Infrared problem in perturbative QED

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Quantum electrodynamics

- QED one of the most important physical theories. Very accurate predictions. First model of relativistic QFT [Dirac, Jordan, Pauli (1927-28)].
- Naive computations of corrections to the scattering amplitudes or cross sections are plagued by divergences of two types:
  - Ultraviolet problem – short distances and large energies.
  - Infrared problem – large distances and low energies.
- In order to deal with the UV problem the renormalization techniques were developed [Tomanaga, Schwinger, Feynman, Dyson (1946-49)].
- By now these techniques are standard $\rightsquigarrow$ UV problem completely solved.
- IR problem still <u>not</u> fully understood.

Infrared problem in perturbative QED

- IR problem in the construction of objects such as the **net of local algebras of interacting fields** or the **Green or Wightman functions** completely under control $\rightsquigarrow$ there are **no problems** with the perturbative definition of QED.
- Problematic property of QED: **long-range interactions** mediated by massless photons $\rightsquigarrow$ evolution of particles is substantially different from the free evolution even long after or before the collision $\rightsquigarrow$ **difficulties** in the construction of **scattering operator** and **differential cross section**.

In the talk:

- Infrared problem in description of scattering of particles in perturbative QED.
- Method of the construction of the IR-finite S-matrix using the technique of adiabatic switching of the interaction [Bogoliubov] and a modified reference dynamics [Dollard, Kulish, Faddeev].

I. Physical origin of infrared problem

II. Gupta-Bleuler formulation of QED

III. Perturbative construction of scattering matrix

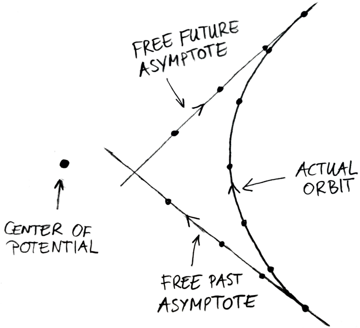
IV. Physical interpretation of construction

Scattering in classical mechanics – short-range potential

- A non-relativistic classical particle in a **short-range potential** V, i.e. $|V(\vec{x})| \leqslant \frac{\text{const}}{1+|\vec{x}|^{1+\delta}}$ with $\delta > 0$ (decays faster than the Coulomb potential).
- Let $\vec{x}(t)$ be the position of the particle in space $\mathbb{R}^3$ as a function of time.
- Assume that the energy of the particle is positive $\rightsquigarrow$ scattering situation.
- One shows that there are constants $\vec{x}_{\text{out}}, \vec{v}_{\text{out}} \in \mathbb{R}^3$ such that

$$\lim_{t\to\infty} |\vec{x}(t) - \vec{x}_{\text{out}} - t\,\vec{v}_{\text{out}}| = 0.$$

- We say that the trajectory $t \mapsto \vec{x}(t)$ of the particle is asymptotic in the future to the trajectory of the free particle $t \mapsto \vec{x}_{\text{out}} + t\,\vec{v}_{\text{out}}$.



Scattering in classical mechanics – Coulomb potential

- A non-relativistic particle of mass m moving in the repulsive **Coulomb potential** $V(\vec{x}) = \frac{e^2}{4\pi}\frac{1}{|\vec{x}|}$.
- The velocity of the particle $\dot{\vec{x}}(t)$ aquires the value $\vec{v}_{\mathrm{out}}$ in the limit $t \to \infty$ $\rightsquigarrow$ the orbit has a free asymptote.
- However, $|\dot{\vec{x}}(t) - \vec{v}_{\mathrm{out}}| = O(\frac{1}{|t|})$ for large $|t|$ $\rightsquigarrow$ the time parametrization of the actual orbit differs significantly from the time parametrization of the free asymptote $\rightsquigarrow$ particle on the interacting orbit lags behind the free particle.
- It holds
$$\lim_{t\to\infty} |\vec{x}(t) - \vec{x}_{\mathrm{out}}(t)| = 0,$$
where
$$\vec{x}_{\mathrm{out}}(t) := \vec{x}_{\mathrm{out}} + \vec{v}_{\mathrm{out}}t - \frac{e^2}{4\pi m}\frac{\vec{v}_{\mathrm{out}}}{|\vec{v}_{\mathrm{out}}|^3}\log|t|.$$
- The trajectory $t \mapsto \vec{x}(t)$ is not asymptotic in the future or past to any trajectory of a free particle.
- Similar problem appears in the scattering of two-particles interacting via the Coulomb potential $\rightsquigarrow$ IR problem in the construction of S-matrix in QED.

