An operator-algebraic approach to Yang-Mills theory in two dimensions

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Content

Operator-algebraic approaches to lattice-gauge theory

Hamiltonian formulation [Kogut, Susskind; 1975]

Operator-algebraic formulations

- Mathematical framework
  - $\rightarrow$ fixed finite lattices [Kijowski, Rudolph; 2002]
  - $\rightarrow$ fixed infinite lattice [Grundling, Rudolph; 2013]
  - $\rightarrow$ inductive limit over finite lattices [Arici, Stienstra, van Suijlekom; 2017] (loop quantum gravity approach, e.g. [Thiemann, 2002],[AS, Thiemann, 2016])
- Common aspect
  - $\rightarrow$ Replace the classical edge phase space T^*G by the C^*-algebra $C(G) \rtimes G$ (G-CCR).

Problem

$C(G) \rtimes G$ is not unital. This complicates constructions.

Observation

Equivariant Duflo-Weyl quantization is related to $C(G) \rtimes G$ as well. It requires a unital extension to be well-defined.

CFTs and unitary representations of Thompson's groups

Reconstruction of CFTs from subfactors [Jones; 2014]

1+1 dimensional chiral CFTs

- $\{\mathcal{A}(I)\}_{I\subset S^1}$ (conformal net of type III factors)
- $\mathcal{A}(I) \subset \mathcal{B}(I)$, extensions give subfactors
  $\rightarrow$ Characterized by algebraic data (planar algebras).

Main idea [Jones; 2014]

Use planar-algebra data to reconstruct CFTs from subfactors.

$\rightarrow$ Define a functor from binary planar forest to Hilbert spaces (tensor networks).

$$\underbrace{\mathsf{Y}}_{\text{basic forest}} \longmapsto \underbrace{(\mathcal{H}_1 \to \mathcal{H}_2)}_{\text{“spin doubling”}}$$

$\rightarrow$ Gives discrete-CFT models (Thompson group symmetry).

Observation

These discrete-CFT models fit into the same framework as lattice-gauge theories defined by equivariant Duflo-Weyl quantization.

$$\text{Functor} \longleftrightarrow \text{Inductive limit over lattices/graphs}$$

Construction for a lattice with a single edge

The classical phase space of time-zero gauge fields

Basic ingredients

- The gauge-field phase space Γ will be modeled on T^*G (cf. [Creutz, 1983]).
  $\rightarrow$ $T^*G \cong G \times \mathfrak{g}$ with the canonical symplectic structure.

The canonical Poisson structure

The following Poisson structure is induced on $C^\infty(T^*G)$:

$$\{\sigma_f, \sigma_{f'}\}_{T^*G} = 0,$$
$$\{\sigma_X, \sigma_f\}_{T^*G} = \sigma_{R_X f'},$$
$$\{\sigma_X, \sigma_Y\}_{T^*G} = -\sigma_{[X,Y]},$$

for $\sigma_f(\theta, g) = f(g)$, $f \in C^\infty(G)$, and $\sigma_X(\theta, g) = \theta(X)$, $X \in \mathfrak{g}$ (momentum map of the Hamiltonian G-action).

Gauge transformations

The gauge transformations are associated with the left and right Hamiltonian G-actions on T^*G. But, there are various forms of gauge groups available depending on the "boundary topology" of the edge (open/closed, finite/infinite, etc.).

