Non-equilibrium steady states for the Klein-Gordon field in 1+3 dimensions

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joint work w/ Rainer Verch

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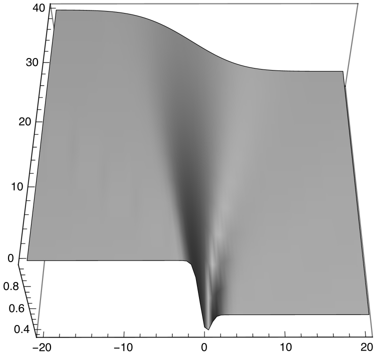
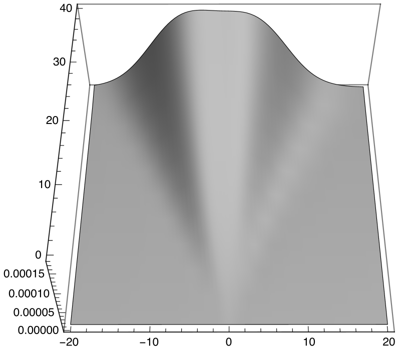
Two pictures say more than thousand words ...



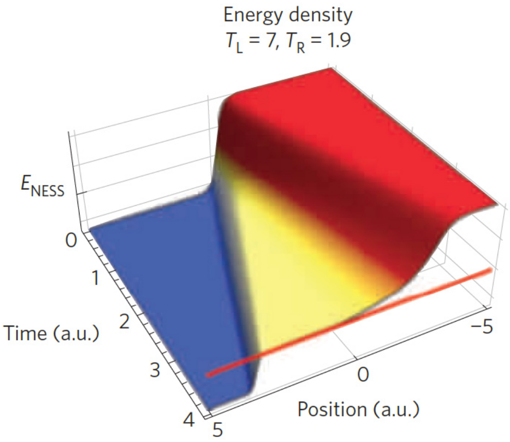


$\rho_G(t, x_1)/\rho_{\beta_1}\,, \quad \rho = T_{00}$ $\qquad$ $j_G(t, x_1)/j_N\,, \quad j = T_{01}$

free $m > 0$ Klein-Gordon in $d = 1 + 3$,

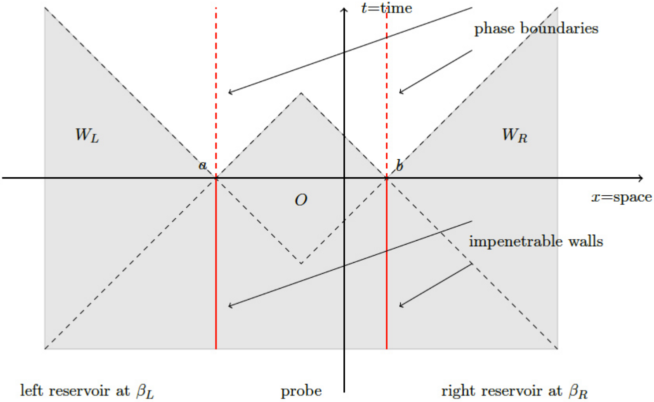
limit state is stationary and homogeneous but not KMS

Other models - conformal hydrodynamics in $d = 1 + 2$



[Bhaseen et al., Nature Phys. 11 (2015) 5]

Other models - CFTs in $d = 1 + 1$



[Bernard & Doyon 2012-2014, Hollands & Longo 2016 (pic.)]