Scattering in quantum mechanics – short-range potential

- A non-relativistic particle of mass m in a **short-range** potential V.
- Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, momentum operator $\vec{p} = -\mathrm{i}\vec{\nabla}$.
- The free and full Hamiltonians:

$$H_{\mathrm{fr}} = \frac{\vec{p}^2}{2m}, \qquad H = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

  and the corresponding evolution operators:

$$U_{\mathrm{fr}}(t) = \exp(-\mathrm{i}tH_{\mathrm{fr}}), \qquad U(t) = \exp(-\mathrm{i}tH).$$

- Let $\Psi \in \mathcal{H}$ be a scattering state $\Rightarrow$ there exist states $\Psi_{\mathrm{out}}, \Psi_{\mathrm{in}} \in \mathcal{H}$ such that:

$$\lim_{t\to+\infty} \|U(t)\Psi - U_{\mathrm{fr}}(t)\Psi_{\mathrm{out}}\| = 0,$$
$$\lim_{t\to-\infty} \|U(t)\Psi - U_{\mathrm{fr}}(t)\Psi_{\mathrm{in}}\| = 0.$$

- Scattering matrix $S\Psi_{\mathrm{in}} = \Psi_{\mathrm{out}}$.

As expected, the above procedure does <u>not</u> work for long-range potentials such as, for example, the Coulomb potential.

Scattering in quantum mechanics – Coulomb potential

- A non-relativistic particle in the repulsive **Coulomb** potential.
- Let
$$H = \frac{\vec{p}^2}{2m} + \frac{e^2}{4\pi}\frac{1}{|\vec{x}|}, \qquad H_{\mathrm{D}}(t) = \frac{\vec{p}^2}{2m} + \frac{e^2}{4\pi}\frac{1}{\frac{|\vec{p}|}{m}|t|}.$$
be the full Hamiltonian and the so-called Dollard Hamiltonian.
- $U(t_2 - t_1)$, $U_D(t_2, t_1)$ – evolution operators – full and **reference** dynamics.
- For every state $\Psi \in \mathcal{H}$ there exist states $\Psi_{\mathrm{out}}, \Psi_{\mathrm{in}} \in \mathcal{H}$ such that:
$$\lim_{t\to+\infty} \|U(t)\Psi - U_{\mathrm{D}}(t,0)\Psi_{\mathrm{out}}\| = 0,$$
$$\lim_{t\to-\infty} \|U(t)\Psi - U_{\mathrm{D}}(t,0)\Psi_{\mathrm{in}}\| = 0.$$
- Modified scattering matrix $S_{\mathrm{mod}}\Psi_{\mathrm{in}} = \Psi_{\mathrm{out}}$.
- The above method was originally proposed by [Dollard (1964)].
- It is applicable to a large class of systems of non-relativistic particles interacting via long-range potentials [Dereziński, Gerard (1997)].

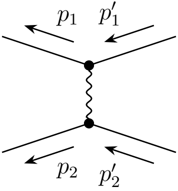
- Two non-relativistic particles interacting via the Coulomb potential.
- In order to define the S-matrix one compares the true evolution of the system

$$t \mapsto U(t)\Psi \in \mathcal{H}$$

with the Dollard reference evolution

$$t \mapsto U_D(t)\Psi = U_{\mathrm{fr}}(t)\ \mathrm{e}^{-\mathrm{i}\frac{e^2 m}{4\pi|\vec{p}_1 - \vec{p}_2|}\log|t|}\Psi \in \mathcal{H}.$$

- The phase factor is called the **Coulomb phase**.
- A similar phase factor appears in the amplitude for the **Møller** (two electrons) or **Bhabha** (electron and positron) scattering in **QED**.



Quantized electromagnetic field coupled to classical current

- Let $F^{\mu\nu}(x)$ be a **quantized** electromagnetic field satisfying the Maxwell equations with some fixed smooth conserved **classical** current $J^{\mu}(x)$ of spatially compact support.
- Let $v \in \mathbb{R}^4$ a unit timelike vector. Assume $J^{\mu}(x)$ has future/past **asymptotes**

$$\lim_{\lambda\to\infty} \lambda^3 J^{\mu}(\lambda v) = v^{\mu}\rho_{\mathrm{out}}(v), \qquad \lim_{\lambda\to\infty} \lambda^3 J^{\mu}(-\lambda v) = v^{\mu}\rho_{\mathrm{in}}(v).$$

- No incoming radiation condition $\rightsquigarrow$ the field $F^{\mu\nu}$ coincides with

$$F^{\mu\nu}_{\mathrm{ret}}(x) = F^{\mu\nu}_{\mathrm{fr}}(x) + 2\int \mathrm{d}^4y\, D^{\mathrm{ret}}(x-y)\, \partial^{[\mu}J^{\nu]}(y),$$

  where $F^{\mu\nu}_{\mathrm{fr}}(x)$ is the standard free quantum field defined in the Fock space.
- Past LSZ limit of the field $F^{\mu\nu}_{\mathrm{ret}}(x)$ coincides with the free quantum field $F^{\mu\nu}_{\mathrm{fr}}(x)$ wheres the future limit gives

$$F^{\mu\nu}_{\mathrm{out}}(x) = F^{\mu\nu}_{\mathrm{fr}}(x) + \begin{pmatrix} \text{radiation field} \\ \text{of the current } J \end{pmatrix}.$$

- $F^{\mu\nu}_{\mathrm{in}}(x)$ and $F^{\mu\nu}_{\mathrm{out}}(x)$ are unitarily related if and only if $\rho_{\mathrm{in}} \equiv \rho_{\mathrm{out}}$ $\rightsquigarrow$ scattering of charged particles is typically accompanied by emission of infinitely many soft photons $\rightsquigarrow$ **IR problem**.

