Mathematical structures underlying the infravacuum picture of the electron

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Wuppertal, 22.06.2018

Sectors

1. $\mathfrak{A}$ - C^*-algebra.
2. $P_{\mathfrak{A}}$ - pure states.
3. $\mathrm{In}\,\mathfrak{A} \subset \mathrm{Aut}\,\mathfrak{A}$ - inner automorphisms.
4. $X := P_{\mathfrak{A}}/\mathrm{In}\,\mathfrak{A}$ - sectors.

Infrared problem: Uncountable families of physically indistinguishable sectors.

Strategy: Form equivalence classes of sectors ('charge classes') [Buchholz 82, Buchholz-Roberts 14]

Question: Can this be done without locality?

(Second) conjugate classes

1. $G \subset \mathrm{Aut}\,\mathfrak{A}$.
2. $X \times G \ni (x,g) \mapsto x \cdot g \in X$ - group action on X.

Definition

1. Fix a 'vacuum' $x_0 \in X$ and 'background' $a \in G$.
2. For $x \in X$ set $G^a_{x,x_0} := \{\, g \in G \,|\, x = x_0 \cdot a \cdot g \,\}$.
3. $\overline{[x]}^a := \{\, x_0 \cdot a \cdot g^{-1} \,|\, g \in G^a_{x,x_0} \,\}$ is called the conjugate class.
4. $\overline{\overline{[x]}}^a := \{\, x_0 \cdot a \cdot (g')^{-1} \,|\, g' \in G^a_{y,x_0},\, y \in \overline{[x]}^a \,\}$ is called the second conjugate class.

Claim: second conjugate classes are meaningful candidates for 'charge classes' in the absence of locality.

(Second) conjugate classes as orbits

- Def. $[x]_H := \{\, x \cdot h \,|\, h \in H \,\}$ denotes the orbit.
- Def. $G_x := \{\, g \in G \,|\, x \cdot g = x \,\}$ denotes the stabilizer group.

Proposition

For $x = x_0 \cdot g_x$ we have

$$\overline{[x]}^{a} = [x_0 \cdot a \cdot g_x^{-1} \cdot a]_{G_{x_0 \cdot a}} \quad \text{and} \quad \overline{\overline{[x]}}^{a} = [x]_{G_{x_0 \cdot a}}.$$

Background is important: For trivial background e.g. $a := e$ second conjugate classes are sensitive to any ‘perturbation’ of the vacuum:

$$x_0 \neq x_0 \cdot g \quad \Leftrightarrow \quad \overline{\overline{[x_0]}}^{e} \neq \overline{\overline{[x_0 \cdot g]}}^{e}.$$

Main general result

Theorem (Cadamuro-W.D. 18)

Let $R \subset S \subset G$ be subgroups. Suppose that

1. $x_0 \cdot r = x_0$ *for all* $r \in R$.
2. $x_0 \cdot s \neq x_0$ *may hold for some* $s \in S$.
3. $a \cdot S \cdot a^{-1} \subset R$.

Then, $\overline{[x_0 \cdot s]}^a = \overline{[x_0]}^a$ *and* $\overline{\overline{[x_0 \cdot s]}}^a = \overline{\overline{[x_0]}}^a$ *for all* $s \in S$.

Definition

The relative normalizer of $R \subset S \subset G$ is defined as

$$N_G(R, S) := \{\, g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \,\}.$$

Geometric meaning of relative normalizer

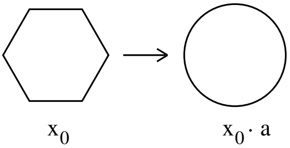
1. Let $R \subset S \subset G$ and $N_G(R,S) := \{\, g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \,\}$.
2. Let $X \times G \ni (x,g) \mapsto x \cdot g \in X$ be a group action on a set X.

