Quantum Field Theory on Random Trees

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York, July 4, 2019

Introduction

Work based on arXiv:1905.12783, with N. Delporte.

Why QFT on random trees ?

- Quantum gravity: space-time as random geometry
- Random trees: simplest example of non-trivial random geometry, with effective dimension $d_s = 4/3$ and large N limit of tensor models (melonic graphs)
- Fluctuations around melons (subleading in N) may be studied as QFT on melons? Higher dimensions accessible via Sachdev-Ye-Kitaev holography?
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Space, Time, Quantum Gravity and All That

At the Planck scale we expect space and time to change drastically as gravity becomes quantized.

There are currently competing attempts to understand what could happen then.

Mainstream proposal ($\geq 35yrs$): superstring theory.

Alternative proposal: space time an emergent fluid (modern ether).

In the first point of view a main problem is to reduce the dimension, typically from 10 to 4.

In the second point of view, a main problem is to increase the dimension, typically from 0 to 4.

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A popular tool to decrease dimension is compactification

A popular tool to increase dimension is holography

Usually holography is implemented as AdS/CFT, eg $N = 4 YM_4/AdS_5$...

However no CFT in $d = 0, 1$, hence first steps are special... is time as emergent as space?

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It is related [Carrozza, Gurau, Klebanov, Tarnopolsky, Witten, 2016-] to the tensor models of the tensor track [R. 2011-], as it crucially uses the same leading melonic graphs. These melonic graphs are also random trees [Gurau, Ryan 2014]. They could provide the crucial step, $d = 0 => d = 4/3$, from where to continue with holography and AdS/CFT.

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Matrix and Tensor Models

They are combinatorial ($d = 0$) quantum field theories which provide a kind of equilateral version of Regge calculus.

Quantum field theories of space time, not on space time

Their Feynman graphs realize a (background independent) sum over piecewise-linear manifolds with discretized Einstein-Hilbert action in arbitrary dimension $d = r$. Matrices correspond to $d = 2$, tensors to $d \geq 3$.

The main analytic tool is the $1/N$-expansion:

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Random Spaces

Probabllity measure on the space of all spaces (Gromov-Hausdorff space).

Right now (2019) probabilists study essentially two universal non-trivial "continuous" random spaces:

- The Continuous Random Tree [Aldous, $\simeq$ 1990]
- The Brownian Sphere [Le Gall, Miermont $\simeq$ 2011]

The second space can be thought of as a set of random labels living on the first.

In practice: these spaces can be discretized as random graphs.

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Finite Galton-Watson trees

Galton-Watson trees have independent branching rates w_i at each vertex.

In the simplest case (binary trees) the critical Galton-Watson process corresponds to offspring probabilities $p_0 = p_2 = \frac{1}{2}$, $p_i = 0$ for $i \neq 0, 2$. .

The generating function for such trees obeys the simple Catalan equation $Z(\zeta) = \zeta(1 + Z^2(\zeta))$, which solves to $Z = \frac{1-\sqrt{1-4\zeta^2}}{2\zeta}$ (Menous?).

Aldous universality class is the Gromov-Hausdorff limit of such critical Galton-Watson trees with fixed branching rate conditioned on non-extinction.

In physics, such random trees are often called branched polymers.

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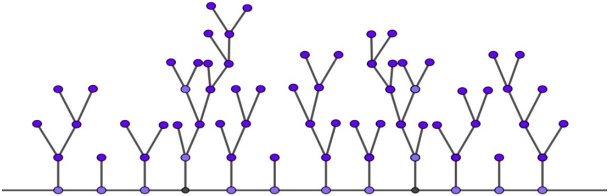

The condition of non-extinction creates an ensemble $\mathcal{T}$ of trees with *a single infinite spine* $\mathcal{S} = \mathbb{N}$ or $\mathbb{Z}$ decorated at each node v by an independent finite Galton-Watson branch T_v. The corresponding measure is

$$d\nu(\Gamma) = \prod_{v \in \mathcal{S}} d\nu_{GW}(T_v)$$

The spectral dimension is $d_{spectral} = 4/3$ [Durhuus, Johnson, Wheather] is defined by

the averaged return probability of Brownian walks. In QFT language it is the infrared scaling of the tadpole.