Quantized electromagnetic field coupled to classical current

Heuristic description of the strategy that will be used in QED.

- Fixed some four-velocity v. Consider modified retarded field

$$F^{\mu\nu}_{\mathrm{ret,mod}}(x) = F^{\mu\nu}_{\mathrm{ret}}(x) + \begin{pmatrix} \text{radiation field of some reference} \\ \text{current depending only on v and } \rho_{\mathrm{in}} \end{pmatrix}$$

- Long-range tail of $F^{\mu\nu}_{\mathrm{ret,mod}}(x)$ in frame of observer moving with velocity v

$$\begin{pmatrix} \text{flux of electric field} \\ \text{in direction } \hat{n} \in S^2 \end{pmatrix} = \lim_{R\to\infty} r^2\, \vec{E}(t, r\hat{n}) = \frac{Q}{4\pi},$$

  where Q is total electric charge of the current J.

- Let $F^{\mu\nu}_{\mathrm{in}}(x)$ and $F^{\mu\nu}_{\mathrm{out}}(x)$ be the LSZ asymptotic fields. Unless the asymptotes of the current J coincide, these fields are <u>not</u> unitarily related.

- The outgoing field $F^{\mu\nu}_{\mathrm{out}}(x)$ contains the radiation emitted by the forward tail of the current which cannot be accommodated in the Fock space.

- The modified S-matrix intertwines the fields

$$F^{\mu\nu}_{\mathrm{in}}(x) + \begin{pmatrix} \text{radiation field} \\ \text{determined by } \rho_{\mathrm{in}} \end{pmatrix} \quad \text{and} \quad F^{\mu\nu}_{\mathrm{out}}(x) - \begin{pmatrix} \text{radiation field} \\ \text{determined by } \rho_{\mathrm{out}} \end{pmatrix}.$$

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- Notation:
  - $\psi(x)$ – massive Dirac spinor field,
  - $A_\mu(x)$ – real massless vector field,
  - $F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ – electromagnetic field tensor,
  - $J^\mu(x) := \overline{\psi}(x)\gamma^\mu\psi(x)$ – spinor current,
  - $\mathcal{L}(x) := J^\mu(x)A_\mu(x)$ – interaction vertex.
- Action of electrodynamics

$$S[A_\mu, \psi] = \int \mathrm{d}^4x \left(\overline{\psi(x)}(\mathrm{i}\not{\partial} - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + e\mathcal{L}(x) \right).$$

- Invariance under gauge transformations $S[A_\mu + \partial_\mu\chi, \psi\exp(\mathrm{i}e\chi)] = S[A_\mu, \psi]$.
- Quadratic part $S_{\mathrm{fr}}[A_\mu, \psi]$ of the action is invariant under free gauge transformations $S_{\mathrm{fr}}[A_\mu + \partial_\mu\chi, \psi] = S_{\mathrm{fr}}[A_\mu, \psi]$ $\rightsquigarrow$ lack of propagators $\rightsquigarrow$ problems with perturbative quantization.
- Solution: Introduce a gauge fixing condition $H(x) = \partial_\mu A^\mu(x)$ and modify the action by adding to it an expression quadratic in H.

Quantization of electrodynamics

- Modified action

$$S_{\mathrm{mod}}[A_\mu, J_\mu] = \int \mathrm{d}^4x \left(\overline{\psi(x)}(\mathrm{i}\partial\!\!\!/ - m)\psi(x) + \frac{1}{2} A_\mu(x) g^{\mu\nu} \Box A_\nu(x) + e\mathcal{L}(x) \right).$$

- If the gauge fixing condition is satisfied, then the equations of motion of the original and modified action coincide.
- Quadratic part of the modified action has a well-defined propagators.

Gupta-Bleuler quantization of QED

- First quantize the free part of the modified action. *Two-point function of A_μ is not positive definite $\rightsquigarrow$ Krein-Fock space.*
- Then, constract the interacting theory perturbatively. *Some IR regulator needed in the intermediate steps: nonzero mass of photon, $\mathrm{i}\epsilon$ prescription for the Feynman propagator with finite $\epsilon > 0$, dimensional regularization, adiabatic cutoff...*
- Finally, impose the gauge fixing condition $H(x) = \partial_\mu A^\mu(x)$. *Construct the physical Hilbert space where this condition is satisfied.*

Quantum electrodynamics

- Wightman and Green functions [Blanchard, Seneor (1975), Lowenstein (1976)].
- Net $\mathfrak{F}(\mathcal{O})$ of local abstract algebras of interacting fields localized in bounded spacetime regions $\mathcal{O}$ and a corresponding net $\mathfrak{A}(\mathcal{O})$ of algebras of gauge-invariant observables. [Dütsch, Fredenhagen (1999)].

