Large gauge transformations, asymptotic charges and soft-photon theorems

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"Mathematics of interacting QFT models"

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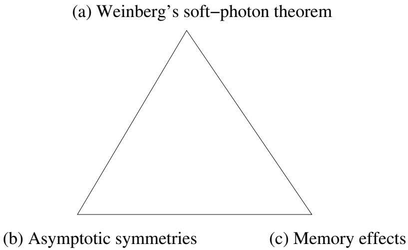
Outline

1. Large gauge transformations in external current QED.
2. Asymptotic charges and soft-photon theorems in classical Maxwell-Lorentz system.
3. Towards soft-photon theorems in non-relativistic QED

Strominger's infrared triangle



Soft-photon theorem:

$$\langle \mathrm{out}|a_{+}^{\mathrm{out}}(q)S|\mathrm{in}\rangle \simeq e\left[\sum_{k=1}^{m}\frac{Q_k^{\mathrm{out}}p_k^{\mathrm{out}}\cdot\varepsilon^{+}}{p_k^{\mathrm{out}}\cdot q}-\sum_{k=1}^{n}\frac{Q_k^{\mathrm{in}}p_k^{\mathrm{in}}\cdot\varepsilon^{+}}{p_k^{\mathrm{in}}\cdot q}\right]\langle \mathrm{out}|S|\mathrm{in}\rangle$$

as a Ward identity w.r.t. an asymptotic symmetry.

Large gauge transformations in external current QED

Asymptotic charges and LGT in this talk

1. By applying the Noether theorem to the global U(1) symmetry, we obtain the electric charge.
2. By applying the Noether theorem to the local U(1) symmetry, we obtain asymptotic charges, e.g.:

$$\Phi(\boldsymbol{n}) := \lim_{r\to\infty} r^2 \boldsymbol{n} \cdot \boldsymbol{E}(r\boldsymbol{n}), \quad \boldsymbol{n} \in S^2,$$

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3. Indeed, the LGT should change the asymptotic charges

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External current QED in Coulomb gauge

For $j = (j_0, 0)$ define:

$$A_{0,\mathrm{C}}(t,\boldsymbol{x}) := -\frac{1}{\Delta} j_0(\boldsymbol{x}),$$

$$\mathbf{A}_{\mathrm{C}}(t,\boldsymbol{x}) := \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \int \frac{d^3\boldsymbol{k}}{\sqrt{2|\boldsymbol{k}|}} \boldsymbol{\varepsilon}_\lambda(\boldsymbol{k}) \big(e^{i|\boldsymbol{k}|t - i\boldsymbol{k}\cdot\boldsymbol{x}} a^*_\lambda(\boldsymbol{k}) + \mathrm{h.c.} \big),$$

$$H_{\mathrm{C}} := H_{\mathrm{fr}} + \frac{1}{2} \int d^3\boldsymbol{x}\, A_{0,\mathrm{C}}(\boldsymbol{x}) j_0(\boldsymbol{x}),\ H_{\mathrm{fr}} := \sum_{\lambda=\pm} \int d^3\boldsymbol{k}\, |\boldsymbol{k}| a^*_\lambda(\boldsymbol{k}) a_\lambda(\boldsymbol{k}),$$

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Electromagnetic fields:

$$E_{\mathrm{C}} = E_{\mathrm{C},\perp} + E_{\mathrm{C},\parallel} = -\partial_t \mathbf{A}_{\mathrm{C}} + \nabla \frac{1}{\Delta} j_0(\boldsymbol{x})$$

$$B_{\mathrm{C}} = \nabla \times \mathbf{A}_{\mathrm{C}}$$

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Remark: The minimal coupling $A_\mu j^\mu$ is not manifest. But

$$\frac{\delta H_{\mathrm{C}}}{\delta j_0(\boldsymbol{x})} = A_{0,\mathrm{C}}(\boldsymbol{x}).$$