Construction for a lattice with a single edge

The C^*-algebra of time-zero gauge fields

Basic ingredients

- The gauge-field C^*-algebra $\mathfrak{A}$ will be based on $C(G) \rtimes G \subset C(G) \vee_{C^*} C^*_\lambda(G)$ (cf. [Creutz, 1983]).
  - $\rightarrow$ The crossed product structure is to be thought of as the "quantum" Poisson structure.
- This is motivated by the following theorem:

Theorem - Duflo-Weyl quantization (generalization of [Landsman; 1993]))

$$Q_\varepsilon^{DW} : C^\infty_{\mathrm{PW},U}(\mathfrak{g}) \hat{\otimes} C^\infty(G) \subset C^\infty(T^*G) \longrightarrow \mathcal{K}(L^2(G)) \cong C(G) \rtimes G$$

is a non-degenerate, strict deformation quantization on $(0, 1]$ w.r.t. to the canonical Poisson structure on T^*G. Furthermore, the G-CCR are satisfied:

$$Q_\varepsilon^{DW}(\{\sigma_f, \sigma_{f'}\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_f), Q_\varepsilon^{DW}(\sigma_{f'})] = 0,$$

$$Q_\varepsilon^{DW}(\{\sigma_X, \sigma_f\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_X), Q_\varepsilon^{DW}(\sigma_f)] = R_X f,$$

$$Q_\varepsilon^{DW}(\{\sigma_X, \sigma_Y\}_{T^*G}) = \tfrac{i}{\varepsilon}[Q_\varepsilon^{DW}(\sigma_X), Q_\varepsilon^{DW}(\sigma_Y)] = i\varepsilon R_{[X,Y]}.$$

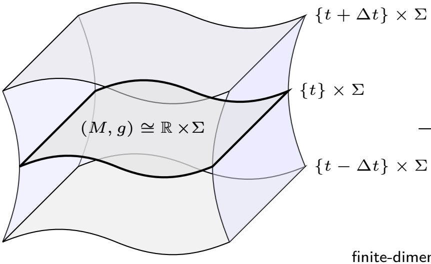
The Weyl form of the G-CCR corresponds to the crossed product relations.

1. Motivation
   - Operator-algebraic approaches to lattice-gauge theory
   - CFTs and unitary representations of Thompson's groups

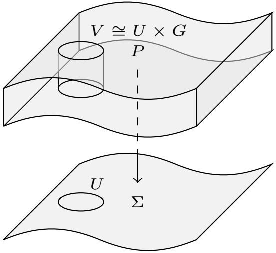
2. Construction for a lattice with a single edge

3. A projective phase space for lattice-gauge theories
   - An example: YM_2 on a space-time cylinder

A projective phase space for lattice-gauge theories



initial data formulation
temporal gauge



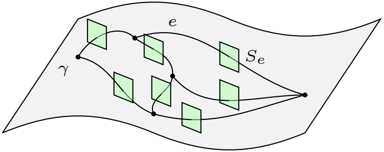
finite-dimensional
projections

Hamiltonian formulation:

- $M \cong \mathbb{R} \times \Sigma$ - Cauchy foliation

Hamiltonian gauge-field formulation:

- Σ - Cauchy surface
- G - structure group (compact)
- A, E - gauge field, conjugate electric field
- $D_A E = 0$ - Gauß constraint



Basic functionals:

- $g_e(A)$ - Holonomy
- $P^e_X(A, E; S_e)$ - Flux

Phase space:

$\Gamma \subset \overline{\Gamma} = \varprojlim_\gamma \Gamma_\gamma$, cp. [Federbush, 1987]

$\Gamma_\gamma = T^* G^{|E(\gamma)|}$

A projective phase space for lattice-gauge theories

Structure of the finite-dimensional phase spaces

The induced Poisson structure, e.g. [Thiemann, 2002]

Using a suitable regularization of the infinite-dimensional Poisson structure, the basic functionals w.r.t. a given graph γ generate the G-CCR of $T^*G^{|E(\gamma)|}$:

$$\{f(g_e), f'(g_{e'})\}_\gamma(A,E) = 0,$$
$$\{P^e_X, f'(g_{e'})\}_\gamma(A,E) = \delta^{e,e'}(R_X f')(g_{e'}(A)),$$
$$\{P^e_X, P^{e'}_Y\}_\gamma(A,E) = -\delta_{e,e'} P^e_{[X,Y]}(A,E)$$