- QED is a **well-defined model** of perturbative QFT.
- However, because of long-range interactions there are **difficulties** in the construction of objects that depend on **long-distance** properties.
- In particular, the standard definition of the S-matrix is <u>not</u> applicable because of **non-standard** behavior of the **Green functions**.

LSZ reduction formulas in massive models

Consider for a moment some model of interacting QFT <u>*without*</u> *long-range interactions containing an interacting scalar field $\psi_{\mathrm{int}}(x)$ of physical mass m.*

LSZ procedure [Lehman, Symanzik, Zimmermann (1955)], [Hepp (1965)]

1. Construct the Green functions:

$$G_n(x_1,\ldots,x_n) = (\Omega|\,\mathrm{T}(\psi_{\mathrm{int}}(x_1),\ldots,\psi_{\mathrm{int}}(x_n))\Omega).$$

2. Compute the amputated Green functions:

$$\tau_n(x_1,\ldots,x_n) = (\Box_{x_1}+m^2)\ldots(\Box_{x_n}+m^2)G_n(x_1,\ldots,x_n).$$

3. Elements of the scattering matrix are given by

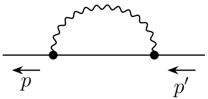
$$(p_1,\ldots,p_k|S|p_{k+1},\ldots,p_n) = \mathcal{Z}^{-\frac{n}{2}}\,\tilde{\tau}_n(p_1,\ldots,p_k,-p_{k+1},\ldots,-p_n)\big|_{p_j^2=m^2,p_j^0>0}$$

where $\mathcal{Z}$ is the residue of the interacting Feynman propagator on the **mass shell** (i.e. at $p^2=m^2$)

$$\tilde{\mathrm{G}}_2(p) \sim \frac{\mathrm{i}\mathcal{Z}}{p^2-m^2+\mathrm{i}0}, \quad \text{where} \quad \tilde{G}_2(p_1,p_2) = (2\pi)^4\delta(p_1+p_2)\,\tilde{\mathrm{G}}_2(p_1).$$

Behavior of electron self-energy in QED

$$\Sigma(p) \;=\; e^2$$



$$+\ \dots$$

- Perturbative corrections to $\Sigma(p)$ are continuous in p close to the mass shell $\rightsquigarrow$ one can renormalize $\Sigma(p)$ such that it vanishes on-shell $\rightsquigarrow$ m is the physical mass.
- However, the corrections are <u>not</u> differentiable on-shell $\rightsquigarrow$ residue $\mathcal{Z}$ is ill defined $\rightsquigarrow$ **IR divergences in S-matrix elements in perturbation theory.**
- It is possible to determine approximate form of corrections to $\Sigma(p)$ close to the mass shell at each order e^n.
- Formal summation gives the following form of the **interacting Feynman propagator** in QED close to mass shell [Kibble (1968), Zwanziger (1976)]

$$\tilde{\mathrm{G}}_2(p) = \frac{\mathrm{i}}{\not{p} - m - \Sigma(p) + \mathrm{i}0} \sim \mathrm{const}\frac{\not{p} + m}{(p^2 - m^2)^{1-\frac{e^2}{4\pi^2}}}.$$

- In non-perturbative QED no residue on the mass shell expected $\rightsquigarrow$ **infraparticle problem** [Schroer (1963)] $\rightsquigarrow$ **scattering amplitudes between Fock states** with finite number of photons **vanish.**

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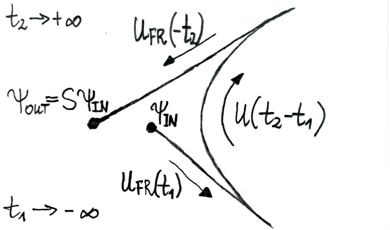
Methods of dealing with IR problem in perturbative QED

1. Inclusive cross-sections [Yennie, Frautschi, Suura (1961), Weinberg (1965), ...].
   - Sum over all outgoing photon configurations with total energy less than some **threshold**. Threshold is fixed and corresponds to the **sensitivity** of the detector.
   - States with different content of soft photons are difficult to discriminate experimentally and have to be all taken into account as possible final states.
2. Modified LSZ procedure [Zwanziger (1974), Papanicolaou (1976), ...].
   - Standard LSZ limit of the interacting spinor field is ill-defined.
   - Construct S-matrix elements using some modified LSZ procedure which takes into account the non-standard asymptotic behavior of this field.
3. Enlargement of the state space [Chung (1965), Kibble (1968), ...].
   - Define S-matrix in a Hilbert space that contains a large class of coherent states and accommodates radiation typically emitted by scattered charged particles.
4. Modified S-matrix [Kulish, Faddeev (1970), Jauch, Rohlich (1976), ...]
   - Compare the full dynamics of the system with some **non-trivial reference dynamics** and construct the modified S-matrix.
   - Approach used in the talk. Similarities to [Morchio, Strocchi (2016)].

