Solution of all quartic matrix models

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joint work with Harald Grosse and Alexander Hock [arXiv:1906.04600]

and Erik Panzer [arXiv:1807.02945]

In memory of Daniel Kastler (1926–2015)

In 1998/99, Daniel was trying to catch three fishes:
la truite, la truite saumonée, le saumon

They stand for flavours of theories in which the standard model is a representation of $SU_q(2)$, with q a third root of unity.

The fish

- Since "Progress in solving a noncommutative quantum field theory in four dimensions" [arXiv:0909.1389] with Harald Grosse I am telling you:
  There is a fish in the ϕ^4-QFT model on noncommutative Moyal space.
- We saw the fish in 2012. It is big.
- I repeated over all the years that the fish is there. But I was not able to show it to you. I understand you think I am crazy.
- Last year, with Erik Panzer, we caught a little fish:
  ϕ^4 on two-dimensional Moyal space is solved by the Lambert-W function.
  This made clear: The big fish is there.
- On the 10th of May, I caught the fish.
  It is real, it is big. I describe it in this talk.

The sea

Consider a familiy of matrix integrals over the space of self-adjoint $\mathcal{N} \times \mathcal{N}$-matrices

$$\mathcal{Z}(E,\lambda) = \frac{\displaystyle\int_{M^*_{\mathcal{N}}} d\Phi \,\exp\Big(-\mathcal{N}\mathrm{Tr}\big(E\Phi^2 + \tfrac{\lambda}{p}\Phi^p\big)\Big)}{\displaystyle\int_{M^*_{\mathcal{N}}} d\Phi \,\exp\Big(-\mathcal{N}\mathrm{Tr}\big(E\Phi^2\big)\Big)}$$

They depend on a positive matrix E and a scalar λ.

$p = 3$ is the Kontsevich model – a gigantic fish

$\log \mathcal{Z}(E, \mathrm{i})$ is formal power series in $t_n = -(2n-1)!!\mathrm{Tr}(E^{-(2n+1)})$. Its coefficients are intersection numbers of Chern classes on the moduli space of complex curves. It defines a QFT.

$p = 4$ is our baby

Today we know: It is structurally identical to $p = 3$.

Theorem (the fish)

- Let $0 < E_1 < \cdots < E_d$ be the eigenvalues of E, of multiplicities $r_1, \ldots, r_d$.
- Take solutions $\{\varepsilon_k, \varrho_k\}$ with $\lim\limits_{\lambda\to 0} \varepsilon_k = E_k$, $\lim\limits_{\lambda\to 0} \varrho_k = r_k$ of

$$\boxed{E_l = \varepsilon_l - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + \varepsilon_l}\,, \qquad 1 = \frac{r_l}{\varrho_l} - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{\varrho_k}{(\varepsilon_k + \varepsilon_l)^2}}$$

Then the planar two-point function of the quartic matrix model is

$$G^{(0)}_{ab} = \frac{1}{\varepsilon_a + \varepsilon_b} \cdot \frac{\prod\limits_{k,l=1}^{d} \Big(1 + \frac{\sigma_k(E_a) + \sigma_l(E_b)}{\varepsilon_k + \varepsilon_l}\Big)}{\prod\limits_{k,l=1}^{d} \Big(1 + \frac{\sigma_k(E_a)}{\varepsilon_k + \varepsilon_l}\Big) \prod\limits_{k,l=1}^{d} \Big(1 + \frac{\sigma_l(E_b)}{\varepsilon_k + \varepsilon_l}\Big)}$$

where $\{\varepsilon_a, -\varepsilon_1 - \sigma_1(E_a), \ldots, -\varepsilon_d - \sigma_d(E_a)\}$ are all solutions z of

