Curing the infrared problem in nonrelativistic QED

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York
1 July 2019

Problem:

System of nonrelativistic QED:
one "slow" spinless electron interacting with a cloud of photons

- Algebra of observables of the system electron + photons
- Coherent states $\omega_{\boldsymbol{P}}$ (ground states of an Hamiltonian $H_{\boldsymbol{P}}$, $\boldsymbol{P}$ a total momentum of the system)

  $\Rightarrow$ they induce inequivalent representations of the algebra

problem of velocity superselection

Consequences:

- states of single electrons with different momenta $\boldsymbol{P}$ cannot be coherently superimposed
- electron is an infraparticle (no definite mass)
- scattering theory of many electrons seems problematic

The model

- Hamiltonian:

$$H = \frac{1}{2}(-i\nabla_{\boldsymbol{x}} + \tilde{\alpha}^{1/2}\boldsymbol{A}(\boldsymbol{x}))^2 + H_{\text{photon}}$$

  selfadjoint on dense domain in $\mathcal{H} = \mathcal{H}_{\text{electron}} \otimes \mathcal{F}_{\text{photon}}$,
  $\boldsymbol{A}$ in Coulomb gauge with UV cutoff.
- total momentum $\boldsymbol{P} := -i\boldsymbol{\nabla}_{\boldsymbol{x}} + \boldsymbol{P}_{\text{photon}}$, $[H, \boldsymbol{P}] = 0$

$$\Rightarrow \quad H = \Pi^*\Big(\int^{\oplus} H_{\boldsymbol{P}}\, d^3\boldsymbol{P}\Big)\Pi, \quad \Pi \text{ a unitary identification}$$

Ground states of the Hamiltonians $H_{\boldsymbol{P}}$

- Absence of ground states:
  - $H_{\boldsymbol{P}}$ do not have ground states (eigenvectors) for $\boldsymbol{P} \neq 0$ at least for small $\tilde{\alpha}$ and for $\boldsymbol{P} \in S = \{\boldsymbol{P} \in \mathbb{R}^3 \ : \ |\boldsymbol{P}| < \frac{1}{3}\}$.
  - This is a feature of the infraparticle problem
- Introduce an infrared cutoff:

$$H_{\boldsymbol{P},\sigma} := \frac{1}{2}(\boldsymbol{P} - \boldsymbol{P}_{\text{photon}} + \tilde{\alpha}^{1/2}\boldsymbol{A}_{[\sigma,\kappa]}(0))^2 + H_{\text{photon}}$$

selfadjoint on dense domain in Fock space $\mathcal{F}$ over $L^2_{\text{tr}}(\mathbb{R}^3;\mathbb{C}^3)$; denote creators/annihilators as $a^*_\lambda(\boldsymbol{k})$, $a_\lambda(\boldsymbol{k})$.

$$A_{[\sigma,\kappa]}(\boldsymbol{x}) = \sum_{\lambda=\pm}\int \frac{d^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\chi_{[\sigma,\kappa]}(|\boldsymbol{k}|)\boldsymbol{\epsilon}_\lambda(\boldsymbol{k})\big(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}a^*_\lambda(\boldsymbol{k})+e^{i\boldsymbol{k}\cdot\boldsymbol{x}}a_\lambda(\boldsymbol{k})\big)$$

(κ: UV cutoff, σ: IR cutoff)

Ground states with IR cutoff

- Fact: For any $\sigma > 0$, the operator $H_{\boldsymbol{P},\sigma}$ has a ground states (eigenvector) $\Psi_{\boldsymbol{P},\sigma} \in \mathcal{F}$ with isolated eigenvalues $E_{\boldsymbol{P},\sigma}$.
  - $\Psi_{\boldsymbol{P},\sigma}$ tend weakly to zero as $\sigma \to 0$.
- Hence ground states exist at fixed cutoff.
- However, we will need to remove the cutoff to describe the physical system.
- This will be done by considering suitable states on a CCR algebra.