- Reminder: S-matrix in QM with short-range potentials

$$S = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} U_{\mathrm{fr}}(-t_2) U(t_2 - t_1) U_{\mathrm{fr}}(t_1) = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \mathrm{Texp}\left(-\mathrm{i}e \int_{t_1}^{t_2} \mathrm{d}t \; H^I_{\mathrm{int}}(t) \right),$$

where $H^I_{\mathrm{int}}(t) = U_{\mathrm{fr}}(-t) H_{\mathrm{int}} U_{\mathrm{fr}}(t)$ is the interaction part of the Hamiltonian in the interaction picture.



- In QED on the heuristic level: $H^I_{\mathrm{int}}(t) = \int \mathrm{d}^3\vec{x} \; {:}\mathcal{L}(t,\vec{x}){:} = \int \mathrm{d}^3\vec{x} \; {:}\overline{\psi} \not{A} \psi(t,\vec{x}){:}$.
- Bogoliubov method: choose a switching function $g \in \mathcal{S}(\mathbb{R}^4)$ and replace

$$\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3\vec{x} \; {:}\mathcal{L}(t,\vec{x}){:} \quad \rightsquigarrow \quad \int \mathrm{d}^4x \, g(x) \, {:}\mathcal{L}(x){:} \, .$$

Scattering matrix in models with short-range interactions

- Bogoliubov S-matrix with adiabatic cutoff

$$S(g) = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\mathcal{L}(x)\right)$$
$$= \sum_{n=0}^{\infty}\frac{\mathrm{i}^n e^n}{n!}\int \mathrm{d}^4x_1\,\ldots\mathrm{d}^4x_n\, g(x_1)\ldots g(x_n)\,\mathrm{T}(\mathcal{L}(x_1),\ldots,\mathcal{L}(x_n)).$$

- Switching function $g \in \mathcal{S}(\mathbb{R}^4)$ plays the role of an infrared regulator.
- For any $g \in \mathcal{S}(\mathbb{R}^N)$ such that $g(0) = 1$ we define a one-parameter family of switching functions:

$$g_\epsilon(x) = g(\epsilon x) \quad \text{for} \quad \epsilon > 0.$$

- Physical S-matrix S is obtained by taking the adiabatic limit

$$S\Psi = \lim_{\epsilon\to 0} S(g_\epsilon)\Psi.$$

- The above procedure is applicable to massive models [Epstein, Glaser (1977)] but <u>not</u> to QED.

Modified scattering matrix in quantum mechanics

- Modified scattering matrix is obtained by comparing the evolution of the system governed by full Hamiltonian $H = H_{\rm fr} + eH_{\rm int}$ with the evolution governed by some Dollard Hamiltonian $H_{\rm D}(t) = H_{\rm fr} + eH_{\rm D,int}(t)$:

$$S_{\rm mod} = \lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}} U_{\rm D}(0,t_2)U(t_2-t_1)U_{\rm D}(t_1,0),$$

- The above expression can be rewritten in the form

$$\begin{aligned} S_{\rm mod} = \lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}} &\overline{\rm T}\exp\left(+{\rm i}e\int_0^{t_2} {\rm d}t\ H^I_{\rm D,int}(t)\right) \\ &\times {\rm T}\exp\left(-{\rm i}e\int_{t_1}^{t_2} {\rm d}t\ H^I_{\rm int}(t)\right)\overline{\rm T}\exp\left(+{\rm i}e\int_{t_1}^{0} {\rm d}t\ H^I_{\rm D,int}(t)\right), \end{aligned}$$

where $H^I_{\rm int}(t)$ and $H^I_{\rm D,int}(t)$ are the interaction parts of the full Hamiltonian and the Dollard Hamiltonian in the interaction picture.

Modified scattering matrix in QFT

- Modified scattering matrix with adiabatic cutoff

$$S_{\mathrm{mod}}(g) = \begin{pmatrix} \text{outgoing} \\ \text{Dollard modifier} \end{pmatrix} \times \begin{pmatrix} \text{Bogoliubov} \\ \text{S-matrix} \end{pmatrix} \times \begin{pmatrix} \text{incoming} \\ \text{Dollard modifier} \end{pmatrix}.$$

- Physical scattering matrix is obtained by taking the adiabatic limit

$$(\Psi|S_{\mathrm{mod}}\Psi') := \lim_{\epsilon\searrow 0}(\Psi|S_{\mathrm{mod}}(g_\epsilon)\Psi').$$

- Dollard modifiers have to be defined in such a way that:
  - they have a simple form,
  - they are well-defined for any switching function $g \in \mathcal{S}(\mathbb{R}^4)$,
  - adiabatic limit of $S_{\mathrm{mod}}(g)$ exists.