Lemma

Suppose that $x_0 \in X$ and $a \in N_G(R,S)$. Then

$$x_0 \cdot R = x_0 \quad \Rightarrow \quad (x_0 \cdot a) \cdot S = (x_0 \cdot a).$$

Proof. $(x_0 \cdot a) \cdot s = x_0 \cdot \underbrace{(a \cdot s \cdot a^{-1})}_{r} \cdot a = x_0 \cdot a.$ $\square$



Existence of relative normalizers for $R \subsetneq S$

1. 'Tension': $R \subsetneq S$ vs $N_G(R,S) := \{\, g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \,\}$.
2. Hence relative normalizers are empty for
   - abelian groups,
   - finite groups,
   - finite-dimensional Lie groups (under some assumptions).
3. However, we show that $\mathrm{ISp}(\mathcal{L})$ over an infinite dim. space $\mathcal{L}$ admits non-empty relative normalizers.
4. Their elements are Kraus-Polley-Reents symplectic maps $\hat{T}$, known as infravacua.
5. Also the resulting Bogolubov transformations $\alpha_{\hat{T}}$ are elements of relative normalizers in $\mathrm{Aut}(\mathfrak{A})$, where $\mathfrak{A} = \mathrm{CCR}(\mathcal{L})$.

Inhomogeneous symplectic group

Symplectic group:

1. $\mathfrak{h} := L^2_{\mathrm{tr}}(\mathbb{R}^3; \mathbb{C}^3)$ - single-photon space.
2. $\mathfrak{h}_\varepsilon := \{\mathsf{f} \in \mathfrak{h} \,|\, \mathsf{f}(\mathsf{k}) = 0 \text{ for } |\mathsf{k}| \leq \varepsilon\}$.
3. $\mathcal{L} := \bigcup_{\varepsilon > 0} \mathfrak{h}_\varepsilon$ symplectic space with $\boldsymbol{\sigma}(\,\cdot\,,\cdot\,) = \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle$.
4. $\mathrm{Sp}(\mathcal{L}) := \{\, T \in \mathrm{GL}(\mathcal{L}) \,|\, \boldsymbol{\sigma}(T\mathsf{f}_1, T\mathsf{f}_2) = \boldsymbol{\sigma}(\mathsf{f}_1, \mathsf{f}_2), \quad \mathsf{f}_1, \mathsf{f}_2 \in \mathcal{L}\,\}$.

Inhomogeneous symplectic group:

1. $\mathcal{L}^*$ - algebraic dual. We write $\boldsymbol{v}(\mathsf{f}) = (\boldsymbol{v}, \mathsf{f})$ for $\boldsymbol{v} \in \mathcal{L}^*$, $\mathsf{f} \in \mathcal{L}$.
2. For $T : \mathcal{L} \to \mathcal{L}$ we have the transposition $T^t : \mathcal{L}^* \to \mathcal{L}^*$.
3. $\mathrm{ISp}(\mathcal{L}) := \mathcal{L}^* \rtimes_\varphi \mathrm{Sp}(\mathcal{L})$, where $\varphi(T) := (T^{-1})^t$.

Relative normalizers in $\mathrm{ISp}(\mathcal{L})$

1. $\mathrm{ISp}(\mathcal{L}) := \mathcal{L}^* \rtimes_\varphi \mathrm{Sp}(\mathcal{L})$, where $\varphi(T) := (T^{-1})^t$.
2. We define $\mathcal{L}_R^* \subset \mathcal{L}_S^* \subset \mathrm{ISp}(\mathcal{L})$ as follows:

$$\mathcal{L}_R^* := L^2_{\mathrm{tr}}(\mathbb{R}^3; \mathbb{C}^3)_{\mathbb{R}},$$

$$\mathcal{L}_S^* := \mathcal{L}_R^* + \mathrm{Span}_{\mathbb{R}}\{\, \boldsymbol{v_P} \,|\, |\boldsymbol{P}| \leq \boldsymbol{P}_{\max} \,\},$$

