Real resummations in Quantum Field Theories

Pierre J. Clavier

Institut für Mathematik
Universität Potsdam

Mathematics of interacting QFT models

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References

General introductions to resurgence:

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Real resummations:

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4. Marc P. Bellon and Pierre J. Clavier; **Analyticity domain of a quantum field theory and accelero-summation** 2018.

Pertubation theory

Perturbative series:

$$O(a) \simeq A_0 + A_1 a + A_2 a^2 + A_3 a^3 + A_4 a^4 + A_5 a^5 + A_6 a^6 + \cdots$$

Fixing all kinematic parameters:

$$O(a) \in \mathbb{C}[[a]]$$

Issues with perturbative approach:

- Soon very tricky!
- $O(a)$ typically not convergent; nor Borel summable.
- When $a \sim 1$: perturbative approach not efficient.

Beyond pertubation theory

$$O(a) \simeq A_0 + A_1 a + A_2 a^2 + A_3 a^3 + A_4 a^4 + A_5 a^5 + A_6 a^6 + \cdots$$

Open questions:

- Can we give a meaning to $O(a)$?
- Can we reach non-perturbative data?

Non-perturbative $\sim$ fonctions with vanishing Taylor development.

Exemple: instantons $e^{-1/a}$

(Hopefully) YES with resurgence theory

Borel-Laplace resummation I

Borel transform:

$$\mathcal{B} : (z^{-1}\ \mathbb{C}[[z^{-1}]], .) \longrightarrow (\mathbb{C}[[\xi]], \star)$$
$$\tilde{f}(z) = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{c_n}{z^n} \longrightarrow \hat{f}(\xi) = \sum_{n=0}^{+\infty}\frac{c_n}{n!}\xi^n.$$

Definition:

A formal series $\tilde{f}(z) = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{c_n}{z^n}$ is **1-Gevrey** if

$$\exists A, B > 0 : \ |a_n| \leq AB^n n!\ \forall n \in \mathbb{N}.$$

We write $\tilde{f}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1$.

Theorem:

The Borel transform $\hat{f}$ of a formal series $\tilde{f}$ is convergent if and only if $\tilde{f}$ is 1-Gevrey.

Borel-Laplace resummation II

Laplace integral:

$$\mathcal{L}^{\theta}[\hat{f}](z) = \int_0^{e^{i\theta}\infty} \hat{f}(\zeta) e^{-\zeta z} d\zeta.$$

Well-defined if:

- $\hat{f}(\zeta) \in \mathbb{C}\{\zeta\} \iff \tilde{f}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1$;
- existence of an analytic continuation in the direction θ.
- in the direction $\theta \in [0, 2\pi[$, $\hat{f}$ bounded by an exponential.

Resummation operator:

$$S_{\theta} = \mathcal{L}^{\theta} \circ \mathcal{B}.$$

Possible obstructions:

1. $\hat{f}$ not subexponential $\implies$ Accelero-summation (not today);
2. singularities in the direction $\theta \implies$ Alien calculus.

Resurgent functions I

Definition:

$\Omega \subset \mathbb{C}$ non-closed, discret, closed. $\hat{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ is an Ω-**continuable germ** if it is continuable along any path in $\mathbb{C} \setminus \Omega$.

$$\widehat{\mathcal{R}}_\Omega := \{\text{all } \Omega\text{-continuable germs}\} \subset \mathbb{C}\{\zeta\}.$$

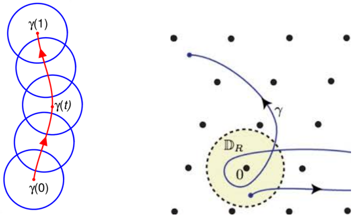

Figure: Continuation along a path & Resurgent functions

Resurgent functions II

Theorem

The convolution product $\star$ extends to $\widehat{\mathcal{R}}_\Omega$ and $(\widehat{\mathcal{R}}_\Omega, \star)$ is an algebra if, and only if, $(\Omega, +)$ is a semigroup.