- Separation of IR and UV problem:
  - no UV problem in the construction of the Dollard modifiers,
  - UV problem in the Bogolibov S-matrix is solve using standard renormalization techniques,
  - to solve IR problem one has to show the existence of adiabatic limit.

Notation

Krein-Fock space

$$\mathcal{H} = \mathcal{H}_{\mathrm{photons}} \otimes \mathcal{H}_{\mathrm{fermions}}$$

- Ω – the vacuum state,
- $a^*_\mu(k)$, $a_\mu(k)$ – photon creation/annihilation operators,
- $b^*(p)$, $b(p)$, $d^*(p)$, $d(p)$ – electron/positron creation/annihilation operators,
- $\mathrm{d}\mu_m(p)$ invariant measure on $H_m = \{p^2 = m^2, p^0 > 0\} \subset \mathbb{R}^4$.

Free fields

$$A_\mu(x) = \int \mathrm{d}\mu_0(k) \,\left(a^*_\mu(k)\mathrm{e}^{\mathrm{i}k\cdot x} + a_\mu(k)\mathrm{e}^{-\mathrm{i}k\cdot x}\right),$$

$$\psi_a(x) = \int \mathrm{d}\mu_m(p) \,\left(b^*(p)u_a(p)\mathrm{e}^{\mathrm{i}p\cdot x} + d(p)v_a(p)\mathrm{e}^{-\mathrm{i}p\cdot x}\right).$$

Asymptotic interaction in QED – heuristics

- Standard interaction vertex:

$$\mathcal{L}(x) = J^\mu(x)\, A_\mu(x), \quad J^\mu(x) = :\overline{\psi}\gamma^\mu\psi(x): .$$

- Hamiltonian $H^I_{\mathrm{int}}(t) = \int \mathrm{d}^3\vec{x}\ \mathcal{L}(t,\vec{x})$ *formally* coincides with

$$\int \mathrm{d}\mu_m(p)\mathrm{d}\mu_m(p')\mathrm{d}\mu_0(k)\ b^*(p)b(p')a^*_\mu(k)$$
$$\times\ (2\pi)^3\delta(\vec{p}-\vec{p}' + \vec{k})\ \mathrm{e}^{\mathrm{i}\sqrt{\vec{p}^2+m^2}\,t-\mathrm{i}\sqrt{\vec{p}'^2+m^2}\,t+\mathrm{i}|\vec{k}|t}\ \overline{u}(p)\gamma^\mu u(p') + \dots$$

- Dollard Hamiltonian $H_D(t)$ is formally given by [Kulish, Faddeev (1970)]

$$\int \mathrm{d}\mu_m(p)\mathrm{d}\mu_0(k)\ b^*(p)b(p)a^*_\mu(k)\,\mathrm{e}^{\mathrm{i}|\vec{k}|t}\,p^\mu + \dots$$

  - It describes the emission or absorption of a photon by an electron or positron whose momentum is unchanged in the process (**no recoil**).
  - Unfortunately, it is <u>not</u> UV finite and has to be appropriately modified.

Asymptotic interaction in QED – definition

Asymptotic interaction vertices

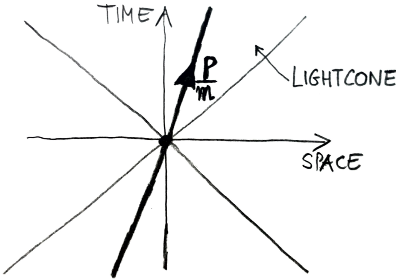
$$\mathcal{L}_{\mathrm{out/in}}(x) = J^{\mu}_{\mathrm{out/in}}(x)\, A_{\mu}(x),$$

where the asymptotic currents $J^{\mu}_{\mathrm{out/in}}(x)$ are given by

$$J^{\mu}_{\mathrm{out/in}}(x) = \int \mathrm{d}\mu_m(p)\, j^{\mu}_{\mathrm{out/in}}(\eta, p; x)\, \big(b^*(p)b(p) - d^*(p)d(p)\big)$$

and $j^{\mu}_{\mathrm{out/in}}(\eta, p; x)$ is defined as follows:

- Consider part of current of free particle of velocity $\frac{p}{m}$ in forward/backward lightcone.
- Convolute above current with a smooth charge density $\eta \in \mathcal{S}(\mathbb{R}^4)$, $\int \mathrm{d}^4x\, \eta(x) = 1$.