Operations on graphs

The basic functionals behave naturally w.r.t. operations on graphs:

$e = e_2 \circ e_1 : g_e(A) = g_{e_2}(A) g_{e_1}(A),$ (composition)

$e \mapsto e^{-1} : g_{e^{-1}}(A) = g_e(A)^{-1},\ P^{e^{-1}}_X(A,E) = -P^e_{Ad_{g_e(A)}(X)}(A,E),$ (inversion)

$e \mapsto \emptyset$: drop dependence. (removal)

Composition for fluxes

The behavior of fluxes w.r.t. composition is more complicated:

A projective phase space for lattice-gauge theories

Some inductive constructions

Action of the gauge group

The gauge group $\mathcal{G}$ has a natural action on the finite-dimensional phase spaces.

→ Gauge transformations act at the vertices of the graphs.

→ The action on $\mathcal{L}(C^\infty(\Gamma_\gamma))$ is induced by the action on convolution kernels:

$$\alpha_\gamma(\{g_v\}_{v\in V(\gamma)})(F)(\{(h_e,g_e)\}_{e\in E(\gamma)}) = F(\{(\alpha_{g_{e(1)}^{-1}}(h_e), g_{e(1)}^{-1} g_e g_{e(0)})\}_{e\in E(\gamma)}).$$

A non-commutative analog of Γ

Construct an inductive system of C^*-algebras $\{\mathfrak{A}_\gamma\}_\gamma$, $\mathfrak{A} = \varinjlim_\gamma \mathfrak{A}_\gamma$.

- First try: $\mathfrak{A}_\gamma = (C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|} \cong \mathcal{K}(L^2(G^{|E(\gamma)|}))$
  → Does not work (non-unital).
- Second try: $\mathfrak{A}_\gamma = M((C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|}) \cong \mathcal{B}(L^2(G^{|E(\gamma)|}))$
  → Works and has nice extension properties:
  (a) Unique extension of morphisms,
  (b) Embedding of $C(G^{|E(\gamma)|})$ and $G^{|E(\gamma)|}$,
  (c) Recovery of states on $(C(G)\rtimes G)^{\hat{\otimes}|E(\gamma)|}$ as strictly-continuous states of $\mathfrak{A}_\gamma$.

A projective phase space for lattice-gauge theories

Some questions

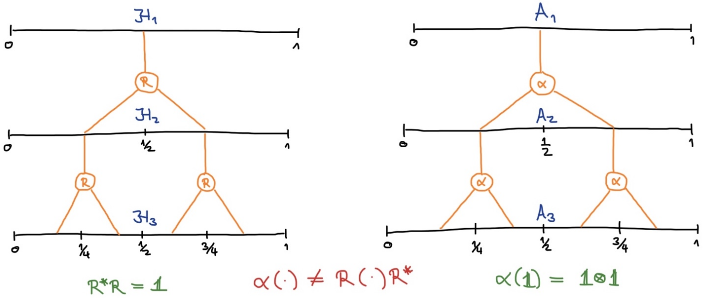
Some related questions

- Different choices of $\mathfrak{A}_\gamma$? Unital extensions of $(C(G) \rtimes G)^{\hat{\otimes}|E(\gamma)|}$?
- Control on the state space of the inductive-limit algebra?

$\rightarrow$ The natural representation on $L^2(\varprojlim_\gamma G^{|E(\gamma)|}, d\mu_0) = \varinjlim_\gamma L^2(G^{|E(\gamma)|}, d\mu_G^{\times|E(\gamma)|})$ is the GNS representation of the Ashtekar-Isham-Lewandowski state.

- Extensions to quantum groups?
- More refined block spin transformations (cp. [Balaban et al., Federbush, 1980's])?