Changing gauge

1. $\boldsymbol{x} \mapsto \chi_\epsilon(\boldsymbol{x})$ operator valued distribution, $[\chi_\epsilon(\boldsymbol{x}), \chi_\epsilon(\boldsymbol{x}')] = 0$.
2. Def: $W_\epsilon := e^{-i\chi_\epsilon(j_0)}$, $\quad \chi_\epsilon(t, \boldsymbol{x}) := e^{itH_C}\chi_\epsilon(\boldsymbol{x})e^{-itH_C}$.
3. Gauge transformation of the potential:
$$\mathbf{A}_\epsilon(t, \boldsymbol{x}) := W_\epsilon(\mathbf{A}_C(t, \boldsymbol{x}) - \nabla\chi_\epsilon(t, \boldsymbol{x}))W_\epsilon^*,$$
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4. The resulting electromagnetic fields:
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5. $H_\epsilon := W_\epsilon H_{\mathrm{C}} W_\epsilon^*$ satisfies $\frac{\delta H_\epsilon}{\delta j_0(\boldsymbol{x})} = A_{0,\epsilon}(\boldsymbol{x})$.

Example: axial gauge

1. Recall $\mathbf{A}_\epsilon(\boldsymbol{x}) := W_\epsilon(\mathbf{A}_\mathrm{C}(\boldsymbol{x}) - \nabla\chi_\epsilon(\boldsymbol{x}))W_\epsilon^*$
2. Impose $\boldsymbol{e}\cdot\mathbf{A}_\epsilon(\boldsymbol{x}) \to 0$ for $\epsilon \to 0$

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3. Solve for χ_ϵ:

$$\chi_{\boldsymbol{e},\epsilon}(\boldsymbol{x}) = \frac{1}{\boldsymbol{e}\cdot\nabla - \epsilon}\boldsymbol{e}\cdot\mathbf{A}_\mathrm{C}(\boldsymbol{x}).$$

4. This gives $A^\mu_{\boldsymbol{e},\epsilon} = W_\epsilon A^\mu_{\mathrm{MSY},\epsilon} W_\epsilon^* + O(\epsilon)$, where

$$A^\mu_{\mathrm{MSY},\epsilon}(x) = \int_0^\infty ds\, e^{-\epsilon s} F^{\mu\nu}_\mathrm{C}(x+\boldsymbol{e}s)\varepsilon_\nu$$

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$$\boldsymbol{E}_{\boldsymbol{e},\epsilon}(\boldsymbol{x}) := \boldsymbol{E}_{\mathrm{C},\perp}(\boldsymbol{x}) + \frac{\boldsymbol{e}}{\boldsymbol{e}\cdot\nabla+\epsilon} j_0(\boldsymbol{x}),$$
$$\boldsymbol{B}_{\boldsymbol{e},\epsilon}(\boldsymbol{x}) := \boldsymbol{B}_{\mathrm{C}}(\boldsymbol{x}).$$

2. $A_{0,\boldsymbol{e},\epsilon}(t,\boldsymbol{x})$ and $H_{\boldsymbol{e},\epsilon}$ diverge as $\epsilon \to 0$.

3. One can improve the situation by MSY-type angular smearing:

$$\chi_{g,\epsilon}(\boldsymbol{x}) := \int d\Omega(\boldsymbol{e})\, g(\boldsymbol{e}) \frac{1}{\boldsymbol{e}\cdot\nabla-\epsilon} \boldsymbol{e}\cdot \mathbf{A}_{\mathrm{C}}(\boldsymbol{x})$$

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LGT from Coulomb to axial gauge

1. The LGT from Coulomb to axial is implemented by:

$$W_{\boldsymbol{e},\epsilon} = e^{-i\chi_\epsilon(j_0)} = \exp(-i\frac{1}{\boldsymbol{e}\cdot\nabla-\epsilon}\boldsymbol{e}\cdot\mathbf{A}_{\mathrm{C}}(j_0)) = \exp(i\int_0^\infty ds\, e^{-\epsilon s}(\boldsymbol{e}\cdot\mathbf{A}_{\mathrm{C}})(j_0)(s\boldsymbol{e})).$$

