Cosmological Solutions to the Semiclassical Einstein Equation with Minkowski-like Vacua

Class. Quantum Gravity: Vol. 39, No 12 • arXiv: 2112.15050

joint work with H. Gottschalk & D. Siemssen

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04 April, 2025

Outline

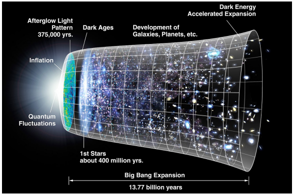
Classical Cosmology

Cosmological model

- Cosmological principle:
  On the largest observable scales the universe appears (spatially) homogeneous and isotropic.
- Space-time: $M = I_t \times \mathbb{R}^3$, $I_t \subset \mathbb{R}$ time interval, with metric: $g = -\mathrm{d}t^2 + a(t)^2 g_{\mathbb{R}^3}$
- Cosmological model: ODE for a, usually by Einstein equation

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \kappa\, T_{\mu\nu}$$

with $T_{\mu\nu} = T_{\mu\nu}[\,(g_{\alpha\beta})\,,\,\dots]$, typically IV's are set today

e.g.: 

Source:

Wikipedia

Cosmological_constant

Friedmann's cosmology

- Simplest (relevant) cosmological model
- From the cosmological principle $(T^{\mu}{}_{\nu}) = \mathrm{diag}(-\varrho, p, p, p)$
- Assume that p, ϱ are functions $I_t \to \mathbb{R}$
  (SE tensor of perfect fluid)

  $\longrightarrow$ Friedmann equations:

$$\frac{(\dot{a})^2}{a^2} = \frac{\kappa}{3}\varrho + \frac{\Lambda}{3}\,, \qquad \frac{\ddot{a}}{a} = -\frac{\kappa}{6}\big(\varrho + 3p\big) + \frac{\Lambda}{3}$$

- State equation: $\frac{p}{\varrho} = \gamma$ (and $\Lambda = 0$)

$$\Rightarrow \quad a \propto (t - t_{\mathrm{BB}})^{\frac{2}{3(\gamma+1)}} \quad \text{if } \gamma \neq -1$$

$$a \propto \exp(Ht) \quad \text{if } \gamma = -1$$

$$\Rightarrow \quad \varrho \propto \frac{1}{a^{3(\gamma+1)}}$$

ΛCDM model

- Ansatz: Plug

$$\varrho = T_{00} = \sum_\gamma \frac{\Omega_\gamma}{a^{3(\gamma+1)}}\,, \qquad \Omega_\gamma \in [0,1]\,, \quad \sum_\gamma \Omega_\gamma = 1$$

into Friedmann (energy) equation $\frac{(\dot a)^2}{a^2} = \frac{\kappa}{3}\varrho + \frac{\Lambda}{3}$

- ΛCDM: $\gamma \in \left\{\frac{1}{3}, 0, -1\right\}$,

$$\frac{(\dot a)^2}{a^2} = H_0^2 \left(\frac{\Omega_{\text{rad}}}{a^4} + \frac{\Omega_{\text{dust}}}{a^3} + \Omega_{\text{DE}} \right)$$

Standard values (PDG):

$$\begin{aligned} \Omega_{\text{rad}} &= 5.38 \cdot 10^{-5}\,, & \Omega_{\text{dust}} &= 0.315\,, \\ \Omega_{\text{DE}} &= 0.685\,, & H_0 &= 2.2 \cdot 10^{-18}\,\tfrac{1}{\text{s}} \end{aligned}$$

- Note: $\Lambda > 0\,,\ \Omega_{\text{DE}} = 0 \quad \longleftrightarrow \quad \Lambda = 0\,,\ \Omega_{\text{DE}} > 0$

Matter type coefficient Γ

- Introduce: $\Gamma[a] := -\frac{1}{3}\left(2\frac{a\,\ddot{a}}{\dot{a}^2}+1\right)$
- The ODEs $\Gamma[a] \overset{!}{=} \gamma$ are solved by

$$\begin{aligned} a &\propto (t - t_{\mathrm{BB}})^{\frac{2}{3(\gamma+1)}} && \text{if } \gamma > -1 \\ a &\propto \quad \exp(Ht) && \text{if } \gamma = -1 \end{aligned}$$