Velocity superselection

Algebra of observables of the system “electron + photon cloud”:

- Weyl (CCR) algebra $\mathfrak{A}$ generated (up to closure in C^*-norm) by $W(\boldsymbol{f})$, $\boldsymbol{f} \in \mathcal{L} := \bigcup_{\epsilon>0} L^2_{\mathsf{tr},\epsilon}(\mathbb{R}^3;\mathbb{C}^3)$, symplectic form $\sigma(\cdot,\cdot) := \mathrm{Im}\langle\cdot,\cdot\rangle$.
- Vacuum representation: $\pi_{\mathsf{vac}}(W(\boldsymbol{f})) = e^{a^*(\boldsymbol{f})-a(\boldsymbol{f})}$

State: Given any $A \in \mathfrak{A}$, define

$$\omega_{\boldsymbol{P}}(A) := \lim_{\sigma\to 0}\langle\Psi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}(A)\Psi_{\boldsymbol{P},\sigma}\rangle$$

- state on $\mathfrak{A}$, describes plane-wave configurations of the electron with velocity $\boldsymbol{P}$

Representations: States $\omega_{\boldsymbol{P}}$ have irreducible GNS representations $\pi_{\boldsymbol{P}}$.

- Fact: $\pi_{\boldsymbol{P}} \not\simeq \pi_{\boldsymbol{P}'}$ for any $\boldsymbol{P} \neq \boldsymbol{P}'$ “velocity superselection”

Cause of the superselection problem

Analyze the phenomenon closely:

- introduce auxiliary vectors $\Phi_{\boldsymbol{P},\sigma} = W(\boldsymbol{v}_{\boldsymbol{P},\sigma})\Psi_{\boldsymbol{P},\sigma}$, where

$$\boldsymbol{v}_{\boldsymbol{P},\sigma}(\boldsymbol{k}) := \tilde{\alpha}^{1/2} P_{\mathsf{tr}} \frac{\chi_{[\sigma,\kappa]}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^{3/2}} \frac{\nabla E_{\boldsymbol{P},\sigma}}{1 - \hat{\boldsymbol{k}} \cdot \nabla E_{\boldsymbol{P},\sigma}}.$$

  Fact: $\Phi_{\boldsymbol{P}} := \lim_{\sigma\to 0} \Phi_{\boldsymbol{P},\sigma}$ exists in norm for suitable choice of the phases of $\Psi_{\boldsymbol{P},\sigma}$.

$$\omega_{\boldsymbol{P}}(W(\boldsymbol{f})) = \lim_{\sigma\to 0} \langle \Phi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}\big(\underbrace{W(\boldsymbol{v}_{\boldsymbol{P},\sigma})W(\boldsymbol{f})W(\boldsymbol{v}_{\boldsymbol{P},\sigma})^*}_{:=\alpha_{\boldsymbol{v}_{\boldsymbol{P},\sigma}}(W(\boldsymbol{f}))}\big)\Phi_{\boldsymbol{P},\sigma}\rangle$$

$$\alpha_{\boldsymbol{v}_{\boldsymbol{P},\sigma}}(W(\boldsymbol{f})) = e^{-2i\,\mathrm{Im}\langle \boldsymbol{v}_{\boldsymbol{P},\sigma}, \boldsymbol{f}\rangle} W(\boldsymbol{f})$$

- For $\sigma > 0$, we have $\pi_{\boldsymbol{P},\sigma} \simeq \pi_{\mathsf{vac}}$, but $\pi_{\boldsymbol{P}} \not\simeq \pi_{\mathsf{vac}}$

A possible solution: regularize the map $\alpha_{\boldsymbol{v}_{\boldsymbol{P},\sigma}}$ $\Rightarrow$ Infravacuum state

Infravacuum state

Walter Kunhardt: DHR theory for the free massless scalar field

- automorphisms γ of the algebra of the free massless scalar field: similar structure to $\alpha_{v_{\boldsymbol{P}}}$
- γ have poor localization property in front of the vacuum:

$$\pi_{\mathsf{vac}} \circ \gamma\big|_{\mathfrak{A}(\mathcal{O}')} \not\simeq \pi_{\mathsf{vac}}\big|_{\mathfrak{A}(\mathcal{O}')}, \quad \pi_{\mathsf{vac}} \circ \gamma\big|_{\mathfrak{A}(\mathcal{C}')} \not\simeq \pi_{\mathsf{vac}}\big|_{\mathfrak{A}(\mathcal{C}')}$$