2. We can change regularisation from $\epsilon \to 0$ to $L \to \infty$:

$$W_{\boldsymbol{e},L} := \exp(i\int_0^L ds\,(\boldsymbol{e}\cdot\mathbf{A}_{\mathrm{C}})(j_0)(s\boldsymbol{e})).$$

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LGT between two different axial gauges

1. The LGT from $\boldsymbol{e}$ to $\boldsymbol{e}'$ is given by

$$W_{\boldsymbol{e}',\boldsymbol{e},L} := W_{\boldsymbol{e}',L} W^*_{\boldsymbol{e},L}$$
$$= \exp(i \int_0^L ds' \, (\boldsymbol{e}' \cdot \mathbf{A}_{\mathrm{C}})(j_0)(s'\boldsymbol{e}') + i \int_{-L}^0 ds \, ((-\boldsymbol{e}) \cdot \mathbf{A}_{\mathrm{C}})(j_0)(s(-\boldsymbol{e}))).$$

2. We can close the contour without effect on the limit $L \to \infty$

$$\widetilde{W}_{\boldsymbol{e}',\boldsymbol{e},L} := \exp(i \int_{\partial S_L} \mathbf{A}_{\mathrm{C}}(j_0)(\boldsymbol{r}) \cdot d\boldsymbol{r})$$
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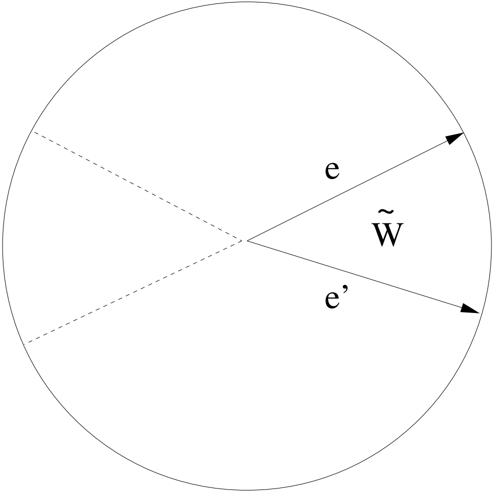
e
W̃
e'

Theorem (Wegener-WD)

For any fixed unit vectors $\boldsymbol{e} \neq \boldsymbol{e}'$ and all smearing functions $\boldsymbol{f}_{\mathrm{el}}, \boldsymbol{f}_{\mathrm{m}}$ we have

$$\lim_{L\to\infty} W_{\boldsymbol{e},L} e^{i(\boldsymbol{E}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{m}}))} W^*_{\boldsymbol{e},L} = e^{i(\boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{m}}))},$$

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$$W_{\boldsymbol{e},L} := \exp(i\int_0^L ds\,(\boldsymbol{e}\cdot\mathbf{A}_{\mathrm{C}})(j_0)(s\boldsymbol{e}))$$

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Theorem (Wegener-WD)

For any fixed unit vectors $\boldsymbol{e} \neq \boldsymbol{e}'$ *and all smearing functions* $\boldsymbol{f}_{\mathrm{el}}, \boldsymbol{f}_{\mathrm{m}}$ *we have*

$$\lim_{L\to\infty} W_{\boldsymbol{e},L} e^{i(\boldsymbol{E}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{m}}))} W^*_{\boldsymbol{e},L} = e^{i(\boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{m}}))},$$

$$\lim_{L\to\infty} \widetilde{W}_{\boldsymbol{e}',\boldsymbol{e},L} e^{i(\boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{m}}))} \widetilde{W}^*_{\boldsymbol{e}',\boldsymbol{e},L} = e^{i(\boldsymbol{E}_{\boldsymbol{e}'}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\boldsymbol{e}'}(\boldsymbol{f}_{\mathrm{m}}))}.$$

However, there are no unitaries U, $\tilde{U}$ *s.t.*

$$U e^{i(\boldsymbol{E}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\mathrm{C}}(\boldsymbol{f}_{\mathrm{m}}))} U^* = e^{i(\boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{el}})+\boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{f}_{\mathrm{m}}))},$$

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That is different gauges are not unitarily equivalent.