- $\Gamma[a] = -\frac{{G[a]^j}_j}{{G[a]^0}_0}$, i.e. if $({T^\mu}_\nu) = \mathrm{diag}(-\varrho, p, p, p)$

  and a solves $G_{\mu\nu} = \kappa\, T_{\mu\nu}$, then $\Gamma[a] = \frac{p}{\varrho}$
- $\Gamma[a] \to -1$ as $a \to \infty$ is interpreted as Dark Energy-dominated late-time

Dark Energy Behavior

- Wald proved:

$$\begin{aligned} T_{\mu\nu} t^\mu t^\nu &\geq 0 \\ \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^\sigma{}_\sigma\right) t^\mu t^\nu &\geq 0 \end{aligned} \quad \text{and} \quad \Lambda > 0$$

  for any future-directed vector field t^μ

$$\Rightarrow \quad a(t) \cdot \exp\left(- \sqrt{\frac{\Lambda}{3}}\, t \right) \to \text{const.} \quad \text{as} \quad t \to \infty$$

$$\left(\Rightarrow \quad \Gamma[a] \to -1 \right)$$

- Recall that $a \propto \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$ solves $\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3}$.
- $\Omega_{\mathsf{DE}} \approx 0.7$ in ΛCDM $\Rightarrow$ Dark Energy should be present!

Semiclassical Einstein Equation

SCE

$$G_{\mu\nu} \quad \Big(\quad + \Lambda g_{\mu\nu} \quad \Big) = \kappa \; T_{\mu\nu}[\,(g_{\alpha\beta})\,,\,\ldots]\big\langle T^{\mathrm{ren}}_{\mu\nu} \big\rangle_\omega$$

- $T^{\mathrm{ren}}_{\mu\nu}$: Renormalized stress-energy tensor of a quantum field ϕ
- $\langle \cdot \rangle_\omega$: Expectation value in state ω
- Choose free scalar field

$$-\nabla^\mu \nabla_\mu \phi + m^2 \phi + \xi R \phi = 0$$

$\longrightarrow$ AQFT approach of quantizing ϕ

Point-splitting regularization for $\left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega$

- Classical stress-energy tensor $\frac{\delta \mathcal{S}_{\mathrm{KG}}}{\delta g_{\mu\nu}}$:

$$\begin{aligned} T_{\mu\nu} = {} & (1-2\xi)(\nabla_\mu \phi)(\nabla_\nu \phi) \\ & - \tfrac{1}{2}(1-4\xi) g_{\mu\nu} (\nabla^\sigma \phi)(\nabla_\sigma \phi) \\ & - \tfrac{1}{2} g_{\mu\nu} m^2 \phi^2 \\ & + \xi \big(G_{\mu\nu} \phi^2 - 2\phi \nabla_\mu \nabla_\nu \phi + 2 g_{\mu\nu} \phi \nabla^\sigma \nabla_\sigma \phi \big) \end{aligned}$$

- Introduce quantum field and (quasi-free) Hadamard state ω
- Determine ω's two-point function ω_2 and its singular part H (Hadamard parametrix)
- For convenience, set $\omega_1 = 0$
- Replace ϕ, its derivatives and products thereof by quantum analogs, e.g.

$$\phi^2 \quad \mapsto \quad \left\langle :\phi^2: \right\rangle_\omega (x) := \lim_{y \to x} \Big[\omega_2(x,y) - H(x,y) \Big]$$

Point-splitting regularization for $\left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega$

Two more ingredients:

- Since $\nabla^\mu G_{\mu\nu} = 0$, also $\nabla^\mu \left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega = 0$ must hold.

  $\longrightarrow$ Add trace anomaly term (or Moretti/Hollands & Wald)

- "Correct way" to regularize ω_2?

  $\longrightarrow$ Add renormalization freedoms

Point-splitting regularization for $\left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega$