Resurgent functions II

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Example: $\hat{f}_1(\zeta) = \frac{1}{\zeta-\omega_1}$, $\hat{f}_2(\zeta) = \frac{1}{\zeta-\omega_2}$, then:

$$(\hat{f}_1 \star \hat{f}_2)(\zeta) := \int_0^\zeta \hat{f}_1(\eta)\hat{f}_2(\zeta-\eta)d\eta$$

$$= \frac{1}{\zeta-\omega_1-\omega_2}\left[\int_0^\zeta \frac{d\eta}{\eta-\omega_1} + \int_0^\zeta \frac{d\eta}{\eta-\omega_2}\right]$$

Lateral alien derivatives

Definition:

For $\omega \in \Omega$ the **lateral alien derivatives** $\Delta_\omega^\pm : \widehat{\mathcal{R}}_\Omega \longrightarrow \widehat{\mathcal{R}}_\Omega$ are

$$\left(\Delta_\omega^\pm \hat{f}\right)(\zeta) = \overline{(\mathrm{cont}_{\gamma_\pm(\omega)}\hat{f})(\zeta+\omega)}$$

with $\gamma_\pm(\omega)$ the path from 0 to ω contourning every singularities from the left (resp. from the right).

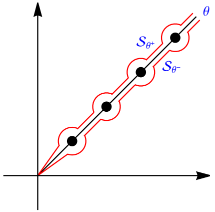

Figure: The paths $\gamma_\pm(\omega)$

Alien derivatives

Take $\Omega = \omega\mathbb{N}^*$.

Definition:

For $n\omega \in \Omega$ the (standard) **alien derivative** $\Delta_{n\omega} : \widehat{\mathcal{R}}_\Omega \longrightarrow \widehat{\mathcal{R}}_\Omega$ is

$$\Delta_{n\omega} = \sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} \sum_{n_1+\cdots+n_p=n} \Delta^+_{n_1\omega} \circ \cdots \circ \Delta^+_{n_p\omega}.$$

Properties:

- $\Delta_{n\omega}(\hat{f} \star \hat{g}) = (\Delta_{n\omega}\hat{f}) \star \hat{g} + \hat{f} \star (\Delta_{n\omega}\hat{g})$.
- $\hat{f}$ regular in $n\omega \implies \Delta_{n\omega}\hat{f} = 0$.
- Reciprocal not true!

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Example: $\hat{f}_1(\zeta) = \frac{1}{\zeta-\omega_1}$, $\hat{f}_2(\zeta) = \frac{1}{\zeta-\omega_2}$.

$$\Delta_{\omega_1+\omega_2}(\hat{f}_1 \star \hat{f}_2) = (\Delta_{\omega_1+\omega_2}\hat{f}_1) \star \hat{f}_2 + \hat{f}_1 \star (\Delta_{\omega_1+\omega_2}\hat{f}_2) = 0$$

$$(\Delta_{\omega_1} \circ \Delta_{\omega_2})(\hat{f}_1 \star \hat{f}_2) = (\Delta_{\omega_1}\hat{f}_1) \star (\Delta_{\omega_2}\hat{f}_2) \neq 0$$

Dotted alien operators I

Take $\Omega = \mathbb{N}^* \subset \mathbb{R}_+$.
Resurgent functions:

$$\widehat{\mathcal{R}}_\Omega \ni \hat{f} : \mathbb{R}//\Omega \longrightarrow \mathbb{C}$$
$$\zeta^{\varepsilon_1 \cdots \varepsilon_n} \longmapsto \hat{f}(\zeta^{\varepsilon_1 \cdots \varepsilon_n}).$$

with

$$\mathbb{R}//\Omega := \text{Ramified line above } \mathbb{R}_+ \subset \mathbb{C}//\Omega,$$
$$\mathbb{C}//\Omega := \{\text{homotopy classes } [\gamma] \text{ of rectifiable paths } \gamma : [0,1] \longrightarrow \mathbb{C} \setminus \Omega\}.$$

Observation:

Need to extend the Borel transform to include nonperturbative terms e^{-nz}.