Spacelike flux of the electric field

1. Def. $\Phi(\boldsymbol{n}) := \lim_{r\to\infty} r^2 \boldsymbol{n}\cdot \boldsymbol{E}(r\boldsymbol{n}), \quad \boldsymbol{n}\in S^2$
2. Def. $\Phi_f(\boldsymbol{n}) := \lim_{r\to\infty} \int d^3\boldsymbol{x}\, f(\boldsymbol{x})\, r^2 \boldsymbol{n}\cdot \boldsymbol{E}((\boldsymbol{x}+\boldsymbol{n})r)$
3. Fact: In the Coulomb gauge

$$\Phi_f^{\mathrm{C}}(\boldsymbol{n}) = \int d^3\boldsymbol{x} f(\boldsymbol{x}) \frac{q\boldsymbol{n}\cdot(\boldsymbol{n}+\boldsymbol{x})}{4\pi|\boldsymbol{n}+\boldsymbol{x}|^2} \simeq \frac{q}{4\pi},$$

where $q := \int d^3\boldsymbol{x}\, j_0(\boldsymbol{x})$.

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Wrong. Exchanging $\lim_{r\to\infty}$ with $\int d\Omega(\boldsymbol{e})$ was not legitimate.

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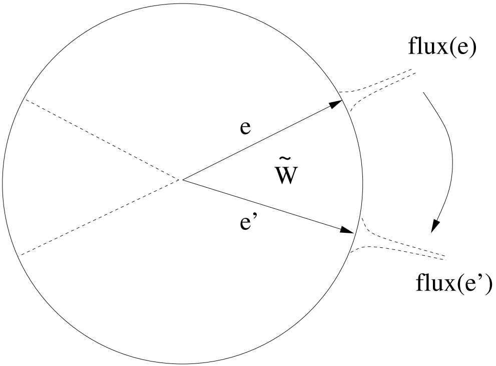
flux(e)
e
W̃
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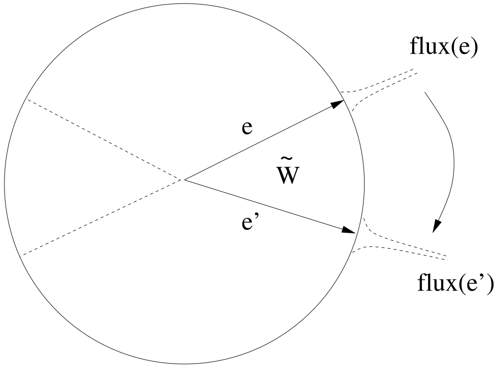

In DHR terms $\widetilde{W} = \exp(i \int \boldsymbol{B}(j_0)(\boldsymbol{r}) \cdot dS)$ is a 'flux carrying field'.

Asymptotic charges in classical Maxwell-Lorentz system

Classical Maxwell-Lorentz system

The classical Maxwell-Lorentz system:

$$\partial_t \boldsymbol{B}(t,\boldsymbol{x}) = -\nabla \times \boldsymbol{E}(t,\boldsymbol{x}), \quad \partial_t \boldsymbol{E}(t,\boldsymbol{x}) = \nabla \times \boldsymbol{B}(t,\boldsymbol{x}) - j(t,\boldsymbol{x}),$$
$$\nabla \cdot \boldsymbol{E}(t,\boldsymbol{x}) = \rho(t,\boldsymbol{x}), \qquad \nabla \cdot \boldsymbol{B}(t,\boldsymbol{x}) = 0,$$
$$\frac{d}{dt}\Big(m\gamma \dot{\boldsymbol{q}}(t)\Big) = e\big(\boldsymbol{E}_\varphi(t,\boldsymbol{q}(t)) + \dot{\boldsymbol{q}}(t) \times \boldsymbol{B}_\varphi(t,\boldsymbol{q}(t))\big),$$