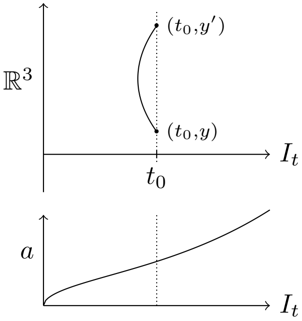
$$
\begin{aligned}
\left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega = {} & (1-2\xi) \left\langle :(\nabla_\mu \phi)(\nabla_\nu \phi): \right\rangle_\omega \\
& - \tfrac{1}{2}(1-4\xi) g_{\mu\nu} \left\langle :(\nabla^\sigma \phi)(\nabla_\sigma \phi): \right\rangle_\omega \\
& - \tfrac{1}{2} g_{\mu\nu} m^2 \left\langle :\phi^2: \right\rangle_\omega \\
& + \xi \Big(G_{\mu\nu} \left\langle :\phi^2: \right\rangle_\omega - 2 \left\langle :\phi \nabla_\mu \nabla_\nu \phi: \right\rangle_\omega \\
& \qquad + 2 g_{\mu\nu} \left\langle :\phi \nabla^\sigma \nabla_\sigma \phi: \right\rangle_\omega \Big) \\
& + \frac{1}{4\pi^2} g_{\mu\nu} [\nu_1] \\
& + c_1\, m^4 g_{\mu\nu} + c_2\, m^2 G_{\mu\nu} + c_3\, I_{\mu\nu} + c_4\, J_{\mu\nu}
\end{aligned}
$$

- $[\nu_1]$ st. $\nabla^\mu \left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle_\omega = 0$
- c_j: Renormalization constants
- $I_{\mu\nu} := 2RR_{\mu\nu} - 2\nabla_\mu \nabla_\nu R - \frac{1}{2} g_{\mu\nu}(R^2 + 4\nabla^\sigma \nabla_\sigma R)$
  $J_{\mu\nu} := 2R^{\sigma\varrho} R_{\sigma\mu\varrho\nu} - \nabla_\mu \nabla_\nu R - \nabla^\sigma \nabla_\sigma R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R_{\sigma\varrho} R^{\sigma\varrho} + \nabla^\sigma \nabla_\sigma R)$

Cosmological SCE with Minkowski Vacuum-like States

State at fixed time?

- A cosmological model requires something like $\omega_2(t, y, t, y')$.
- The Hadamard parametrix expresses the singularities of ω_2 in terms of geodesic distance between (t, y) and (t, y').
- But:



Follow Gottschalk & Siemssen ('21):

- Assume ω "cosmological" (spatially homogeneous & isotropic), $\quad \omega_2(t,y,t',y') = \omega_2(t,t',|y-y'|)$
- Construct new (non-geometric/cosmological) parametrix $\widetilde{H}$
- Insert "fruitful zero":

$$\begin{aligned} \omega_2^{\text{reg}} &:= \omega_2 - H \\ &= \underbrace{\omega_2 - \widetilde{H}}_{\to\, \mathcal{m} \in \mathcal{B}} \quad + \quad \underbrace{\widetilde{H} - H}_{\substack{\to\, a,\dot{a},\ddot{a},a^{(3)},a^{(4)},\log(a) \\ \text{(for all Hadamard states)}}} \end{aligned}$$

"Moments" $\mathcal{m}$:

- Sequences in Banach space $\mathcal{B}$
- Somewhat like radial, even-order Taylor coefficients of $\omega_2 - \widetilde{H}$ at some coincidence time t
- Only "concrete information" on the state which enters the SCE

Theorem: (Gottschalk - Siemssen, '21)

The SCE can be written in the form

$$\begin{cases} \dot{A}(t) &= F\big(A(t), m(t)\big) \\ \dot{m}(t) &= G\big(A(t)\big) \cdot m(t) \end{cases} \qquad (*)$$

with $A := (a, \dot{a}, \ddot{a}, a^{(3)})$. The dynamical system $(*)$ has a solution for all $m_0 = m(0)$ and almost all $A_0 = A(0)$.

Note:

- Linear evolution for m
- Evolution equation for m is derived from KG, not (!) from the full SCE
- Warning: Not clear whether arbitrarily selected m_0 belongs to a state
- Rather: (SCE) $\Rightarrow (*)$ and $(*) +$ positivity of $\omega_2 \Rightarrow$ (SCE)

Lemma:

On Minkowski spacetime

$$A(t) = (1,0,0,0), \quad t \in \mathbb{R}$$

the Minkowski vacuum for the massless, free Klein-Gordon field has moments

$$m(t) = 0, \quad t \in \mathbb{R}\,.$$

- First try: $m_0 = 0$ in $(*)$ equation (with $m = 0$)
- Linearity of m's evolution equation $\Rightarrow$ $m(t) = 0$ for all t
- The SCE reduces to an ODE for a

e.g.
$$-R = g^{\mu\nu}\kappa\langle T^{\text{ren}}_{\mu\nu}\rangle_\omega\Big|_{m=0}$$

constrained by
$$G_{00} = \kappa\langle T^{\text{ren}}_{00}\rangle_\omega\Big|_{m=0}$$