$$\boldsymbol{v_P}(\mathbf{k}) := \Big(\frac{\tilde{\alpha}}{2(2\pi)^3}\Big)^{1/2} P_{\mathrm{tr}} \frac{\chi_{[0,\kappa]}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^{3/2}} \frac{\nabla E_{\boldsymbol{P}}}{1 - \hat{\boldsymbol{k}} \cdot \nabla E_{\boldsymbol{P}}},$$

where $\boldsymbol{P} \mapsto E_{\boldsymbol{P}}$ is the 'dispersion relation of the electron'.

Proposition

$T \in \mathrm{Sp}(\mathcal{L})$ belongs to $N_{\mathrm{ISp}(\mathcal{L})}(\mathcal{L}^*_{\mathrm{R}}, \mathcal{L}^*_{\mathrm{S}})$ iff $(T^{-1})^t\, \mathcal{L}^*_{\mathrm{S}} \subset \mathcal{L}^*_{\mathrm{R}}$.
Kraus-Polley-Reents infravacuum maps $\hat{T}$ satisfy this condition.

Kraus-Polley-Reents infravacuum map

1. Let $\varepsilon_i := 2^{-(i-1)}\kappa$ and $b_i := \frac{1}{i}$ for $i \in \mathbb{N}$.
2. Let $\xi_i(|\mathbf{k}|) = \frac{\chi_{[\varepsilon_{i+1},\varepsilon_i]}(|\mathbf{k}|)}{|\mathbf{k}|^{3/2}} \in L^2(\mathbb{R}_+, |\mathbf{k}|^2 d|\mathbf{k}|)$.
3. Define orthogonal projections on $\mathfrak{h} = L^2_{\mathrm{tr}}(\mathbb{R}^3; \mathbb{C}^3)$:

$$\boldsymbol{Q}_i := \frac{|\xi_i\rangle\langle\xi_i|}{\langle\xi_i|\xi_i\rangle} \otimes \sum_{0\le\ell\le i} \sum_{m=-\ell}^{\ell} \sum_{\lambda=\pm} |\boldsymbol{Y}_{\ell m\lambda}\rangle\langle\boldsymbol{Y}_{\ell m\lambda}|$$

4. Set $\hat{T}\mathbf{f} = \hat{T}_1(\mathrm{Re}\,\mathbf{f}) + i\,\hat{T}_2(\mathrm{Im}\,\mathbf{f})$, where $\mathbf{f} \in \mathcal{L}$ and

$$\hat{T}_1 := \mathbf{1} + \operatorname*{s-lim}_{n\to\infty} \sum_{i=1}^{n} (b_i - 1)\boldsymbol{Q}_i, \quad \hat{T}_2 := \mathbf{1} + \operatorname*{s-lim}_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{b_i} - 1\right)\boldsymbol{Q}_i.$$

Relative normalizers in $\mathrm{Aut}(\mathfrak{A})$

1. Let $\mathfrak{A}$ be the C^*-algebra generated by symbols $\{W(\mathbf{f})\}_{\mathbf{f}\in\mathcal{L}}$ s.t.

$$W(\mathbf{f}_1)W(\mathbf{f}_2) = e^{-i\boldsymbol{\sigma}(\mathbf{f}_1,\mathbf{f}_2)}W(\mathbf{f}_1+\mathbf{f}_2), \quad W(\mathbf{f})^* = W(-\mathbf{f}).$$

2. Let $\alpha: \ \mathrm{ISp}(\mathcal{L}) \to \mathrm{Aut}(\mathfrak{A})$ be the group homomorphism s.t.

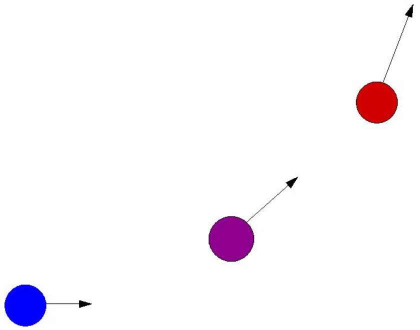
$$\alpha_{(\boldsymbol{v},T)}(W(\mathbf{f})) = e^{-2i(\boldsymbol{v},T\mathbf{f})}W(T\mathbf{f})$$