1. Here $\rho(t,\boldsymbol{x}) := e\varphi(\boldsymbol{x}-\boldsymbol{q}(t)), \quad j(t,\boldsymbol{x}) := e\varphi(\boldsymbol{x}-\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t),$

$$\boldsymbol{E}_\varphi(t,\boldsymbol{q}(t)) := \int d^3\boldsymbol{x}\,\varphi(\boldsymbol{q}(t)-\boldsymbol{x})\boldsymbol{E}(t,\boldsymbol{x}),$$

2. Initial conditions: smooth fields s.t. for large $|\boldsymbol{x}|$

$$|\boldsymbol{E}(\boldsymbol{x})| + |\boldsymbol{B}(\boldsymbol{x})| + |\boldsymbol{x}|(|\nabla \boldsymbol{E}(\boldsymbol{x})| + |\nabla \boldsymbol{B}(\boldsymbol{x})|) \leq C(|\boldsymbol{x}|)^{-2}.$$

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Soliton solutions

1. By minimizing the total energy of the system at fixed total momentum, $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{v})$ and for $\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \rho(\boldsymbol{x})$, $\nabla \cdot \boldsymbol{B}(\boldsymbol{x}) = 0$:

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Scattering theory

Theorem (Imaikin, Komech, Kunze, Spohn 1997, 2002)

Let $t \mapsto (\boldsymbol{E}(t,\,\cdot\,), \boldsymbol{B}(t,\,\cdot\,), \boldsymbol{q}(t))$ *be a solution of the Maxwell-Lorentz system. Then, for* $|e|$ *sufficiently small,*

$$\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{E}_{\dot{\boldsymbol{q}}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)) + \boldsymbol{E}_{\mathrm{sc}}(t,\boldsymbol{x}) + R_1(t,\boldsymbol{x}),$$
$$\boldsymbol{B}(t,\boldsymbol{x}) = \boldsymbol{B}_{\dot{\boldsymbol{q}}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)) + \boldsymbol{B}_{\mathrm{sc}}(t,\boldsymbol{x}) + R_2(t,\boldsymbol{x}),$$

where

1. $(\boldsymbol{E}_{\mathrm{sc}}, \boldsymbol{B}_{\mathrm{sc}})$ *solve the free Maxwell equations. (Emited radiation).*
2. *The limit* $\boldsymbol{v}_{\infty} := \lim_{t\to\infty} \dot{\boldsymbol{q}}(t)$ *exists.*
3. $\lim_{t\to\infty} \int d^3\boldsymbol{x}\, |R(t,\boldsymbol{x})|^2 = 0.$

Asymptotic constants of motion

Theorem (Duc Viet Hoang, WD)

For a solution of the Maxwell-Lorentz system, the limit

$$\mathcal{E}(t, \boldsymbol{x}) := \lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^2 \boldsymbol{E}(t, \boldsymbol{x})$$

is independent of t and depends only on the direction $\hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$.

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\end{aligned}$$

The last equality can be considered a soft-photon theorem for this system.

Asymptotic constants of motion

Theorem (Duc Viet Hoang, WD)

For a solution of the Maxwell-Lorentz system, the limit

$$\hat{\boldsymbol{\mathcal{E}}}(t, \boldsymbol{k}) := \lim_{|\boldsymbol{k}| \to 0} |\boldsymbol{k}| \tilde{\boldsymbol{E}}(t, \boldsymbol{k})$$

is independent of t and depends only on the direction $\hat{\boldsymbol{k}} := \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$.