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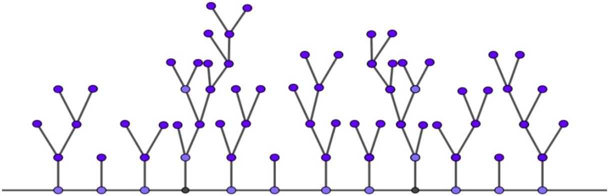

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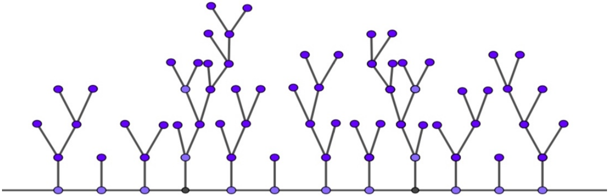

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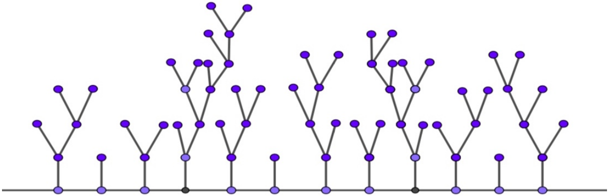

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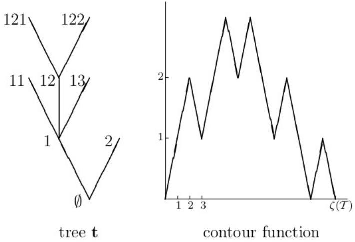
Turning around the tree

One can understand the metric properties of a large random tree via a nice one-to-one map.

The Dyck walk turns around the tree to identify the tree to its contour function quotiented by an equivalence relation. The contour function is exactly a Brownian excursion.



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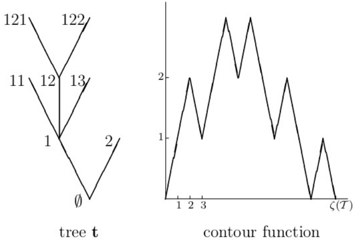

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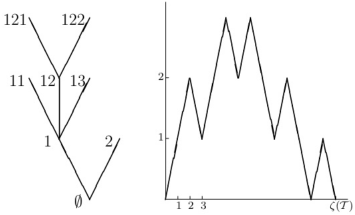

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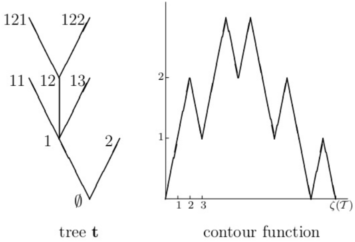

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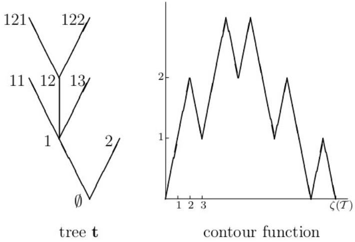
The Dyck walk turns around the tree to identify the tree to its contour function quotiented by an equivalence relation. The contour function is exactly a Brownian excursion.



This map is the heart of the proof that $d_{Hausdorff} = 2$ and $d_{spectral} = 4/3$.

QFT on a Graph

On a graph Γ

- we have no longer translation and rotation or Lorenz invariance and the notion of momenta is lost
- what remains: the Laplace operator. $\mathcal{L}_\Gamma = D_\Gamma - A_\Gamma$ (D_Γ: degree matrix ; A_Γ: incidence matrix). It inverse has the random path expansion:

$$C^m_\Gamma(x,y) = \sum_{\omega : x \to y} \prod_{v \in \Gamma} \left[\frac{1}{d_v + m^2} \right]^{n_v(\omega)} \sim \int_0^\infty \mathrm{d}t \; e^{-m^2 t} p_t(x,y),$$

Spectral dimension d_s: if $p_t(x)$ is the probability for a random walk starting at x to be at x in a time t, then

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Field theory and Observables