An example: YM_2 on a space-time cylinder



Construction of time-zero data

A local C^*-algebra $\mathfrak{A}(I)$ is given as inductive limit over dyadic partitions of $I \subset [0,1]$:

$$\mathfrak{A}(I) = \{[\tfrac{t}{a}] : t \text{ a binary tree, } a \in \bigotimes_{J \in P_t(I)} \mathfrak{A}_J \otimes 1\},$$

$P_t(I)$ is the partition given by t subordinate to I. $\mathfrak{A}_J$ is the algebra corresponding to the leaf in J.

- $\mathfrak{A} = \mathfrak{A}([0,1]) = \varinjlim_t \mathfrak{A}_t$, $\mathcal{H} = \varinjlim_t \mathcal{H}_t$,
- $\mathcal{A} = \mathfrak{A}''$, $\mathcal{A}(I) = \mathfrak{A}(I)''$ (requires a state).

An example: YM_2 on a space-time cylinder

Locally thermal states

Consider the β-KMS states associated with H_N:

$$\omega_\beta^{(N)} = (\omega_\beta^{(1)})^{\otimes n}, \qquad \omega_\beta^{(1)}(\ .\ ) = Z_\beta(a_1^{-1} g_1^2)^{-1}\,\mathrm{tr}(\exp(-\beta H_1)\ .\ ).$$

State consistency

The requirement that the β-KMS states are consistent

$$\omega_\beta^{(N)} \circ \alpha_{N-1}^N = \omega_\beta^{(N-1)},$$

leads to (renormalization group flow):

$$g_{N-1}^2 = 2 g_N^2 \Rightarrow \frac{g_N^2}{a_N} = \frac{g_1^2}{L} = \underbrace{g_0^2}_{\text{bare coupling}} L.$$

- $\rightarrow$ The maps $\alpha_{N-1}^N : A_{N-1} \to A_N$ are non-trivial (block-spin transformations).
- $\rightarrow$ The state on the field algebra $\mathcal{A}_\beta$ has a Thompson-group symmetry (discrete CFT).
- $\rightarrow$ The β-limit Hamiltonian $H_\beta^{(\infty)}$ is given by the modular Hamiltonian of $\omega_\beta^{(\infty)}$.

An example: YM_2 on a space-time cylinder

Field algebra

The net of gauge-field algberas $\{\mathcal{A}_\beta(I)\}_{I\subset S^1}$ forms a local, Thompson-covariant net:

(a) $[\mathcal{A}_\beta(I), \mathcal{A}_\beta(J)] = \{0\}$ if $I \cap J = \emptyset$,

(b) $\rho_g(\mathcal{A}_\beta(I)) = \mathcal{A}_\beta(gI)$,

(c) $\omega_\beta^{(\infty)} \circ \rho_g = \omega_\beta^{(\infty)}$.

The algebras are expected to be generically of type III by an argument related to he construction of the Powers factors [Powers, 1967].

Observable algebra

Implementing gauge-invariance, i.e. constructing $\mathcal{A}_\beta^{\mathcal{G}}$, $\mathcal{H}_\beta^{\mathcal{G}}$, gives

$$\mathcal{H}_{\beta=0}^{\mathcal{G}} = L^2(G)^{Ad_G}, \qquad \mathcal{H}_\beta^{\mathcal{G}} = \mathcal{H}\,\mathcal{S}(L^2(G))^{Ad_G}, \qquad H = -\tfrac{1}{2}g_0^2 L \Delta_G,$$

as expected. The Hamiltonian and the "area law" can be read of from the "state sum":

$$Z_\beta(a_1^{-1}g_1^2) = \sum_{\pi\in\hat{G}} d_\pi\ e^{-\frac{\beta}{2}g_0^2 L\lambda_\pi} \xrightarrow{L\to\infty} \begin{cases} \text{“}\delta_e^{(G)}(e)\text{”} & : \quad \beta = 0 \\ 1 & : \quad \beta \in (0,\infty] \end{cases}$$

Thank you for your attention!