Remark 2: By scattering theory

$$\hat{\mathcal{E}}(0, \hat{\boldsymbol{k}}) = \hat{\mathcal{E}}_{\boldsymbol{v}_\infty}(\hat{\boldsymbol{k}}) + \hat{\mathcal{E}}_{\mathrm{sc}}(+\infty, \hat{\boldsymbol{k}}).$$

Choose purely longitudinal initial data, then

$$\lim_{|\boldsymbol{k}| \to 0} |\boldsymbol{k}| \tilde{\boldsymbol{E}}_{\mathrm{sc}}(+\infty, \boldsymbol{k}) = \hat{\mathcal{E}}_{\mathrm{sc}}(+\infty, \hat{\boldsymbol{k}}) = -P_{\mathrm{tr}} \hat{\mathcal{E}}_{\boldsymbol{v}_\infty}(\hat{\boldsymbol{k}}) = -ie \left(\frac{(P_{tr} \cdot \boldsymbol{v}_\infty)(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_\infty)}{1 - (\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_\infty)^2} \right.$$

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Classical origin of infrared problems

1. We found that $\tilde{\boldsymbol{E}}_{\mathrm{sc}}(\boldsymbol{k}) \sim -\frac{ie}{|\boldsymbol{k}|}\left(\frac{(P_{tr}\cdot\boldsymbol{v}_{\infty})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_{\infty})}{1-(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_{\infty})^2}\right)$ for small $|\boldsymbol{k}|$.
2. It is often stated, that 'such fields escape from Fock space'.
3. For $|g\rangle := e^{a^*(g)-a(g)}|0\rangle$ impose for small $\boldsymbol{k}$

$$\tilde{\boldsymbol{E}}_{\mathrm{sc}}(\boldsymbol{k}) = \langle g|\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k})|g\rangle.$$

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$$g(\boldsymbol{k}) = -\frac{e\tilde{\varphi}(\boldsymbol{k})}{\sqrt{2}|\boldsymbol{k}|^{3/2}}\frac{\boldsymbol{v}_{\infty}}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_{\infty}}.$$

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Towards soft-photon theorems in non-relativistic QED

Non-relativistic QED in Coulomb gauge

1. Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathcal{F}$.
2. Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} \otimes 1 - eA_{\perp,\varphi}(\boldsymbol{q}))^2 + \frac{1}{2}\int d^3\boldsymbol{x} \left\{ :(1 \otimes E_{\perp}(\boldsymbol{x}))^2: + :(1 \otimes \nabla_{\boldsymbol{x}} \times A_{\perp}(\boldsymbol{x}))^2: \right\}$$

Scattering theory

Rigorous scattering theory for the model was developed by Chen-Fröhlich-Pizzo [2010].

Can be related to Faddeev-Kulish approach [W.D. *Nucl. Phys. B*, 2017].

$$|\Psi_{\mathrm{el}}^{\mathrm{out}}\rangle = \lim_{t\to\infty} e^{iHt} e^{-iH_0 t} \int d^3\boldsymbol{p}\, U^{\mathrm{D}}_{\boldsymbol{p}}(t) \Psi_{\mathrm{el}}(\boldsymbol{p})(|\boldsymbol{p}\rangle \otimes |0\rangle).$$

The Dollard modifier $U^{\mathrm{D}}_{\boldsymbol{p}}(t)$ is constructed as follows:

1. $V = -e\frac{\boldsymbol{p}}{m} A_{\perp,\varphi}(\boldsymbol{q}) \quad \to \quad V^{as}(\tau) = -e \boldsymbol{v}_{\boldsymbol{p}} A_{\perp,\varphi}(v_{\boldsymbol{p}}\tau)$.
2. $V^{\mathrm{as},I}_{\boldsymbol{p}}(\tau) := e^{i\tau H_0} V^{as}(\tau) e^{-i\tau H_0} = -\int d^3\boldsymbol{x}\, A_{\perp}(\tau, \boldsymbol{x}) j_{\boldsymbol{v}_{\boldsymbol{p}}}(\tau, \boldsymbol{x})$.
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Scattering theory