$$E_a = z - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z}$$

Dyson-Schwinger equation

The two-point function is

$$ZG^{(0)}_{ab} = \left[\frac{1}{\mathcal{N}} \log \frac{\displaystyle\int_{M^*_{\mathcal{N}}} d\Phi\ \Phi_{ab}\Phi_{ba} \exp\Big(-\mathcal{N}\mathrm{Tr}\big(E\Phi^2 + \tfrac{\lambda}{4}\Phi^4\big)\Big)}{\displaystyle\int_{M^*_{\mathcal{N}}} d\Phi\ \exp\Big(-\mathcal{N}\mathrm{Tr}\big(E\Phi^2 + \tfrac{\lambda}{4}\Phi^4\big)\Big)}\right]_{\text{up to } \frac{1}{\mathcal{N}}}$$

Using the Ward identity of [Disertori, Gurau, Magnen, Rivasseau 06] one derives a closed equation for $G^{(0)}_{ab}$:

Theorem [Grosse, W 09]

$$(E_a + E_b)ZG^{(0)}_{ab} = 1 - \lambda \sum_{n=1}^{\mathcal{N}} \left(ZG^{(0)}_{ab}\, ZG^{(0)}_{an} - \frac{ZG^{(0)}_{nb} - ZG^{(0)}_{ab}}{E_n - E_a} \right)$$

Today we can solve this or a limit $\mathcal{N} \to \infty$ to unbounded operators E with $\sum_{n=2}^{\infty}(E_n - E_1)^{-3} < \infty$.

The latter requires renormalisation $Z(\mathcal{N})$ and $E_1 = \frac{1}{2}\tilde{\mu}^2(\mathcal{N})$.

Extension to sectionally holomorphic functions

- Define $\rho_0(t) = \frac{1}{\mathcal{N}}\sum_{n=1}^{\mathcal{N}}\delta(t-(E_n-E_1))$, with $E_1=\frac{1}{2}\tilde{\mu}^2$, $E_{\mathcal{N}}-E_1=\Lambda^2$
- Then $G^{(0)}_{ab} = G(E_a - E_1, E_b - E_1)$ for

$$
\begin{aligned}
&(x+y+\tilde{\mu}^2)ZG(x,y)\\
&= 1-\lambda\int_0^{\Lambda^2} dt\ \rho_0(t)\Big(ZG(x,y)\,ZG(x,t) - \frac{ZG(t,y)-ZG(x,y)}{t-x}\Big)
\end{aligned}
$$

- This is the analogue of

$$
(W(x))^2 + \lambda^2\int_0^{\Lambda^2} dt\ \rho_0(t)\frac{W(t)-W(x)}{t^2-x^2} = x
$$

in Kontsevich model, solved by [Makeenko, Semenoff 91].

Hilbert transform

- Temporarily assume that ρ_0 is Hölder-continuous.
- Ansatz $ZG(a,b) = \dfrac{e^{\mathcal{H}_a[\tau_b(\bullet)]}\sin\tau_b(a)}{\lambda\pi\rho_0(a)} = \dfrac{e^{\mathcal{H}_b[\tau_a(\bullet)]}\sin\tau_a(b)}{\lambda\pi\rho_0(b)}$

  where $\mathcal{H}_a[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\big(\int_0^{a-\epsilon}+\int_{a+\epsilon}^{\Lambda}\big)\frac{dt\, f(t)}{t-a}$ is Hilbert transform
- Gives

$$\Big(\tilde{\mu}^2+a+b+\lambda\pi\mathcal{H}_a[\rho_0(\bullet)] + \frac{1}{\pi}\int_0^{\Lambda^2} dt\; e^{\mathcal{H}_t[\tau_a(\bullet)]}\sin\tau_a(t)\Big)ZG(a,b)$$
$$= 1 + \mathcal{H}_a\Big[e^{\mathcal{H}_\bullet[\tau_b]}\sin\tau_b(\bullet)\Big]$$