Proposition

1. *Let $\mathcal{L}^*_{\mathrm{R}} \subset \mathcal{L}^*_{\mathrm{S}} \subset \mathrm{ISp}(\mathcal{L})$ as before.*
2. *Let $R := \alpha_{\mathcal{L}^*_{\mathrm{R}}}$, $S := \alpha_{\mathcal{L}^*_{\mathrm{S}}}$ and $\alpha_{\mathrm{ISp}(\mathcal{L})} \subset G \subset Aut(\mathfrak{A})$.*

If $T \in N_{\mathrm{ISp}(\mathcal{L})}(\mathcal{L}^*_{\mathrm{R}}, \mathcal{L}^*_{\mathrm{S}})$ then $\alpha_T \in N_G(R,S)$.
In particular $\alpha_{\hat{T}} \in N_G(R,S)$, where $\hat{T}$ is the KPR map.

Problem of velocity superselection



Free electromagnetic field

1. Single-photon space: $\mathfrak{h} := \{ \mathsf{f} \in L^2(\mathbb{R}^3; \mathbb{C}^3) \,|\, \boldsymbol{k} \cdot \boldsymbol{f}(\boldsymbol{k}) = 0 \}$.
2. Fock space of multi-photon states: $\mathcal{F}_{\mathrm{ph}} := \Gamma(\mathfrak{h})$.
3. Energy-momentum operators of photons:

$$H_{\mathrm{ph}} = \sum_{\lambda=\pm} \int d^3k\, |\mathbf{k}|\, a^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}), \quad \boldsymbol{P}_{\mathrm{ph}} = \sum_{\lambda=\pm} \int d^3k\, \mathbf{k}\, a^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}).$$

4. Electromagnetic potential in the Coulomb gauge:

$$\boldsymbol{A}(\boldsymbol{x}) := \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{1}{2|\mathbf{k}|}} \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \big(e^{i\mathbf{k}\cdot\boldsymbol{x}} a_\lambda(\mathbf{k}) + e^{-i\mathbf{k}\cdot\boldsymbol{x}} a^*_\lambda(\mathbf{k}) \big),$$

where $\boldsymbol{\varepsilon}_+(\mathbf{k}), \boldsymbol{\varepsilon}_-(\mathbf{k}), \hat{\mathbf{k}}$ is an orthonormal basis in $\mathbb{R}^3$ for each $\mathbf{k}$.

Free electromagnetic field

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4. Electromagnetic potential in the Coulomb gauge:

$$\boldsymbol{A}_{[\sigma,\kappa]}(\boldsymbol{x}) := \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{1}{2|\mathbf{k}|}} \chi_{[\sigma,\kappa]}(|\mathbf{k}|) \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \big(e^{i\mathbf{k}\cdot\mathbf{x}} a_\lambda(\mathbf{k}) + \cdots$$

where $\boldsymbol{\varepsilon}_+(\mathbf{k}), \boldsymbol{\varepsilon}_-(\mathbf{k}), \hat{\mathbf{k}}$ is an orthonormal basis in $\mathbb{R}^3$ for each $\mathbf{k}$.