Partition function on a graph Γ:

$$Z(\Gamma;\lambda) = \int e^{-\lambda \sum_{x \in V_\Gamma} \phi^q(x)} d\mu_{C_\Gamma}(\phi) = \int d\nu_\Gamma(\phi).$$

Correlation functions:

$$S_N(\Gamma; z_1, \ldots, z_N) = \int \phi(z_1) \ldots \phi(z_N)\, d\nu_\Gamma(\phi) = \sum_{V=0}^{\infty} \frac{(-\lambda)^V}{V!} \sum_G A_G(\Gamma; z_1, \ldots, z_N).$$

The spine is common to all $\Gamma \in \mathcal{T}$. Hence we can define the observables as averaged Schwinger functions with arguments $\{z_1, \ldots, z_N\} \in \mathcal{S}$ on this spine

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Summary of Results

We developed the essential elements for a full perturbative renormalization group analysis of this type of QFT on random trees.

- We identify the fractional power of the Laplacian which makes the theory just-renormalizable
- We introduce a multiscale analysis by slicing the propagator according to the time of its random path representation
- We combine this analysis with precise heat kernel estimates of [Barlow, Kumagai]
- We obtain uniform bounds on convergent graphs and localization estimates for the divergent subgraphs which require renormalization
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Propagating the matter field

The propagator is the inverse of the Lapacian

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with an IR regulator m.
We then use the Euler β-function identity:

$$\mathcal{L}^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mathrm{d}m \frac{2m^{1-2\alpha}}{\mathcal{L} + m^2},$$

$(0 < \alpha \leq 1)$ to define the rescaled propagator as

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Divergence degree

The standard degree of divergence for a ϕ^q Feynman graph G in dimension d (V vertices, E internal edges and N external legs, $qV = 2E + N$) is:

$$\omega(G) = (d - 2\alpha)E - d(V - 1) = (d - 2\alpha)(qV - N)/2 - d(V - 1),$$

The just-renormalizable case occurs for

$$\alpha = \frac{d}{2} - \frac{d}{q}$$

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RG: multiscale analysis (towards the IR)

(1) Decompose the propagators into "proper time" scales $I_j = [M^{2(j-1)}, M^{2j}]$: $C = \sum_{j=0}^{\rho} C^j$ (note that $j = 0$ is the UV scale in our setting; for simplicity, external propagators are taken at IR cutoff scale ρ).
Each amplitude becomes a sum over all scale assignments μ.

(2) Identify superficial degree of divergence ω and divergent graphs.
Given μ, high subgraphs (quasi-local) control the divergences:

$$HS: \quad \text{(scales of internal legs)} < \text{(scales of external legs)}$$

$$|A_{G,\mu}| \leq \prod_{G_i \in HS} M^{\omega(G_i)}.$$

(3) Expand the divergent subgraphs around some reference point (localization of external propagators). Kill the first diverging terms by (local) counterterms.

(4) A renormalizable theory is defined at scale i by a finite number of parameters, with all parameters associated to lower scales $j < i$ having been integrated out. ($\rightarrow$ RG flow)

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Probabilistic estimates

For a parameter $\lambda \geq 1$, the ball $B(x, r)$ is said λ–good if (essentially):

$$r^2 \lambda^{-2} \leq |B(x, r)| \leq r^2 \lambda.$$

Crucially, [Barlow, Kumagai] showed that λ–good balls occur more and more likely for larger and larger λ:

$$\mathbb{P}[B(x, r) \text{ is not } \lambda\text{–good}] \leq c_1 e^{-c_2 \lambda}.$$

Then, they obtained the (quenched) bounds:
Given $r > 0$ and that $B(x, r)$ is λ–good, if $t \in [r^3 \lambda^{-6}, r^3 \lambda^{-5}]$, then

- for any $K \geq 0$ and any $y \in T$ with $d(x, y) \leq K t^{1/3}$

$$p_t(x, y) \leq c \left(1 + \sqrt{K}\right) t^{-2/3} \lambda^3 ,$$

- for any $y \in T$ with $d(x, y) \leq c_1 r \lambda^{-19}$

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Lemma (Single Line)

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- $cM^{-2j/3} \leq \mathbb{E}\left[C^j_T(x,x)\right] \leq c' M^{-2j/3}$,
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Interpretation: a typical volume integration corresponds to $d = 4/3$ while in proper time t the propagator scales as $t^{-1/3}$. c, c' are dummy names for inessential constants.