- [Tricomi 57] $\mathcal{H}_a\Big[e^{\mathcal{H}_\bullet[f]}\sin f(\bullet)\Big] = e^{\mathcal{H}_a[f]}\cos f(a) - 1$
- [Panzer, W 18] $\int_0^{\Lambda^2} dt\; e^{\mathcal{H}_t[f(\bullet)]}\sin f(t) = \int_0^{\Lambda^2} dt\; f(t)$

The τ-equation

$$\tau_a(p) = \arctan\left(\frac{\lambda\pi\rho_0(p)}{\tilde{\mu}^2 + a + p + \lambda\pi\mathcal{H}_p[\rho_0(\bullet)] + \frac{1}{\pi}\int_0^{\Lambda^2} dt\, \tau_p(t)}\right)$$

Solution in case of $\rho_0(x) \equiv 1$ [Panzer, W 18]

$$\tau_a(p) = \operatorname{Im}\log\left(a + I(p+\mathrm{i}\epsilon)\right)$$

$$I(z) := \lambda W_0\left(\frac{1}{\lambda}e^{\frac{1+z}{\lambda}}\right) - \lambda\log\left(1 - W_0\left(\frac{1}{\lambda}e^{\frac{1+z}{\lambda}}\right)\right)$$

where W_0 is the principal branch of Lambert-W and
$\tilde{\mu}^2 = 1 - 2\lambda\log(1+\Lambda^2)$.

In April, Alexander Hock told me that this solution has a remarkable structure:

$I(z) = f - \lambda\log(1-f)$ where f solves $1 + z = f - \lambda\log f$

Such expressions arise in topological recursion.

The fishing-rod

What escaped for 10 years, was possible to catch in one week:

Ansatz

$\tau_a(p) = \mathrm{Im}\log\big(a + I(p+\mathrm{i}\epsilon)\big)$ with $I(z) = -J(-\mu^2 - J^{-1}(z))$, where

$$\boxed{J(z) = z - \lambda(-z)^{D/2}\int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \rho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}}$$

- Take smallest $D \in \{0,2,4\}$ for which this converges.
- ρ_λ is NOT the same as ρ_0.
- $\mu = \tilde{\mu}$ for $D = 0$, otherwise a free parameter
- $J : \{\mathrm{Re}(z) > -\frac{2\mu^2}{3}\} \to U \subseteq \mathbb{C}$ is biholomorphic

Solution of all quartic matrix models

Use

- [Cauchy 1831] residue theorem
- [Lagrange 1770] inversion theorem
- [Bürmann 1799] formula

Theorem

The ansatz $J(z) = z - \lambda(-z)^{D/2} \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \rho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$ solves the τ-equation provided that

- ρ_λ is implicit solution of $\rho_0(J(t)) = \rho_\lambda(t)$.
- $\tilde{\nu} = J^{-1}(0)$, $\tilde{\Lambda}^2 = J^{-1}(\Lambda^2)$,
- $\tilde{\mu}^2 = \mu^2 - 2\lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \rho_\lambda(t)}{(\mu^2+t)}$ for $D = 2$,
  $\tilde{\mu}^2 = \mu^2 \Big(1 - \lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \rho_\lambda(t)}{(\mu^2+t)^2}\Big) - 2\lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \rho_\lambda(t)}{(\mu^2+t)}$ for $D = 4$.

Evaluating the Hilbert transform

Remains to evaluate $G(a,b) = Z^{-1}\dfrac{e^{\mathcal{H}_b[\tau_a(\bullet)]}\sin\tau_a(b)}{\lambda\pi\rho_0(b)}$.

For $D=4$ need $Z = Z_0 e^{\mathcal{H}_r[\tau_r(\bullet)]}$ to remove divergences.