Model of non-relativistic QED

1. Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\mathrm{ph}}$.
2. Energy-momentum operators:

$$H_\sigma = \frac{1}{2}\big(-i\nabla_{\boldsymbol{x}} \otimes 1 + \tilde{\alpha}^{1/2}\boldsymbol{A}_{[\sigma,\kappa]}(\boldsymbol{x})\big)^2 + 1 \otimes H_{\mathrm{ph}},$$
$$\hat{\boldsymbol{P}} = -i\nabla_{\boldsymbol{x}} \otimes 1 + 1 \otimes \boldsymbol{P}_{\mathrm{ph}}.$$

3. Fiber decomposition: $H_\sigma = I^*\big(\int^{\oplus} d^3P\, H_{\boldsymbol{P},\sigma}\big)I$, where

$$H_{\boldsymbol{P},\sigma} = \frac{1}{2}\big(\boldsymbol{P} - \boldsymbol{P}_{\mathrm{ph}} + \tilde{\alpha}^{1/2}\boldsymbol{A}_{[\sigma,\kappa]}(0)\big)^2 + H_{\mathrm{ph}}$$

are operators on $\mathcal{F}$.

Spectrum of the fiber Hamiltonians $H_{P,\sigma}$

Let $\mathcal{S} = \{\boldsymbol{P} \in \mathbb{R}^3 \,|\, |\boldsymbol{P}| < \frac{1}{3}\}$ and $\tilde{\alpha}$ small.

Lemma (Hasler-Herbst 08, Chen-Fröhlich-Pizzo 09)

Let $E_{\boldsymbol{P},\sigma} := \inf \operatorname{sp}(H_{\boldsymbol{P},\sigma>0})$ and $E_{\boldsymbol{P}} := \inf \operatorname{sp}(H_{\boldsymbol{P},\sigma=0})$.

1. *$E_{\boldsymbol{P},\sigma}$ is a (simple) eigenvalue with eigenvector $\Psi_{\boldsymbol{P},\sigma}$.*
2. $\mathrm{w} - \lim_{\sigma\to 0} \Psi_{\boldsymbol{P},\sigma} = 0$ *and $E_{\boldsymbol{P}}$ is not an eigenvalue if $\boldsymbol{P} \neq 0$.*
3. $\lim_{\sigma\to 0} W_0(\boldsymbol{v}_{\boldsymbol{P},\sigma})\Psi_{\boldsymbol{P},\sigma}$ *exists and is non-zero, where*

$$W_0(\boldsymbol{v}_{\boldsymbol{P},\sigma}) := e^{a^*(\boldsymbol{v}_{\boldsymbol{P},\sigma}) - a(\boldsymbol{v}_{\boldsymbol{P},\sigma})},$$

$$\boldsymbol{v}_{\boldsymbol{P},\sigma}(\boldsymbol{k}) := \left(\frac{\tilde{\alpha}}{2(2\pi)^3}\right)^{\frac{1}{2}} \frac{\chi_{[\sigma,\kappa]}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^{3/2}} P_{tr} \frac{\nabla E_{\boldsymbol{P},\sigma}}{1 - \hat{\boldsymbol{k}} \cdot \nabla E_{\boldsymbol{P},\sigma}}.$$

Velocity superselection

Thm (Fröhlich 74, Chen-Fröhlich-Pizzo 09, Könenberg-Matte 14)

For any $\boldsymbol{P} \in \mathcal{S}$ *the following limits exist and define states on* $\mathfrak{A}$

$$\omega_{\boldsymbol{P}}(A) := \lim_{\sigma \to 0} \langle \Psi_{\boldsymbol{P},\sigma}, \pi_0(A) \Psi_{\boldsymbol{P},\sigma} \rangle, \qquad A \in \mathfrak{A}.$$

The corresponding sectors are mutually disjoint i.e.

$$[\omega_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}} \neq [\omega_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}} \quad \text{for} \quad \boldsymbol{P}_1 \neq \boldsymbol{P}_2.$$

Theorem (Cadamuro-W.D. 18)