  ($\mathcal{O}$ a double cone, $\mathcal{C}$ a spacelike cone)
- improve the localization property: infravacuum state

$$\omega_T(W(f)) = e^{-\frac{1}{4}\|Tf\|^2}$$

- Fact: $\pi_T \circ \gamma\big|_{\mathfrak{A}(\mathcal{C}')} \simeq \pi_T\big|_{\mathfrak{A}(\mathcal{C}')}$
- automorphism of the algebra α_T: $\alpha_T(W(f)) = W(Tf)$

The symplectic map T

- Recall $\mathcal{L} := \bigcup_{\epsilon>0} L^2_{\mathsf{tr},\epsilon}(\mathbb{R}^3;\mathbb{C}^3)$
- $T : \mathcal{L} \to \mathcal{L}, \quad T = T_1 \frac{1+J}{2} + T_2 \frac{1-J}{2}$

$$T_1 := \mathbf{1} + \text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} (b_i - 1)\boldsymbol{Q}_i, \quad T_2 := \mathbf{1} + \text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \big(\frac{1}{b_i} - 1\big)\boldsymbol{Q}_i$$

  - $\boldsymbol{Q}_i$ orthogonal projectors on $L^2_{\mathsf{tr}}(\mathbb{R}^3;\mathbb{C}^3)$, $\sum_i \boldsymbol{Q}_i = 1$
  - i large means "low energy"
  - b_i decay with i large
- T modify the low energy behaviour of wave functions in $\mathcal{L}$, and in particular of $\boldsymbol{v}_{\boldsymbol{P},\sigma}$, in such a way that $\lim_{\sigma\to 0} T\boldsymbol{v}_{\boldsymbol{P},\sigma} \in L^2_{\mathsf{tr}}(\mathbb{R}^3;\mathbb{C}^3)$.

Infravacuum state

- Idea: Instead of $\omega_{\boldsymbol{P}}$, consider a modified state $\omega_{\boldsymbol{P},T}$ defined by

$$\begin{aligned}\omega_{\boldsymbol{P},T}(A) &:= \lim_{\sigma\to 0}\langle \Phi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}\big(\alpha_T(\alpha_{\boldsymbol{v}_{\boldsymbol{P},\sigma}}(A))\big)\Phi_{\boldsymbol{P},\sigma}\rangle \\ &= \lim_{\sigma\to 0}\langle \Phi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}\big(\alpha_T(W(\boldsymbol{v}_{\boldsymbol{P},\sigma})AW(\boldsymbol{v}_{\boldsymbol{P},\sigma})^*)\big)\Phi_{\boldsymbol{P},\sigma}\rangle \\ &= \lim_{\sigma\to 0}\langle \Phi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}\big(W(T\boldsymbol{v}_{\boldsymbol{P},\sigma})\alpha_T(A)W(T\boldsymbol{v}_{\boldsymbol{P},\sigma})^*\big)\Phi_{\boldsymbol{P},\sigma}\rangle\end{aligned}$$

- Fact: $\lim_{\sigma\to 0} T\boldsymbol{v}_{\boldsymbol{P},\sigma} := T\boldsymbol{v}_{\boldsymbol{P}} \in L^2_{\mathsf{tr}}(\mathbb{R}^3;\mathbb{C}^3)$
- Result: $\pi_{\boldsymbol{P},T} \simeq \pi_{\boldsymbol{P}',T}$ for any $\boldsymbol{P} \neq \boldsymbol{P}'$.

Restriction to the light cone

Alternative approach:

- Arbitrariness in the choice of the algebra $\mathfrak{A}$ as long as it acts irreducibly on $\mathcal{F}$ and the states $\omega_{\boldsymbol{P}}$ are well-defined.
- choose $\mathfrak{A}$ to be the algebra of observables of the free electromagnetic field $\quad\rightarrow$ local and relativistic

$$\mathfrak{A}(\mathcal{O}) :=$$
$$C^*\{e^{i(\boldsymbol{E}(\boldsymbol{f}_e)+\boldsymbol{B}(\boldsymbol{f}_b))} \mid \operatorname{supp}\boldsymbol{f}_e, \operatorname{supp}\boldsymbol{f}_b \subset \mathcal{O}, \boldsymbol{f}_{e,b} \in \mathcal{D}(\mathbb{R}^4, \mathbb{R}^3)\}$$