Minkowski-like vacua

Theorem: (Gottschalk - R. - Siemssen, '22)

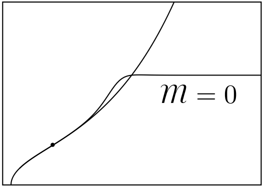
On any cosmological space-time such that a solves

$$\begin{aligned} -R &= g^{\mu\nu}\kappa\langle T^{\text{ren}}_{\mu\nu}\rangle_\omega\Big|_{m=0} \\ G_{00} &= \kappa\langle T^{\text{ren}}_{00}\rangle_\omega\Big|_{m=0} \end{aligned} \qquad (*)$$

there exists a Hadamard state ω for the massless, free KG field with $m(t) = 0\ \forall t$ such that (a, ω) is a solution for the SCE.

Minkowski-like vacua

Idea of proof: Tow-in argument



- Choose initial values for $a, \dot{a}, \ddot{a}, a^{(3)}$
- Solution a of $(*)$ (e.g. numerically)
- Smoothly deform into Minkowski (a_{tow})
- Solve KG on a_{tow} ($m_0 = 0$)
- Solve KG on a $\rightarrow$ still $m_0 = 0$

Our model

- Trace equation:

$$0 = \big(k_2 \log(\lambda_0\, a) - k_1\big)\left(\frac{a^{(4)}}{a} + 3\frac{\dot a a^{(3)}}{a^2} + \frac{\ddot a^2}{a^2} - 5\frac{\dot a^2 \ddot a}{a^3}\right)$$
$$+ \frac{k_2}{2}\left(4\frac{\dot a a^{(3)}}{a^2} + 3\frac{\ddot a^2}{a^2} + 12\frac{\dot a^2 \ddot a}{a^3} - 3\frac{\dot a^4}{a^4}\right) - k_3\frac{\dot a^2 \ddot a}{a^3} + k_4\left(\frac{\dot a^2}{a^2} + \frac{\ddot a}{a}\right)$$

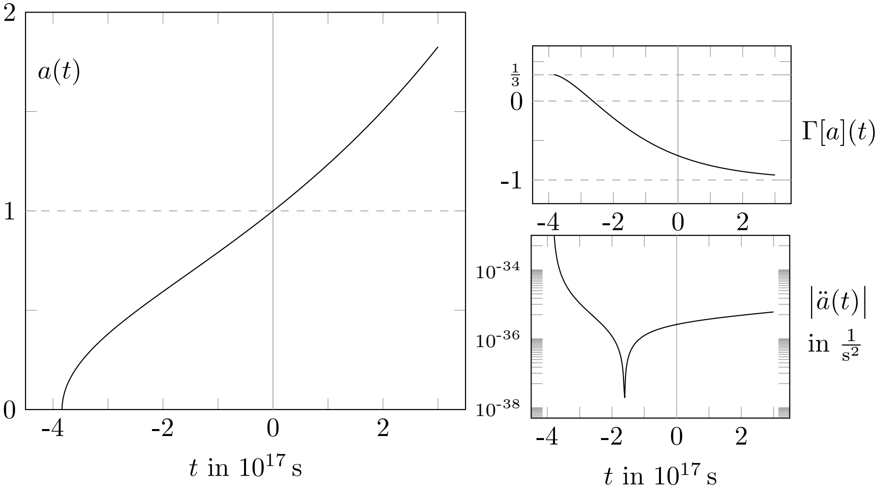
- Energy constraint:

$$0 = -\big(k_2 \log(\lambda_0\, a) - k_1\big)\left(\dot a a^{(3)} - \frac{1}{2}\ddot a^2 + \frac{\dot a^2 \ddot a}{a} - \frac{3}{2}\frac{\dot a^4}{a^2}\right)$$
$$- k_2\left(\frac{\dot a^2 \ddot a}{a} + \frac{\dot a^4}{a^2}\right) + \frac{k_3}{4}\frac{\dot a^4}{a^2} - \frac{k_4}{2}\dot a^2$$

where $$k_1 = 12(3c_3 + c_4) + \frac{1}{480\pi^2} - \frac{6\xi - 1}{48\pi^2},$$

$$k_2 = \frac{(6\xi-1)^2}{16\pi^2} \geq 0, \qquad k_3 = \frac{1}{240\pi^2} > 0, \qquad k_4 = \frac{6}{\kappa} > 0.$$