Plan of the talk

- ... and other models / initial states, axiomatic analysis in quant. stat. mech. by *[Ruelle 2000]*, different NESS-setup in 1+3 QFT *[Drago, Faldino, Pinamonti 2017]*
- *[Doyon et. al 2014]* have studied the free Klein-Gordon field w/ $m \geq 0$ in $d = 1 + n$, "semi-box" Fock space picture, sharp contact surface
- we analyse the case of a "smooth contact" to have better regularity of the initial state (Hadamard state), the limit NESS is the same as the one of *[Doyon et. al 2014]*
- with better regularity of the initial state, we can also construct and analyse interacting case (at first order)
- the interacting NESS does not appear to be closer to equilibrium than the free NESS, but both are stable w.r.t. spatially localised perturbations

NESS for the free Klein-Gordon field

Basic idea

- basic idea: construct an initial state ω_G by gluing together initial data of Wightman correl. fct. of ω_{β_1}, ω_{β_2}, $\beta_1 = (k_B T_1)^{-1} \neq \beta_2 = (k_B T_2)^{-1}$, evolve initial data in time

- (technical) problem: positivity (unitarity) of correl. fct. for observables in the contact region and cross-correl.

$$\Delta_{+,G}(\overline{f}, f) \doteq \omega_G(\phi(\overline{f})\phi(f)) \overset{!}{\geq} 0 \qquad f \text{ any test function}$$

$$\phi(f) \doteq \int dx\, \phi(x) f(x)$$

- consider ω_i, $i = 1, 2, 3$ with $\Delta_{+,i} \geq \Delta_{+,3}$, $i = 1, 2$, μ_i initial data for $\Delta_{+,i}$, $1 = \chi_1(x_1) + \chi_2(x_1)$ smooth part. of 1 for x_1-axis $\Rightarrow$ initial data for $\Delta_{+,G} \geq 0$ constructed by

$$\mu_G \doteq (\chi_1 \otimes \chi_1)\mu_1 + (\chi_2 \otimes \chi_2)\mu_2 + 2(\chi_1 \otimes \chi_2 + \chi_2 \otimes \chi_1)\mu_3 \quad \geq \quad \mu_3$$

Thermal domination

- we choose $\dot\chi_1 = -\dot\chi_2$ of compact support = "contact region", $\dot\chi_1$ determines contact profile, ω_3 is "state in contact region and for cross-correlations"
- correl. fct. of thermal (KMS) state

$$\Delta_{+,\beta}(x_1,x_2) = \Delta_{+,\infty}(x_1,x_2)$$

$$+\frac{1}{(2\pi)^{d-1}}\int \frac{d\boldsymbol{p}}{\omega_{\boldsymbol{p}}}\,\frac{\cos\left(\omega_{\boldsymbol{p}}(t_1-t_2)-\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)\right)}{\exp(\beta\omega_{\boldsymbol{p}})-1} \qquad \omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2+m^2}$$

- $\Delta_{+,\beta_1} \geq \Delta_{+,\beta_2}$ for $\beta_1 \leq \beta_2 \Rightarrow \omega_3$ can be chosen e.g. as mixture of ω_β with $\beta \geq \max(\beta_1,\beta_2)$, for practical computations vacuum is simplest choice
- physically we expect that contact state can also be "hotter" than left/right reservoirs

Generalisation of gluing procedure

- consider a smooth partition of unity $1 = \psi + (1-\psi)$ of the time-axis such that $\dot{\psi}$ has compact support and ψ vanishes for large negative times
- $K \doteq \Box + m^2$, $g \doteq [K,\psi] = \ddot{\psi} + 2\dot{\psi}\partial_t$, $\Delta \doteq \Delta_R - \Delta_A$

$$\tau\phi \doteq \Delta g\phi = \int dy \Delta(x,y)[K,\psi(t_y)]\phi(y)$$

  is well-defined for any smooth function ϕ and the identity on solutions of the KGE, $g\phi$ is "thickened initial data" for ϕ localised in supp $\dot{\psi}$