- Result: if $\mathfrak{A} := \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^4} \mathfrak{A}(\mathcal{O})}$ (quasi-local algebra), then $\pi_{\boldsymbol{P}} \not\simeq \pi_{\boldsymbol{P}'}$, but with $\mathfrak{A}(V_+) := \overline{\bigcup_{\mathcal{O}\subset V_+} \mathfrak{A}(\mathcal{O})}$,

$$\pi_{\boldsymbol{P}}\big|_{\mathfrak{A}(V_+)} \simeq \pi_{\boldsymbol{P}'}\big|_{\mathfrak{A}(V_+)} \text{ for any } \boldsymbol{P}, \boldsymbol{P}' \in S$$

(V_+: forward light cone)

Restriction to the light cone

Recall that

$$\omega_{\boldsymbol{P}}(A) = \lim_{\sigma \to 0} \langle \Phi_{\boldsymbol{P},\sigma}, \pi_{\mathsf{vac}}\big(W(\boldsymbol{v}_{\boldsymbol{P},\sigma}) A W(\boldsymbol{v}_{\boldsymbol{P},\sigma})^*\big) \Phi_{\boldsymbol{P},\sigma} \rangle$$

Idea of proof:

- Use Huygens principle: $\mathfrak{A}(V_-) \subset \mathfrak{A}(V_+)'$
- Approximate $\boldsymbol{v}_{\boldsymbol{P},\sigma}$ with functions in the symplectic space of the backward light cone V_-.
- Then $W(\boldsymbol{v}_{\boldsymbol{P},\sigma})$ and $A \in \mathfrak{A}(V_+)$ approximately commute, hence $\omega_{\boldsymbol{P}}$ lives in the vacuum representation.
- Hence $\pi_{\boldsymbol{P}}\big|_{\mathfrak{A}(V_+)} \simeq \pi_{\mathsf{vac}} \simeq \pi_{\boldsymbol{P}'}\big|_{\mathfrak{A}(V_+)}$ for any $\boldsymbol{P}, \boldsymbol{P}' \in S$.

Restriction to the light cone

Local approximation of $\boldsymbol{v_P}$:

- The symplectic space for a double cone $O_r + \tau e_0$ is:

$$e^{i|\boldsymbol{k}|\tau}\mathcal{L}_{BJ}(O_r) :=$$
$$e^{i|\boldsymbol{k}|\tau}(1+J)\overline{|\boldsymbol{k}|^{-1/2}(i\boldsymbol{k}\times\tilde{\mathcal{D}}(O_r;\mathbb{R}^3))}+(1-J)\overline{|\boldsymbol{k}|^{1/2}P_{\text{tr}}\tilde{\mathcal{D}}(O_r;\mathbb{R}^3)}$$

- Local approximant: Let $g:\mathbb{R}^3\to\mathbb{R}$ be smooth and compactly supported, $\tilde{g}(\mathbf{0})=1$, consider

$$\hat{\boldsymbol{v}}_{\boldsymbol{P}}(\boldsymbol{k}) := \tilde{\alpha}^{1/2}P_{\text{tr}}\frac{\tilde{g}(\boldsymbol{k})e^{-iu|\boldsymbol{k}|}\nabla E_{\boldsymbol{P}}}{|\boldsymbol{k}|^{3/2}(1-\nabla E_{\boldsymbol{P}}\cdot\hat{\boldsymbol{k}})}$$

- Hence, local approximant for $W(-i\boldsymbol{v_P})$ is

$$W(-i\hat{\boldsymbol{v}}_{\boldsymbol{P},T}) =$$
$$\exp\left(-i\tilde{\alpha}^{1/2}\int_0^T dt\int_t^T d\tau\frac{1}{(2\pi)^{3/2}}\nabla E_{\boldsymbol{P}}\cdot\boldsymbol{E}(g)(-u-\tau,-\nabla E_{\boldsymbol{P}}t)\right)$$

Conclusions and Outlook

- We have investigated the problem of velocity superselection in a non-relativistic QED model.
- Our resolution of this problem rely on two possible methods:
  - infravacuum state
  - restriction of the algebra to the forward light-cone
- It would be interesting to investigate the problem of scattering theory of many electrons in these two approaches.
- A method like Haag-Ruelle scattering theory could be applied:
  - In the first approach one needs to make sense of Haag-Ruelle scattering theory with respect to the infravacuum background state
  - In the second approach only either an outgoing or an incoming particle is available at the same time