Dotted alien operators II

Extend Borel transform:

$$\mathcal{B} : \widetilde{\mathcal{R}}_\Omega[[e^{-nz}]] \longrightarrow \widehat{\mathcal{R}}_\Omega \oplus_{n\geq 0} \mathbb{C}\delta_n.$$

By setting $\mathcal{B}(e^{-nz}) =: \delta_n$.
Characterisation: $\delta_n \star \delta_m = \delta_{n+m}$;

$$(\delta_m \star \hat{f})(\zeta^{\varepsilon_1 \cdots \varepsilon_n}) \begin{cases} \hat{f}(\zeta^{\varepsilon_1 \cdots \varepsilon_{n-m}}) & \text{if } n \geq m \\ 0 & \text{otherwise.} \end{cases}$$

Definition:

For $m\omega \in \Omega$ the **dotted lateral alien derivatives** $\dot{\Delta}^{\pm}_{m\omega}$ are

$$(\dot{\Delta}^{\pm}_m \hat{f})(\zeta^{\varepsilon_1 \cdots \varepsilon_n}) = \left(\delta_m \star (\Delta^{\pm}_m \hat{f})\right)(\zeta^{\varepsilon_1 \cdots \varepsilon_n})$$

Averages

$\widehat{\mathcal{U}}_\Omega$ univariate functions over $\mathbb{C}//\Omega$:

$$\phi \in \widehat{\mathcal{U}}_\Omega \Longleftrightarrow \begin{cases} \phi \in \widehat{\mathcal{R}}_\Omega, \\ \forall \zeta^{\varepsilon_1,\cdots,\varepsilon_n}, \zeta^{\sigma_1,\cdots,\sigma_n} \in \mathbb{C}//\Omega,\ \phi(\zeta^{\varepsilon_1,\cdots,\varepsilon_n}) = \phi(\zeta^{\sigma_1,\cdots,\sigma_n}). \end{cases}$$

Definition:

An **average** is a linear map $\mathbf{m} : \widehat{\mathcal{R}}_\Omega \longrightarrow \widehat{\mathcal{U}}_\Omega$ defined by its weights $\{\mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n}\}$ and

$$(\mathbf{m}\phi)(\zeta) = \sum_{\varepsilon_1,\cdots,\varepsilon_n=\pm} \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} \phi(\zeta^{\varepsilon_1,\cdots,\varepsilon_n})$$

for any $\zeta \in \mathbb{C} \setminus \Omega$ with $\zeta \in]n, n+1[$; with the coherence relations $\mathbf{m}^{\emptyset} = 1$ and

$$\mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_{n-1},+} + \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_{n-1},-} = \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_{n-1}}.$$

Example of averages

- Left lateral average:

$$\mathbf{mur}^{\varepsilon_1\cdots\varepsilon_n} = \begin{cases} 1 & \text{if } \varepsilon_1 = \cdots = \varepsilon_n = + \\ 0 & \text{otherwise.} \end{cases}$$

- Median average:
set $p = \#$ of $+$ in $\{\varepsilon_1, \cdots, \varepsilon_n\}$, $q = \#$ of $-$ in $\{\varepsilon_1, \cdots, \varepsilon_n\}$

$$\mathbf{mun}^{\varepsilon_1\cdots\varepsilon_n} = \frac{(2p)!(2q)!}{4^{p+q}(p+q)!p!q!}$$

- Catalan average:
Ca_n the n-th Catalan number, $Qa_n(x)$ the n-th Catalan polynomial, $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$.
Write $\boldsymbol{\varepsilon} = \varepsilon_1 \cdots \varepsilon_n = (\pm)^{n_1}(\mp)^{n_2} \cdots (\varepsilon_s)^{n_s}$, set

$$\mathbf{man}^{\boldsymbol{\varepsilon}}_{(\alpha,\beta)} = (\alpha\beta)^n Ca_{n_1} \cdots Ca_{n_{s-1}} Qa_{n_s}((\alpha/\beta)^{\varepsilon_n}).$$

Well-behaved averages

Definition:

An average **m** is called **well-behaved** if

- **(P1)** It preserves the convolution $\mathbf{m}(\phi \star \psi) = (\mathbf{m}\phi) \star (\mathbf{m}\psi)$.
- **(P2)** It preserves reality: $\mathbf{m}^{\varepsilon_1 \cdots \varepsilon_n} = \overline{\mathbf{m}}^{\bar{\varepsilon}_1 \cdots \bar{\varepsilon}_n}$, with $\bar{\pm} = \mp$.
- **(P3)** It preserves exponential growths: $\forall \phi \in \widehat{\mathcal{R}}_\Omega, \zeta \in \mathbb{C} \setminus \Omega$

$$|\phi(\zeta^{\pm \cdots \pm})| \leq K e^{c|\zeta|} \implies |(\mathbf{m}\phi)(\zeta)| \leq K e^{c|\zeta|}$$

- **(P1)** $\Rightarrow$ The resummed function solves the initial problem.
- **(P2)** $\Rightarrow$ The resummed function is real.
- **(P3)** $\Rightarrow$ We can take the Laplace transform.