- for ω_i, $i=1,2,3$ with $\Delta_{+,i} \geq \Delta_{+,3}$, $i=1,2$, $\sigma_i \doteq \tau\chi_i = \Delta\chi_i g$

$$\Delta_{+,G} \doteq (\sigma_1 \otimes \sigma_1)\Delta_{+,1} + (\sigma_2 \otimes \sigma_2)\Delta_{+,2} + (\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1)\Delta_{+,3} \quad \geq \quad \Delta_{+,3}$$

  defines Gaussian Hadamard state, $\psi(t) = \Theta(t)$ gluing at $t=0$

The limit NESS

- consider large time limit of $\Delta_{+,G}$

$$\Delta_{+,N}(t_1, \boldsymbol{x}_1, t_2, \boldsymbol{x}_2) \doteq \lim_{\tau\to\infty} \Delta_{+,G}(t_1+\tau, \boldsymbol{x}_1, t_2+\tau, \boldsymbol{x}_2)$$

- for any (admissible) contact state ω_3, any contact/switch-on profiles χ_i, ψ we find

$$\Delta_{+,N}(x_1, x_2) = \Delta_{+,\infty}(x_1, x_2)$$
$$+\frac{1}{(2\pi)^{d-1}} \int \frac{d\boldsymbol{p}}{\omega_{\boldsymbol{p}}} \frac{\cos\left(\omega_{\boldsymbol{p}}(t_1 - t_2) - \boldsymbol{p}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\right)}{\exp(\beta(p_1)\omega_{\boldsymbol{p}}) - 1}$$

$$\beta(p_1) = \beta_1\Theta(p_1) + \beta_2\Theta(-p_1)$$

- convergence is $O(\tau^{-1})$, limit exists for any $m \geq 0$ and $d = 1+n$ (for $m = 0, d = 1 + \{1, 2\}$ limit of derivatives exist)

- $\Delta_{+,N}$ defines Gaussian Hadamard state ω_N

Chemical potentials and condensates

- we obtain analogous results for non-vanishing chemical potentials (complex ϕ)
- ... and for Bose-Einstein condensates

NESS is mode-wise KMS

- form of 2PF suggests: ω_N is a state in which "left/right-movers" are separately in equilibrium at different temperatures β_1, β_2 (compare with Unruh state in Schwarzschild)
- rigorously: define time-translation $\alpha_t^{\beta_1,\beta_2}$, shifts "left/right-moving Fourier modes" by $\beta_1 t$, $\beta_2 t$, well-def. in the sense of exp. val. in states with tempered correl. fcts.
- $\to \omega_N$ is a KMS state w.r.t. $\alpha_t^{\beta_1,\beta_2}$ at inverse temp. $\beta = 1$.

Reasons for non-equilibrium asymptotics

- expectation in the literature: generic state will evolve to generalised Gibbs ensemble (GGE) *[Rigol et al. 2007. ...]*, where entropy is maximised for all conserved quantities I_i of system, formally

$$\rho = \frac{1}{Z}\exp(-\sum \lambda_i I_i)$$

  if only H (and N) conserved, proper thermalisation

- formal density matrix for limit NESS ω_N is *[Doyon et. al 2014]*

$$\rho_N = \frac{1}{Z}\exp(-H_N) \qquad H_N = \beta_1 H_L + \beta_2 H_R = \frac{\beta_1+\beta_2}{2}H + \frac{\beta_1-\beta_2}{2}(P_1+Q)$$

$$H = \int_{\mathbb{R}^{d-1}} d\boldsymbol{x}\, T_{00}(\boldsymbol{x}) \qquad P_1 = \int_{\mathbb{R}^{d-1}} d\boldsymbol{x}\, T_{01}(\boldsymbol{x}) \qquad Q = \int \text{non-local}$$

Improving thermalisation

- expectation *[Doyon et. al 2014]*: in a non-linear QFT only conserved quantities are H and P_i, initial state should thermalise, at least in a different rest frame.
- before looking at non-linear QFT, we analysed linear inhomogeneous models
$$K\phi = 0 \qquad K = \Box + m^2 + U(x^1)$$
- two toy-models: $U = \delta$, "phase-shift at $x^1 = 0$" $\rightarrow$ no (apparent) improved thermalisation in initial rest-frame

