acts* **freely and transitively** *on renormalized determinants :*

$$P \in \mathcal{O}_{loc,\leqslant d} \mapsto \exp\left(P(A)\right)\mathcal{R}\det\left(D_A\right). \tag{8}$$

- *Bijection : holomorphic trivializations of* $\mathcal{L} \simeq$ *solutions* $\mathcal{R}\det$.

Thanks for your attention !