Proposition

$$G(a,b) := \frac{(\mu^2)^{\delta_{D,4}}(\mu^2+a+b)\exp(N(a,b))}{(\mu^2+b+J^{-1}(a))(\mu^2+a+J^{-1}(b))}\,,$$

$$\begin{aligned}N(a,b) := \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} dt\Big\{ &\log\Big(\frac{a-J(-\frac{\mu^2}{2}-\mathrm{i}t)}{a-(-\frac{\mu^2}{2}-\mathrm{i}t)}\Big)\frac{d}{dt}\log\Big(\frac{b-J(-\frac{\mu^2}{2}+\mathrm{i}t)}{b-(-\frac{\mu^2}{2}+\mathrm{i}t)}\Big)\\ &-\delta_{D,4}\log\Big(\frac{J(-\frac{\mu^2}{2}-\mathrm{i}t)}{(-\frac{\mu^2}{2}-\mathrm{i}t)}\Big)\frac{d}{dt}\log\Big(\frac{J(-\frac{\mu^2}{2}+\mathrm{i}t)}{(-\frac{\mu^2}{2}+\mathrm{i}t)}\Big)\Big\}\end{aligned}$$

For ρ discrete, ρ_λ is also discrete (see here), and the N-integral is evaluated by the residues to the right of $\mathrm{Re}(z) < -\frac{\mu^2}{2}$.

The limit $\Lambda \to \infty$ for $D = 4$ and $\lambda > 0$

- $J(z) = z - \lambda z^2 \int_0^\infty \frac{dt\, \rho_\lambda(t)}{(\mu^2+t)^2(t+z)}$ is bounded above on $\mathbb{R}_+$.
- Consequently, $J^{-1}(a)$ needed in $\tau_b(a)$ and $G(a,b)$ on previous slide does not exist (for $D = 4$, $\lambda > 0$, $\Lambda^2 \to \infty$, all a).

Is the Landau ghost back?

Not here! Express $G(a,b) := \frac{\mu^2 \exp(N_4(a,b))}{(\mu^2+a+b)}$ which avoids J^{-1}:

$$
\begin{aligned}
N_4(a,b) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} dt \Big\{ &\log\big(a - J(-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(b - J(-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
&- \log\big(a - (-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(b - (-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
&- \log\big(-J(-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(-J(-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
&+ \log\big(-(-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(-(-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \Big\}
\end{aligned}
$$

4D-Moyal space at infinite noncommutativity

Previously we were mainly interested Φ^4_4-model on Moyal space at infinite noncommutativity.

Defined by $\rho_0(x) = x$; it's the only case where

$$\rho_\lambda(t) = \rho_0(J(t)) =: t(t+\mu^2)\psi(t)$$

solves a Fredholm integral equation of second kind:

$$\psi(t) = \frac{1}{t+\mu^2} - \lambda \int_0^\infty du\, \frac{t}{(t+\mu^2)} \frac{1}{(u+t+\mu^2)} \frac{u}{(u+\mu^2)} \psi(u)$$

Theorem [Seiringer 19]

The integral operator has spectrum $[0,\pi]$. Consequently, the Φ^4_4-model on Moyal space exists for $\lambda > -\frac{1}{\pi}$.

- Proves that $J(t)$ stays positive!
- Solved as convergent(!) power series in hyperlogarithms.
- It is here where number theory meets QFT!

Beyond the 2-point function

Recursive equations for all planar functions [Grosse, W 12]:

$$G^{(0)}_{b_0\dots b_{N-1}} = -\lambda \sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{b_0\dots b_{2l-1}} \cdot G^{(0)}_{b_{2l}\dots b_{N-1}} - G^{(0)}_{b_1\dots b_{2l}} \cdot G^{(0)}_{b_0 b_{2l+1}\dots b_{N-1}}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

Theorem [de Jong, Hock, W 19]

The solution is in 1:1-correspondence with Catalan tables of length $\frac{N}{2}$. There are $d_{(N-2)/2}$ of them, where $d_n = \frac{1}{n+1}\binom{3n+1}{n}$.

Definition (Catalan tuple $\tilde{e} = (e_0, \dots, e_n)$ of length $|\tilde{e}| = n$)

consists of $e_j \in \mathbb{N}$ with $\sum_{j=0}^{n} e_j = n$ and $\sum_{j=0}^{k} e_j > k$ for all $k<n$.