Idea of proof: slice the y-sum into rings and sum over all possible volumes inside with associated λ–not-goodness probability factors.

Propagators

Slicing the propagator into proper time slices $I_j = [M^{2(j-1)}, M^{2j}]$

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Convergent graphs

Theorem ($N > 4$)

For a completely convergent graph (no 2- or 4- point subgraphs) G of order $V(G) = n$, the limit as $\lim_{\rho\to\infty} \mathbb{E}(A_G)$ of the averaged amplitude exists and obeys the uniform bound

$$\mathbb{E}(A_G) \leq c^n (n!)^{\beta}$$

where $\beta = \frac{52}{3}$.

Comment: essentially uses the bounds above, Cauchy-Schwarz and again slicing the space into rings that are asked to be λ-good.

However intersecting rings don't have independent probabilities. This is what leads to the (non-optimal) factorial growth.

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We want to know how an amplitude changes when moving an external leg from one point z to a close point y:

Lemma

Defining $\Delta^j_T(x;y,z) := \left| C^j_T(x,y) - C^j_T(x,z) \right|$, *we obtain*

$$\mathbb{E}[\Delta^j_T(x;y,z)] \leq cM^{-2j/3} M^{-j/3} \sqrt{d(y,z)}.$$

Comment: uniform in x and the factor $M^{-j/3}\sqrt{d(y,z)}$ is the gain, provided $d(y,z) \ll r_j = M^{2j/3}$. The precise inequality for $y,z \in T$ is

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For $j_m \ll j_M$, we want to compare the “bare" amplitude

$$A_T^{bare}(x,z) := \sum_{y \in T} C_T^{j_M}(x,y) C_T^{j_m}(y,z)$$

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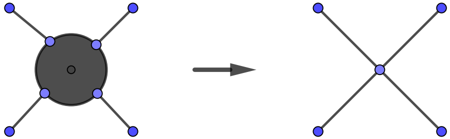
The previous lemma allows to write 4-point subgraphs as a local 4-vertex, plus corrections unseen by the external scale, defining hence an effective amplitude A^{eff}:



Theorem ($N \geq 4$)

For a graph G with $N(G) \geq 4$ and no 2-point subgraph G of order $V(G) = n$, the averaged effective-renormalized amplitude $\mathbb{E}[A^{eff}_G] = \lim_{\rho \to \infty} \mathbb{E}[A^{eff}_{G,\rho}]$ is convergent as $\rho \to \infty$ and obeys the same uniform bound than in the completely convergent case, namely

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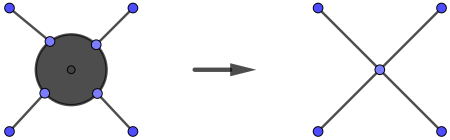

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Divergent graphs IV

In order to deal with multiple subtractions (e.g. for $N = 2$), we need to consider higher-order derivatives. In random or fractal spaces, a well-studied operator useful for transport is the Laplacian.

We can expand in powers of $\mathcal{L}$; Writing $\Delta_{yz} g := g(z) - g(y)$ and $\bar{f}(u) := \frac{1}{d_u} \sum_{v \sim u} f(v)$, we have

$$\begin{aligned} f(z) = &\Big[\bar{f} + \overline{\mathcal{L}f} + \overline{\mathcal{L}^2 f} + \cdots + \overline{\mathcal{L}^p f}\Big](y) \\ &+ \Delta_{yz}\Big[\bar{f} + \overline{\mathcal{L}f} + \overline{\mathcal{L}^2 f} + \cdots + \overline{\mathcal{L}^p f}\Big] + \mathcal{L}^{p+1} f(z). \end{aligned}$$

Lemma

There exists constants c_r, c_p such that uniformly for $t_j \in [M^{2(j-1)}, M^{2j}]$

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- Develop thermal physics in $d = 4/3$ by compactifying the spine
- Define and study SYK and tensor models on random trees
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The Emergent Picture

Graphs on graphs