NESS for the interacting Klein-Gordon field

Finally ... a proper interaction

- consider $d = 1 + 3$ and Klein-Gordon field with homogeneous linear part and interaction
$$V = V(f) = \lambda \int dx\ f(x)\phi^4(x)$$
- $f(x)$ is a coupling-cutoff function, we would like to consider adiabatic limit $f \to 1$
- however direct definition of adiabatic limit is already problematic for equilibrium states at $T > 0$, presumably because interacting field does not behave like a free field at large times *[Buchholz & Bros 2002]*
- solution given by *[Fredenhagen & Lindner 2014]* in perturbative algebraic QFT (pAQFT) based on earlier work by *[Hollands & Wald 2003]*

pAQFT for pedestrians

- in pAQFT *[Brunetti, Dütsch, Fredenhagen, Hollands, Wald, ...]* one defines interacting observables in the algebra of the free theory $\mathscr{A}_0$ which is "the algebra of normal ordered (Wick) polynomials"

- elements of $\mathscr{A}_0$ are functionals of smooth field configurations $\phi : \mathbb{R}^d \to \mathbb{R}$

$$A = A(\phi) = a_0 + \sum_{n=1}^{\infty} \int dx_1 \dots dx_n \, \phi(x_1) \dots \phi(x_n) \, f(x_1, \dots, x_n)$$

corresponding to

$$:A(\phi):= a_0 \mathbf{1} + \sum_{n=1}^{\infty} \int dx_1 \dots dx_n \;\; :\phi(x_1) \dots \phi(x_n): \;\; f_n(x_1, \dots, x_n)$$

f_n symm. distributions with prescribed singularity (wave front set), e.g. $f_n = f(x_1)\delta_n(x_1, \dots, x_n)$, but generic f_n is not localised on diagonal

Products

- the product $\star$ on $\mathscr{A}_0$ is implementing the Wick theorem

$$A \star B \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left\langle \frac{\delta^n A}{\delta \phi^n}, \Delta_{+,\infty}^{\otimes n} \frac{\delta^n B}{\delta \phi^n} \right\rangle}_{2n\text{-fold integration}} = \text{“sum of contractions with } \Delta_{+,\infty}\text{”}$$

corresponding to $:A::B:=:AB:+\ldots$

- $\mathscr{A}_0$ also contains all time ordered product of local ($f_n \propto \delta_n$) observables

$$T(A \otimes B) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\delta^n A}{\delta \phi^n}, \underbrace{\Delta_{F,\infty}^{\otimes n}}_{\text{renormalised}} \frac{\delta^n B}{\delta \phi^n} \right\rangle \simeq T(:A::B:) =:AB:+\ldots$$

$$\Delta_{F,\infty}(x_1, x_2) = \omega_\infty(T(\phi(x_1) \otimes \phi(x_2)))$$

Expectation values and interacting observables

- for a Gaussian Hadamard state ω with two-point fct. $\Delta_{+,\omega}$, the expectation value of $A \in \mathscr{A}_0$ is

$$\omega(A) = \gamma_{W_\omega}(A)|_{\phi=0} \qquad W_\omega \doteq \Delta_{+,\omega} - \Delta_{+,\infty}$$

$$\gamma_{W_\omega} \doteq \exp\left(\left\langle W_\omega(x,y), \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)} \right\rangle\right)$$

$$\text{e.g.} \quad \omega(:\phi^2(x):) \simeq \omega(\phi^2(x)) = \left(\phi^2(x) + W_\omega(x,x)\right)|_{\phi=0} = W_\omega(x,x)$$