Dollard modifiers

- The Dollard modifiers $S^{\mathrm{as}}_{\mathrm{out/in}}(g)$ are formally given by

$$\overline{\mathrm{T}}\exp\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x) J^{\mu}_{\mathrm{out/in}}(x) A_{\mu}(x)\right).$$

- Because the asymptotic currents commute the above expression coincides with

$$\exp\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\, J^{\mu}_{\mathrm{out/in}}(x) A_{\mu}(x)\right)$$
$$\times \exp\left(\mathrm{i}\frac{e^2}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, g(x)g(y)\, g_{\mu\nu} D_0^D(x-y)\, {:}J^{\mu}_{\mathrm{out/in}}(x) J^{\nu}_{\mathrm{out/in}}(y){:}\right).$$

- The first factor is responsible for the generation of **coherent clouds of photons** which depend on the momentum of the electron/positron.
- The second factor is the **relativistic Coulomb phase**.
- Asymptotic currents are conserved $\partial_{\mu} J^{\mu}_{\mathrm{out/in}}(x) = 0$ only in sectors with vanishing electric charge $\rightsquigarrow$ if total charge is non-zero a modification needed.

S-matrix in QED

Modified S-matrix with adiabatic cutoff

$$S_{\mathrm{mod}}(\eta, \mathrm{v}, g) \;=\; S^{\mathrm{as}}_{\mathrm{out}}(\eta, \mathrm{v}, g)\; S(g)\; S^{\mathrm{as}}_{\mathrm{in}}(\eta, \mathrm{v}, g),$$

where $\eta \in \mathcal{S}(\mathbb{R}^4)$, $\int \mathrm{d}^4x\, \eta(x) = 1$, is a charge density and v is a four-velocity.

Physical S-matrix in QED is given by

$$(\Psi | S_{\mathrm{mod}}(\eta, \mathrm{v}) \Psi') = \lim_{\epsilon \searrow 0} (\Psi | S_{\mathrm{mod}}(\eta, \mathrm{v}, g_\epsilon) \Psi')$$

- Above adiabatic limit exists at least in low orders of perturbation theory.
- Physical S-matrix $S_{\mathrm{mod}}(\eta, \mathrm{v})$ is **gauge invariant** and can be consistently restricted to the physical Hilbert space in which the gauge fixing condition $\partial_\mu A^\mu(x) = 0$ is satisfied.
- In sectors with zero total charge $S_{\mathrm{mod}}(\eta, \mathrm{v})$ is v-independent.
- There exist intertwining operators $V(\eta', \eta, \mathrm{v})$ such that:

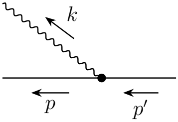
$$S_{\mathrm{mod}}(\eta', \mathrm{v}) = V(\eta', \eta, \mathrm{v})\, S_{\mathrm{mod}}(\eta, \mathrm{v})\, V(\eta, \eta', \mathrm{v}).$$

First and second order of perturbation theory

$$S_{\mathrm{mod}}(g) = S^{\mathrm{as}}_{\mathrm{out}}(g) S(g) S^{\mathrm{as}}_{\mathrm{in}}(g) = \mathbb{1} + e S^{[1]}_{\mathrm{mod}}(g) + e^2 S^{[2]}_{\mathrm{mod}}(g) + \ldots$$

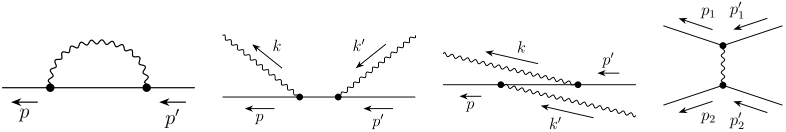
Theorem

$$\lim_{\epsilon \searrow 0} \| S^{[1]}_{\mathrm{mod}}(g_\epsilon)\Psi \| = 0 \quad \text{for all } \Psi \in \mathcal{D}$$



Assume that the self-energy of the photon and its first derivative vanish on-shell and the self-energy of electron vanishes on shell.

$$S^{[2]}_{\mathrm{mod}}\Psi := \lim_{\epsilon \searrow 0} S^{[2]}_{\mathrm{mod}}(g_\epsilon)\Psi \in \mathcal{H} \quad \text{exists for all } \Psi \in \mathcal{D}.$$



I. Physical origin of infrared problem

II. Gupta-Bleuler formulation of QED

III. Perturbative construction of scattering matrix

IV. Physical interpretation of construction

Net of local algebras of interacting fields

- There is a simple prescription for the construction of the **retarded interacting fields** due to Bogolibov.
  - C – polynomial in fields and their derivatives, g – switching function.
  - One first constructs first the extended scattering matrix

$$S(g;h) = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\mathcal{L}(x) + \mathrm{i}\int \mathrm{d}^4x\, h(x)C(x)\right)$$

  - and uses the Bogolibov formula to define the retarded field $C_{\mathrm{ret}}(g;x)$

$$C_{\mathrm{ret}}(g;x) = (-\mathrm{i})\frac{\delta}{\delta h(x)}S(g)^{-1}S(g;h)\Big|_{h=0}.$$

- $C_{\mathrm{ret}}(g,x)$ coincides with the Wick polynomial $:C(x):$ for spacetime points x which are not in the future of $\operatorname{supp} g$.
- Retarded fields can be used to define a **net of local abstract algebras** $\mathfrak{F}(\mathcal{O})$ of interacting fields localized in a spacetime region $\mathcal{O}$.
- The net satisfies the **Haag-Kastler axioms** in the sense of formal power series [Brunetti, Fredenhagen, Rejzner].