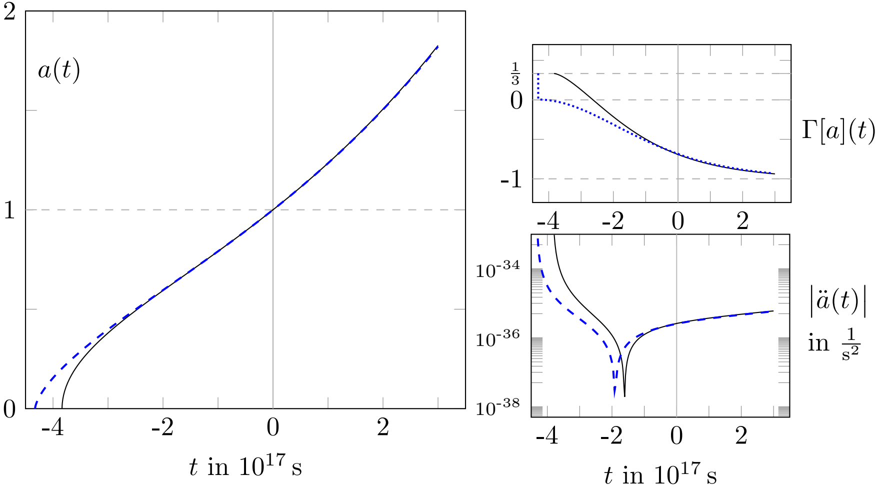
Generic solution:



$$H_0 = 2.2 \cdot 10^{-18}\,\frac{1}{\mathsf{s}},\ q_0 = -\frac{a(0)\ddot{a}(0)}{\dot{a}(0)^2} = -0.538,\ \ \frac{\kappa}{8\pi} = 8 \cdot 10^{40},$$

$$\xi = \frac{1}{12} \text{ and } \varepsilon := 3c_3 + c_4 = 0.5$$

Generic solution:



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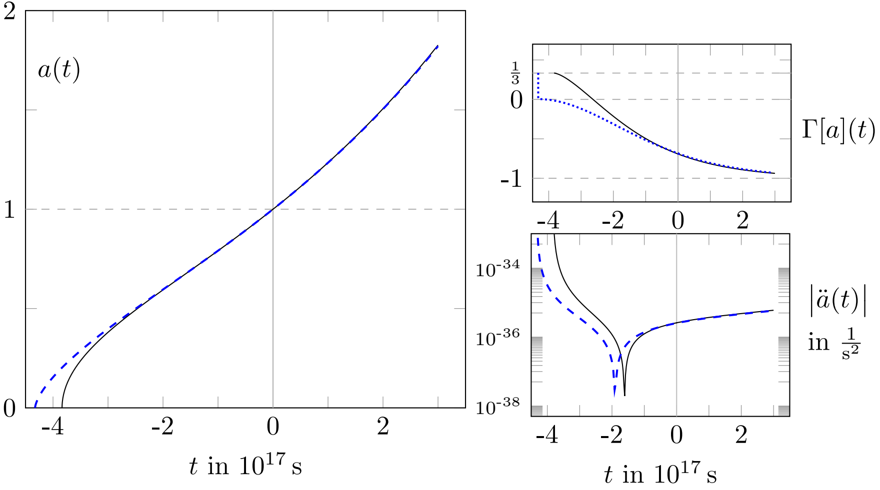
$\xi = \frac{1}{12}$ and $\varepsilon := 3c_3 + c_4 = 0.5$

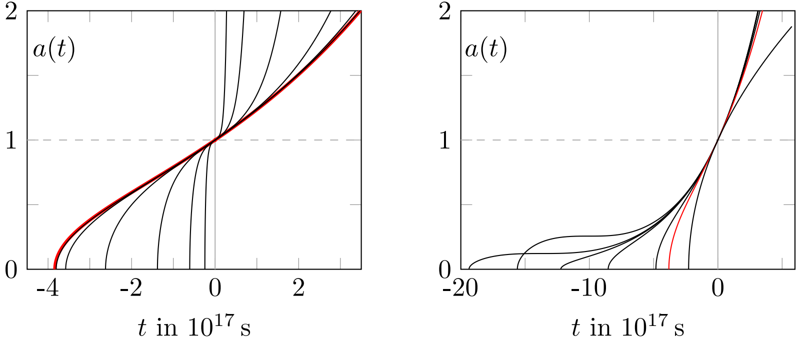
ΛCDM: standard PDG values

Generic solution:

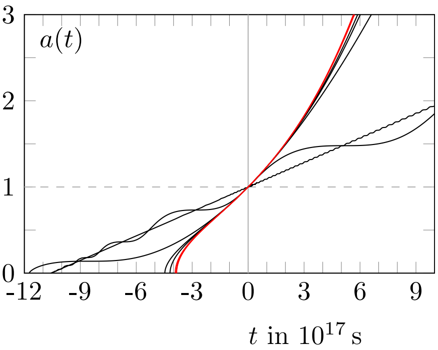


- Radiation dominated early phase & curvature singularity
- Dark Energy dominated late time expansion
- Zero of $\ddot{a}$ in the past

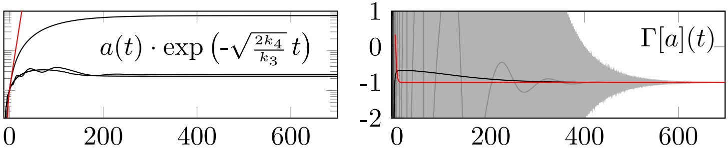
Some more solutions



Some more solutions

- 

- Plotted for longer times:



Attractive de Sitter solutions

De Sitter solutions

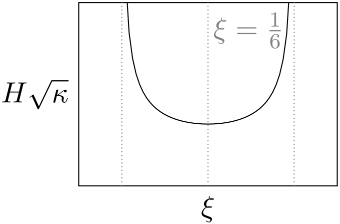
- Plugging the Ansatz $a(t) \propto \exp(Ht)$ into

  $g^{\mu\nu}\big(\langle T^{\text{ren}}_{\mu\nu}\rangle_\omega - \frac{1}{\kappa}G_{\mu\nu}\big)$ and $\langle T^{\text{ren}}_{00}\rangle_\omega - \frac{1}{\kappa}G_{00}$ yields

  $$(8k_2 - k_3)H^4 + 2k_4 H^2 = 0$$

- Consistency equation for parameters

  Solution: $(H^{\mathsf{dS}})^2 = \dfrac{2k_4}{k_3 - 8k_2}$ $\left(> 0 \;\Leftrightarrow\; |\xi - \frac{1}{6}| < 4320^{-1/2} \right)$



De Sitter solutions

Lemma:

Any cosmological solution of $G_{\mu\nu} = \kappa \big\langle T^{\mathrm{ren}}_{\mu\nu} \big\rangle_\omega$ has a strictly increasing a (if $\dot{a}(0) > 0$).

- Substitution: $\mathcal{H} := H \circ a^{-1} = \left(\frac{\dot{a}}{a}\right) \circ a^{-1}$
- $G_{\mu\nu} = \kappa \big\langle T^{\mathrm{ren}}_{\mu\nu} \big\rangle_\omega$ implies

$$
\begin{aligned}
0 \;=\; & -\big(k_2 \log(\lambda_0\, a) - k_1\big)\left(\mathcal{H}^3\mathcal{H}'' + \tfrac{4}{a}\mathcal{H}^3\mathcal{H}' + \tfrac{1}{2}\mathcal{H}^2(H')^2\right) \\
& - k_2\left(\frac{\mathcal{H}^3\mathcal{H}'}{a} + 2\frac{\mathcal{H}^4}{a^2}\right) + \frac{k_3}{4}\frac{\mathcal{H}^4}{a^2} - \frac{k_4}{2}\frac{\mathcal{H}^2}{a^2}\ .
\end{aligned}
$$

De Sitter solutions

Proposition:

If $\xi = \frac{1}{6}$ (i.e. $k_2 = 0$) and $3c_3 + c_4 > -\frac{1}{5760\pi^2}$ (i.e. $k_1 > 0$), then $(H^{\mathrm{dS}}, 0)$ is an asymptotic stable fixpoint in the $(\mathcal{H}, \mathcal{H}')$-plane of

$$0 = k_1 \left(\mathcal{H}^3\mathcal{H}'' + \frac{4}{a}\mathcal{H}^3\mathcal{H}' + \frac{1}{2}\mathcal{H}^2(H')^2 \right) + \frac{k_3}{4}\frac{\mathcal{H}^4}{a^2} - \frac{k_4}{2}\frac{\mathcal{H}^2}{a^2}$$

as $a \to \infty$.