- algebra of interacting observables $\mathscr{A}_V \subset \mathscr{A}_0$ generated via $\star$ by

$$\mathscr{R}_V(A) \doteq T(\exp_\otimes(iV))^{\star -1} \star T(\exp_\otimes(iV)A)\,,$$

$$A = T(A_1 \otimes \cdots \otimes A_n)\,, \quad A_i \text{ local}$$

Definition of interacting thermal states

- construction and adiabatic limit of ω_β^V, KMS state for interaction V *[Fredenhagen & Lindner 2014]*:
- consider $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 1$, $t > \epsilon$, $\psi(t) = 0$, $t < -\epsilon$ and $h \in C_0^\infty(\mathbb{R}^3)$
- construct ω_β^V, $V = V(\psi h)$ for obs. localised where $\psi = 1$, spatial adiabatic limit $h \to 1$ exists for $m > 0$ ($m = 0$ *[Drago, TPH, Pinamonti 2016]*) and is independent of ψ, i.e. temporal cutoff "invisible", temporal adiabatic limit is "implicit"

Definition of interacting thermal states cont.

- for A in $\mathscr{A}_V$, $V = V(\psi h)$, A localised where $\psi = 1$, define

$$\omega_\beta^V(A) \doteq \frac{\omega_\beta(A \star U_V(i\beta))}{\omega_\beta(U_V(i\beta))}$$

- $U_V(t)$ intertwines free and interacting time evolution

$$\alpha_t^V(\mathscr{R}_V(A)) \doteq \mathscr{R}_V(\alpha_t(A)) = U_V(t)\, \alpha_t(\mathscr{R}_V(A))\, U_V^{-1}(t)$$

$$U_V(t) = 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, \alpha_{t_n}(K_V) \star \cdots \star \alpha_{t_1}(K_V)$$

$$K_V \doteq \mathscr{R}_V(V(\dot{\psi} h)) \qquad \alpha_t(\phi(x^0, \boldsymbol{x})) \doteq \phi(x^0 + t, \boldsymbol{x})$$

- formally $U_V(t) = \exp(it(H_0 + V))\exp(-itH_0)$

Gluing of interacting KMS states

- in order to define a prescription of gluing interacting KMS states, consider ω_G for a free field with mass $m^2 + \delta m^2$ and expand correl. fcts. perturbatively in δm^2
- result should correspond to glued state ω_G^V in interacting theory with interaction $V = \int dx \frac{\delta m^2}{2} \phi(x)^2$
- $\to$ we define ω_G^V – for general polynomial V – perturbatively by prescribing "Feynman rules"
- consider data β_i, $i = 1, 2, 3$, χ_i, $i = 1, 2$, ψ used for defining ω_G in free theory as before (ω_{β_3} is contact state), $V = V(\psi h)$

Gluing Feynman rules

- consider Feynman graphs for $\omega_\beta^V(\mathscr{R}_V(A_1) \star \cdots \star \mathscr{R}_V(A_n))$, where β is considered as a dummy variable

- these graphs contain propagators of ω_β, and can be computed using *[Fredenhagen & Lindner 2014]*

$$\omega_\beta^V(A) = \sum_{n=0}^{\infty}(-1)^n \int_{\beta S_n} \omega_\beta^{\text{conn.}}\left(A \otimes \alpha_{iu_1}(K_V) \otimes \cdots \otimes \alpha_{iu_n}(K_V)\right) du_1 \dots du_n,$$

  where S_n is n-dim. unit simplex and α_{iu_i} is considered in the sense of analytic continuation of expectation values

- the graphs have external vertices from the A_i, internal vertices from the V in $\mathscr{R}_V(A_i)$, and internal vertices from $K_V \doteq \mathscr{R}_V(V(\dot{\psi}h))$

Gluing Feynman rules (cont.)