Definition (Catalan table $\langle \tilde{e}^{(0)}, \tilde{e}^{(1)}, \dots, \tilde{e}^{(n)} \rangle$ of length n)

$(n+1)$-tuple of Catalan tuples, such that
$(1 + |\tilde{e}^{(0)}|, |\tilde{e}^{(1)}|, \dots, |\tilde{e}^{(n)}|)$ is itself a Catalan tuple.

Higher topology

Fact: Moments in matrix models have topological expansion

$$\log\left(\frac{\int_{M^*_{\mathcal{N}}} d\Phi\ \Phi_{k_1 l_1}\cdots\Phi_{k_n l_n} e^{-\mathcal{N}\,S(\Phi)}}{\int_{M^*_{N}} d\Phi\ e^{-\mathcal{N}\,S(\Phi)}}\right) = \sum_{g=0}^{\infty}\sum_{B=1}^{n}\mathcal{N}^{2-2g-B} G^{(g)}_{k^1_1\dots k^1_{n_1}|\dots|k^B_1\dots k^B_{n_B}}$$

(the l's are a permutation of k's which has B cycles)

- These $G^{(g)}_{\dots}$ satisfy a hierarchy of Dyson-Schwinger equations, which holomorphically extend to linear singular integral equations.
- There is a general solution theory [Carleman 21]. In some sense, everything is already solved.
- However, we expect that the solution simplifies enormously and shows universal properties: topological recursion.
- This is not yet done. The outcome could be exciting!

Topological recursion [Eynard, Orantin 07]

- Main ingredient is a polynomial equation $\mathcal{E}(x, y) = 0$ for a plane algebraic curve: the (classical) spectral curve.
- Any such curve gives rise to topological invariants.
  It defines a partition function which is a τ-function for Hirota equations: Integrability can be made precise.
- Solutions of $\mathcal{E}(x, y) = 0$ parametrised by meromorphic functions $x(z), y(z)$.
- In the Kontsevich model, $y(z)$ is the expectation value of a single resolvent $W(z) = \mathrm{Tr}((z - M)^{-1})$, and $x(z) = z^2 - c$.
- Starting from a universal 2-form $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$, a family $\omega_{g,s}$ of s-forms on $\overline{\mathbb{C}}^s$ is constructed which satisfy universal recursive equations.

These equations can be solved by residue operations!

What is the spectral curve of the quartic matrix model

- Almost surely: $y(z) = z - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z}$ (because it describes the solution of the 2-point function).
- We don't know yet $x(z)$. The same $x = z^2 - c$ as in the Kontsevich model works, but it would be a little disappointing.
- The 1+1-point function (genus 0) should be decisive. It solves

$$\begin{aligned}&\lambda\pi\rho_0(t)\cot\tau_a(a)\, G(a|c)\\&= \lambda\pi\mathcal{H}_a\big[\rho_0(\bullet)G(\bullet|c)\big] + \lambda\frac{G(a,c)-G(c,c)}{a-c}\,.\end{aligned}$$

- Work in progress, procedure is clear, but lengthy.
- There might be difficulties with the $2+2$-point functions, but most is routine once we have the spectral curve.

Summary

- I hope I convinced you that we have the fish.
- It remains to determine the species.
- I invite you to join us. There is no longer any risk.

Agenda

- Write the dictionary between differential forms in the spectral curve and correlations functions of the quartic matrix model (see Harald's talk).
- Identify Hirota equations and integrability.
- Compute the rational numbers in the partition function. They should be topological invariants: of what?

Final words

For your interacting QFT model:
Think of implicitly defined functions.

If the free theory has variable p^2, you cannot expect that p^2 is a good variable in the interacting model.

It will be an inverse z of $p^2 = z + f(z)$.

Even very simple functions f produce extremely rich inverses z!
This richness makes QFT so interesting.