Representation of the net of algebras of interacting fields

- In order to study **global properties** of interacting fields one has to consider their **representation** in a Hilbert space.
- In massive models the vacuum representation in the standard Fock space can be obtained with the use of the adiabatic limit [Epstein, Glaser (1976)]

$$\pi(C_{\mathrm{ret}}(x))\Psi := \lim_{\epsilon \searrow 0} C_{\mathrm{ret}}(g_\epsilon, x)\Psi.$$

- The above construction does not work in QED.
- Idea: modify the extended scattering matrix

$$S_{\mathrm{mod}}(g;h) \;=\; S^{\mathrm{as}}_{\mathrm{out}}(g)\; S(g,h)\; S^{\mathrm{as}}_{\mathrm{in}}(g)$$

  and define modified retarded fields $C_{\mathrm{ret,mod}}(g_\epsilon, x)$ using Bogoliubov formula.
- Construction of the **vacuum representation** in QED:

$$\pi_{\mathrm{mod}}(C_{\mathrm{ret}}(x))\Psi := \lim_{\epsilon \searrow 0} C_{\mathrm{ret,mod}}(g_\epsilon, x)\Psi$$

  The above adiabatic limit exists in the Fock space for gauge-invariant fields (**observables**) at least in low orders of perturbation theory.

Properties of the first-order correction to $F^{\mu\nu}_{\mathrm{ret,mod}}(x)$

- **Flux** of the electric field measured by an observer moving with velocity v:

$$\lim_{r\to\infty} r^2 \vec{E}(t, r\hat{n}) = \frac{Q}{4\pi},$$

where Q is the electric charge $\rightsquigarrow$ the Gauss law is satisfied.

- **LSZ limit** of electromagnetic field exists [Buchholz (1977)]. One shows that it coincides with the standard expression for the free field but with new **physical creator and annihilator of photons**

$$c^*_\mu(k) = a^*_\mu(k) - e\int \mathrm{d}\mu_m(p)\frac{\tilde{\eta}(k)}{p\cdot k}(b^*(p)b(p) - d^*(p)d(p))$$

which does not coincide with the standard ones $a^*_\mu(k)$, and depend on the creation/annihilation operators of electrons/positrons $b^*(p)$, $d^*(p)$.

- State $b^*(p)\Omega$ with one electron contains infinitely many photons. Electron is dressed with the radiation field which is chosen in such a way that the sum of its Coulomb field and the radiation field has a long-range tail independent of the momentum p of the electron $\rightsquigarrow$ solution of the **infraparticle** problem.

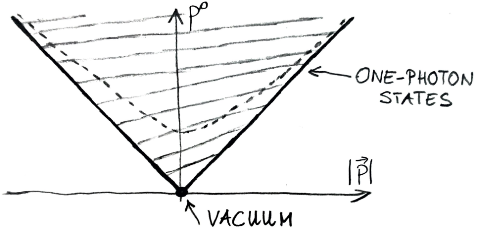
Energy-momentum operators

- Assume that the adiabatic limit of the modified scattering matrix and modified retarded fields exists in each order of perturbation theory.
- There is a **representation of the group of translations** $U_{\mathrm{mod}}(a)$ such that

$$U_{\mathrm{mod}}(a)\, S_{\mathrm{mod}}\, U_{\mathrm{mod}}(-a) = S_{\mathrm{mod}},$$
$$U_{\mathrm{mod}}(a)\, C_{\mathrm{ret,mod}}(x)\, U_{\mathrm{mod}}(-a) = C_{\mathrm{ret,mod}}(x+a).$$

Theorem

1. $U_{\mathrm{mod}}(a)$ *is not unitarily equivalent to the standard Fock representation.*
2. $U_{\mathrm{mod}}(a)$ *is strongly continuous. Generators* P^μ_{mod} *have simple form and are interpreted as* **the physical energy-momentum operators**.
3. *Joint spectrum of* P^μ_{mod} *coincides with the forward lightcone and contains*
   - *unique vacuum state* Ω,
   - *one-particle massless states,*
   - *no one-particle massive states* $\rightsquigarrow$ **electrons/positrons are infraparticles**.



Summary and outlook

Main results:

- Method of construction of IR-finite S-matrix in perturbative QED.
- Physical energy-momentum operators.

Details: arXiv:1906.00940

Open problems:

- Proof of existence of adiabatic limit in arbitrary order of perturbation theory.
- Relation between modified S-matrix and the inclusive cross sections.