1. Scattering states including 'hard' photons:

$$|\Psi^{\mathrm{out}}\rangle = a^*_{\mathrm{out}}(h_1)\dots a^*_{\mathrm{out}}(h_n)|\Psi^{\mathrm{out}}_{\mathrm{el}}\rangle,$$

   where

$$a^*_{\mathrm{out}}(h) := \lim_{t\to\infty} e^{itH} a^*(e^{-i|\boldsymbol{k}|t}h)e^{-itH}$$

2. S-matrix elements:

$$S_{i,j} = \langle \Psi^{\mathrm{out}}_i | \Psi^{\mathrm{in}}_j \rangle$$

3. The asymptotic velocity of the electron is defined via

$$\boldsymbol{v}_{\mathrm{out}} = \lim_{t\to\infty} e^{itH}\left(\frac{\boldsymbol{q}}{t}\right)e^{-itH} = \lim_{t\to\infty}\frac{\boldsymbol{q}(t)}{t}.$$

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Towards soft-photon theorems

Conjecture

Let $\rho \in C_0^\infty(\mathbb{R}^3)$ and $\rho^s(\boldsymbol{k}) := |\boldsymbol{k}|^{3/2} s^3 \rho(s\boldsymbol{k})$. Then the following holds true

$$\lim_{s\to\infty} \langle \Psi_i^{\mathrm{out}} | a_{\mathrm{out}}(\rho^s \boldsymbol{\varepsilon}_\lambda) \Psi_j^{\mathrm{in}} \rangle = -\frac{e}{\sqrt{2}} \int d^3\boldsymbol{k}\, \rho(\boldsymbol{k}) \langle \Psi_i^{\mathrm{out}} | \frac{\boldsymbol{v}^{\mathrm{out}} \cdot \boldsymbol{\varepsilon}_\lambda(\hat{\boldsymbol{k}})}{1 - \hat{\boldsymbol{k}} \cdot \boldsymbol{v}^{\mathrm{out}}} \Psi_j^{\mathrm{in}} \rangle$$

Since for coherent states $a_\lambda(\boldsymbol{k})|g\rangle = (g(\boldsymbol{k}) \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}}))|g\rangle$, we expect

$$\lim_{s\to\infty} \langle \Psi_i^{\mathrm{out}} | a_{\mathrm{in}}(\rho^s \varepsilon_\lambda) \Psi_j^{\mathrm{in}} \rangle = -\frac{e}{\sqrt{2}} \int d^3\boldsymbol{k}\, \rho(\boldsymbol{k}) \langle \Psi_i^{\mathrm{out}} | \frac{\boldsymbol{v}^{\mathrm{in}} \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}})}{1 - \hat{\boldsymbol{k}} \cdot \boldsymbol{v}^{\mathrm{in}}} \Psi_j^{\mathrm{in}} \rangle$$

Now it suffices to show, that

$$\lim_{s\to\infty} \langle \Psi_i^{\mathrm{out}} | \big(a_{\mathrm{out}}(\rho^s \varepsilon_\lambda) - a_{\mathrm{in}}(\rho^s \varepsilon_\lambda)\big) \Psi_j^{\mathrm{in}} \rangle$$

$$= -\frac{e}{\sqrt{2}} \int d^3\boldsymbol{k}\, \rho(\boldsymbol{k}) \langle \Psi_i^{\mathrm{out}} | \frac{\boldsymbol{v}^{\mathrm{out}} \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}})}{1 - \hat{\boldsymbol{k}} \cdot \boldsymbol{v}^{\mathrm{out}}} - \frac{\boldsymbol{v}^{\mathrm{in}} \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}})}{1 - \hat{\boldsymbol{k}} \cdot \boldsymbol{v}^{\mathrm{in}}} \Psi_j^{\mathrm{in}} \rangle.$$