- For $\xi \neq \frac{1}{6}$ similar behavior can be observed numerically and one can show that there exists an attractive direction in the $(\mathcal{H}, \mathcal{H}')$-plane
- The value $H^{\mathrm{dS}} = \sqrt{\frac{2k_4}{k_3 - 8k_2}}$ appears as effective cosmological constant/Dark Energy
  (recall: $\Lambda = 0$ and $\Lambda - m^4 c_1 = 0$)
- For $\xi = \frac{1}{6}$ similarly observed by Pinamonti, Dappiaggi et. al. and Haensel & Verch

Far from $\xi = \frac{1}{6}$: Box-R model

- Note, for $a \neq a_{\text{crit}} := \frac{1}{\lambda_0} \exp\left(\frac{k_1}{k_2}\right)$

$$\frac{g^{\mu\nu}\left(\left\langle T^{\text{ren}}_{\mu\nu}\right\rangle_\omega - \frac{1}{\kappa} G_{\mu\nu}\right)}{k_2 \log(\lambda_0 a) - k_1} = -\frac{1}{6} \nabla^\mu \nabla_\mu R + \frac{r_1\left(a, \dot{a}, \ddot{a}, a^{(3)}\right)}{k_2 \log(\lambda_0 a) - k_1}$$

  and

$$\frac{\left\langle T^{\text{ren}}_{00}\right\rangle_\omega - \frac{1}{\kappa} G_{00}}{k_2 \log(\lambda_0 a) - k_1} = \frac{a^2}{36} I_{00} + \frac{r_2\left(a, \dot{a}, \ddot{a}, a^{(3)}\right)}{k_2 \log(\lambda_0 a) - k_1}$$

- Far from a_{crit} (easily achieved far from $\xi = \frac{1}{6}$) our model is well approximated by

$$I_{\mu\nu} = 0$$

  $\rightarrow$ Box-R model (note $I^\mu{}_\mu = 6 \nabla^\mu \nabla_\mu R$)

- Recall: $\frac{1}{4\pi^2}[\nu_1]$, $c_3 I_{\mu\nu} + c_4 J_{\mu\nu}$ and $\widetilde{H} - H$ contribute into $k_1 - k_2 \log(\lambda_0 a)$-terms

Box-R model

Theorem:

(a) $I_{00} = 0 \Rightarrow I^{\mu}{}_{\mu} = 0$.

(b) Any solution of $I_{00} = 0$ is either Minkowski ($a =$const, if $a(t_0) = \dot{a}(t_0) = \ddot{a}(t_0) = 0$) or strictly increasing/decreasing.

(c) For any stritly increasing solution a of $I_{00} = 0$ we have either

- $a \propto (t - t_{\mathsf{BB}})^{1/2}$ if $\Gamma[a](t_0) = \frac{1}{3}$ (implying $\Gamma[a](t) = \frac{1}{3}\ \forall t$)

or

- a exist for all late times and
$$a(t) \underset{\sim}{\propto} \mathrm{e}^{Ht}$$
as $t \to \infty$ for some $H > 0$ ($\Gamma[a](t) \to -1$).

Box-R model

Proof:
Solve $0 = a^2 I_{00} = \dot{a} a^{(3)} - \frac{1}{2}\ddot{a}^2 + \frac{\dot{a}^2\ddot{a}}{a} - \frac{3}{2}\frac{\dot{a}^4}{a^2}$
Depending on initial values we have (a rescaling of)

- $a(t) =$const.,
- $a(t) \propto t^{1/2}$,
- $a(t) \propto \mathrm{e}^{Ht}$ or
- $a(t) = a_\pm(t) := \chi_\pm^{-1}(t)$,

where

$$\chi_\pm(x) = \frac{1}{6}\log\left(x^2 \pm x(1 \pm x^3)^{1/3} + (1 \pm x^3)^{2/3}\right) - \frac{1}{3}\log\left((1 \pm x^3)^{1/3} \mp x\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}\,x}{x \pm 2(1 \pm x^3)^{1/3}}\right)$$

$$\in \log(x) + \mathcal{O}(1) \qquad \text{as } x \to \infty\,.$$

Box-R model

- Exponential/Dark Energy-dominated late-time is generic (up to null-set of initial values).
- No effective cosmological constant/distingished value for $H(t) = \frac{\dot{a}(t)}{a(t)}$ in the limit $t \to \infty$: Any value is possible!
  $\to$ No cosmological constant problem?

Thanks for your attention!