(Counter-) examples of well-behaved averages

	(P1)	**(P2)**	**(P3)**
mur	N	N	✓
mun	✓	✓	N
man	✓	✓	✓

Theorem (Menous):

The Catalan averages $\mathbf{man}_{(\alpha,\beta)}$ is a well-behaved average.

Remarks:

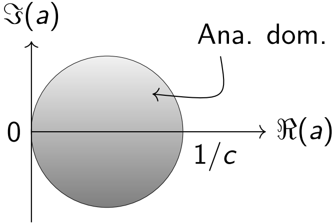
- **(P1)**, **(P2)** and **(P3)** have a tendancy to exclude each other: not many well-behaved averages!
- Physicists use **mun**! Big deal?

Application to the two-point function

$$G(a, L = \log(p^2)) \simeq 1 + A_1(L)a + A_2(L)Ca^2 + A_3(L)a^3 + A_4(L)a^4 + \cdots$$

- Borel transform in $z = 1/a$.
- Show that $\hat{G}(\zeta, L)$ is a resurgent function with exponential bound $\hat{G}(\zeta, L) \leq Ke^{c|\zeta|}$.
- Apply a well-behaved average
- Apply Laplace transform.

$\Rightarrow G^{res}(z = 1/a, L)$, analytic for $\Re(z) > c$



The Wess-Zumino model

We study a model which is:

- Massless,
- Exactly supersymmetric,
- Without vertex renormalisation.

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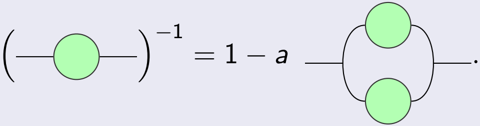
Renormalization Group Equation (RGE):

$$\partial_L G(a, L) = \gamma \left(1 + 3a\partial_a\right) G(a, L)$$

Schwinger–Dyson equation (SDE):



The Wess-Zumino model: results

Theorem (Bellon, C.):

The solution $\hat{G}(\zeta, L)$ of the RGE and SDE written in the Borel plane is a resurgent function with singular locus $\mathbb{Z}^*/3$.

$\hat{\gamma}(\zeta) := \partial_L \hat{G}(\zeta, L)|_{L=0}$.
Assumption (based on numerics): along the path γ_+

$$\exists C, D > 0 : |\hat{\gamma}(\zeta)| \leq C, \ |\hat{\gamma}'(\zeta)| \leq D$$

in an open neighborhood of $[R, +\infty[$.

Proposition:

Under the above assumption, along the path γ_+

$$\exists K, \tau > 0 : |\hat{G}(\xi, L)| \leq K e^{\tau L |\zeta|}$$

in an open neighborhood of $[R, +\infty[$.

The Wess-Zumino model: consequences

$G^{res}(a, L = \log(p^2))$ analytic for

$$a \leq \frac{1}{\tau L} \iff p^2 \leq M_{NP}(a) = e^{1/\tau a}.$$

Observations:

- In computations: pole $\Rightarrow$ $M_{NP}(a) =$ mass.
- Resurgence $\Rightarrow$ NP contributions $\Rightarrow$ mass generation mechanism.
- $M_{NP}(a) \longrightarrow +\infty$ as $a \to 0^+$.

Outlook

Resurgence theory

- allows to deal with divergent perturbative series from physics.
- is a powerful tool to compute NP contributions;
- in QFT, these contributions can give rise to a mass generation mechanism;

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Future projects:

- Characterisation of f^{res}.
- Asymptotically free theories $\Rightarrow$ Accelero-summation.
- More complicated theories (QCD?).

Thank you for your attention.