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Weinberg's soft-photon theorem:

$$\langle \mathrm{out} | a_+^{\mathrm{out}}(\boldsymbol{q}) S | \mathrm{in} \rangle \simeq e \left[\sum_{k=1}^{m} \frac{Q_k^{\mathrm{out}} p_k^{\mathrm{out}} \cdot \varepsilon^+}{p_k^{\mathrm{out}} \cdot q} - \sum_{k=1}^{n} \frac{Q_k^{\mathrm{in}} p_k^{\mathrm{in}} \cdot \varepsilon^+}{p_k^{\mathrm{in}} \cdot q} \right] \langle \mathrm{out} | S | \mathrm{in} \rangle$$

Since for coherent states $a_\lambda(\boldsymbol{k})|g\rangle = (g(\boldsymbol{k}) \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}}))|g\rangle$, we expect

$$\lim_{s\to\infty} \langle \Psi_i^{\mathrm{out}} | a_{\mathrm{in}}(\rho^s \varepsilon_\lambda) \Psi_j^{\mathrm{in}} \rangle = -\frac{e}{\sqrt{2}} \int d^3\boldsymbol{k}\, \rho(\boldsymbol{k}) \langle \Psi_i^{\mathrm{out}} | \frac{\boldsymbol{v}^{\mathrm{in}} \cdot \varepsilon_\lambda(\hat{\boldsymbol{k}})}{1 - \hat{\boldsymbol{k}} \cdot \boldsymbol{v}^{\mathrm{in}}} \Psi_j^{\mathrm{in}} \rangle$$

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Towards soft-photon theorems

Conjecture

Let $\rho \in C_0^\infty(\mathbb{R}^3)$ and $\rho^s(\boldsymbol{k}) := |\boldsymbol{k}|^{3/2} s^3 \rho(s\boldsymbol{k})$. Then the following holds true

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Towards soft-photon theorems

We compute:

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&\big(a_{\rm out}(\rho^s\varepsilon_\lambda)-a_{\rm in}(\rho^s\varepsilon_\lambda)\big)\\
&=\int_{-\infty}^{\infty}d\tau\,\partial_\tau\Big(e^{iH\tau}a(e^{-i|\boldsymbol{k}|\tau}\rho^s\varepsilon_\lambda)e^{-iH\tau}\Big)\\
&=\int_{-\infty}^{\infty}d\tau\,\Big(e^{iH\tau}i[V,a(e^{-i|\boldsymbol{k}|\tau}\rho^s\varepsilon_\lambda)]e^{-iH\tau}\Big)
\end{aligned}
$$

Considering $-e\frac{\boldsymbol{p}}{m}\cdot A_{\perp,\varphi}(\boldsymbol{q})\in V$, this gives

$$
\frac{ie}{\sqrt{2}}\int_{-\infty}^{\infty}d\tau'\,\frac{\boldsymbol{p}(\tau' s)}{m}\cdot\int d^3\boldsymbol{k}\,\varepsilon_\lambda(\hat{\boldsymbol{k}})|\boldsymbol{k}|\tilde{\varphi}(\boldsymbol{k}/s)\rho(\boldsymbol{k})e^{i|\boldsymbol{k}|\tau'(1-\hat{\boldsymbol{k}}\cdot(\boldsymbol{q}(\tau' s)/(\tau' s)))}.
$$

By first taking $s\to\infty$ and then integrating over τ', we get

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-\frac{e}{\sqrt{2}}\int d^3\boldsymbol{k}\,\rho(\boldsymbol{k})\left(\frac{\boldsymbol{v}^{\rm out}\cdot\varepsilon_\lambda(\hat{\boldsymbol{k}})}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{v}^{\rm out}}-\frac{\boldsymbol{v}^{\rm in}\cdot\varepsilon_\lambda(\hat{\boldsymbol{k}})}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{v}^{\rm in}}\right).
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Open problems

1. We have shown that different gauges are unitarily inequivalent. Can we show that this physically doesn't matter?
    1. Are they equivalent in front of an infravacuum?
    2. Are they equivalent after restriction to a lightcone?
2. A DHR-type description of a family of different gauges.
3. A complete proof of soft-photon theorems and their relation to asymptotic symmetries.

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