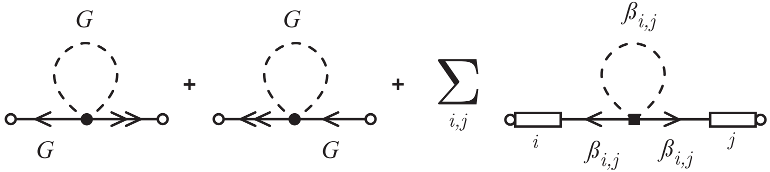
- for any connected subgraph γ a of such graph, consider all subgraphs of γ which are connected and contain only K_V-vertices, where all lines connected to these vertices are taken as part of such a subgraph, i.e. the external vertices of these subgraph are all from the $\mathscr{R}_V(A_i)$
- let $F_\gamma(x_1, \ldots, x_k)$ denote the product of the amplitudes of these subgraphs, where k is the total number of their external vertices
- decompose F_γ as

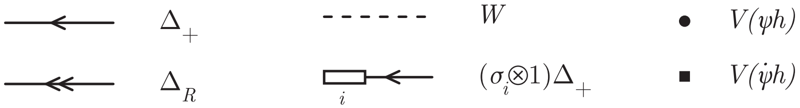
$$F_\gamma = (\sigma_1 + \sigma_2)^{\otimes k} F_\gamma = F_{\gamma,1} + F_{\gamma,2} + F_{\gamma,3}$$

$$F_{\gamma,i} \doteq \sigma_i^{\otimes^n} F_\gamma \quad i = 1, 2 \qquad F_{\gamma,3} \doteq F_\gamma - F_{\gamma,1} - F_{\gamma,2}$$

- replace all propagators in $F_{\gamma,i}$ by those of ω_{β_i}, replace all other propagators in γ by those of ω_G, perform the simplex integral for β_i (factorises for unconnected subgraphs) and sum over i

Gluing Feynman rules - example - 2PF at first order





$\omega_G^V\left(\mathscr{R}_V(\phi(x)) \star \mathscr{R}_V(\phi(y))\right)$ at 1st order, $\beta_{1,1} \doteq \beta_1$, $\beta_{2,2} \doteq \beta_2$, $\beta_{1,2} \doteq \beta_{2,1} \doteq \beta_3$

Properties and large time limit of ω_G^V

- ω_G^V is well-defined, positive in the sense of formal power series, equal to $\omega_{\beta_1}^V$ / $\omega_{\beta_2}^V$ on "left" / "right" observables
- $$\omega_N^V = \lim_{t\to\infty} \omega_G^V \circ \alpha_t^V$$
  exists (at first order), both with and without spatial adiabatic limit $h \to 1$, independent of ψ, χ_i, β_3, convergence is $O(t^{-1})$
- only apparent problem for all-order statement: closed expressions

I-NESS = KMS?

- ω_N^V is again separately KMS for left-/right-movers at β_1, β_2 (at first order)
- ω_N^V can not be KMS in a different rest-frame, because it is not at 0th order (ω_N)
- $\rightarrow$ proper thermalisation is (presumably, and unsurprisingly) a non-perturbative effect
- cross-check: production of "relative entropy" *[..., Jakšić & Pillet 2001-2002, Drago, Faldino & Pinamonti 2017]* between ω_N^V and ω_N is vanishing $\rightarrow$ ω_N^V and ω_N are "thermodynamically similar"

NESS are stable

- *[Drago, Faldino, Pinamonti 2017]* showed that KMS state ω_β^V is stable under spatially localised perturbations.
- ω_N and ω_N^V have the same property! proved at first order, expected to hold at all orders:
- $V = V(\psi h_V)$ with $h_V \to 1$, $W = W(h_W) = W(\psi h_W)$, $h_W \in C_0^\infty(\mathbb{R}^3)$

$$\lim_{t\to\infty} \omega_N^V \circ \alpha_t^{V+W(h_W)} = \omega_N^{V+W(h_W)}$$

$$\lim_{t\to\infty} \omega_N \circ \alpha_t^{W(h_W)} = \omega_N^{W(h_W)}$$

- interpretation: ω_N^V stable w.r.t. $\alpha_t^{\beta_1,\beta_2}$ and $[\alpha_t^{\beta_1,\beta_2}, \alpha_t^{V+W(h_W)}] = 0$.

Thanks a lot for your attention!