Let $\hat{T}$ *be the KPR infravacuum. Then, for all* $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{S}$

$$\overline{[[\omega_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}}]}^{\alpha_{\hat{T}}} = \overline{[[\omega_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}}]}^{\alpha_{\hat{T}}}, \quad \text{and} \quad \overline{\overline{[[\omega_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}}]}}^{\alpha_{\hat{T}}} = \overline{\overline{[[\omega_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}}]}}^{\alpha_{\hat{T}}}.$$

Velocity superselection and conjugate sectors

Theorem (Cadamuro-W.D. 18)

Let $\hat{T}$ be the KPR infravacuum. Then, for all $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{S}$

$$\overline{[[\omega_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}}]}^{\alpha_{\hat{T}}} = \overline{[[\omega_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}}]}^{\alpha_{\hat{T}}}, \quad \text{and} \quad \overline{\overline{[[\omega_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}}]}}^{\alpha_{\hat{T}}} = \overline{\overline{[[\omega_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}}]}}^{\alpha_{\hat{T}}}.$$

1. Any $[\rho_{\boldsymbol{P}}]_{\mathrm{In}\mathfrak{A}} \in \overline{[[\omega_{\boldsymbol{P}}]_{\mathrm{In}\mathfrak{A}}]}^{\alpha_{\hat{T}}}$ is a conjugate sector of $[\omega_{\boldsymbol{P}}]_{\mathrm{In}\mathfrak{A}}$.
2. By the theorem, we can choose $\boldsymbol{P} \mapsto [\rho_{\boldsymbol{P}}]_{\mathrm{In}\mathfrak{A}}$ s.t.
$$[\rho_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}} = [\rho_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}} \quad \text{for all} \quad \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{S}.$$
3. But distinct conjugate sectors $\boldsymbol{P} \mapsto [\tilde{\rho}_{\boldsymbol{P}}]_{\mathrm{In}\mathfrak{A}}$, that is s.t.
$$[\tilde{\rho}_{\boldsymbol{P}_1}]_{\mathrm{In}\mathfrak{A}} \neq [\tilde{\rho}_{\boldsymbol{P}_2}]_{\mathrm{In}\mathfrak{A}} \quad \text{for all} \quad \boldsymbol{P}_1 \neq \boldsymbol{P}_2 \in \mathcal{S}$$

   are also possible.

Velocity superselection and conjugate sectors

Recall that the electron has an ‘additional charge’ whose value depends on velocity.

Velocity superselection and conjugate sectors

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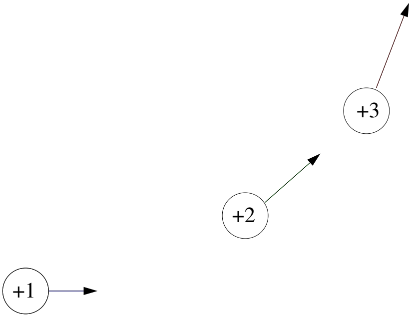
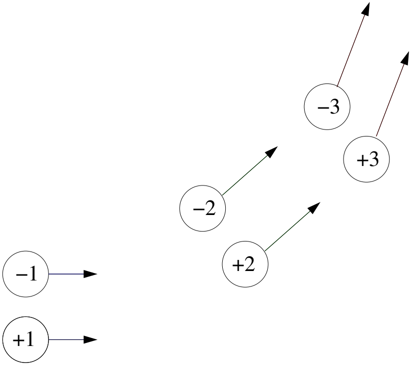


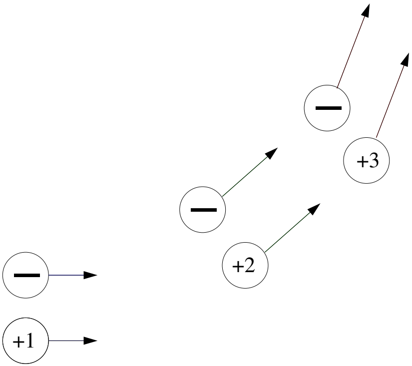
Velocity superselection and conjugate sectors

The conjugate sectors ('compensating charges') can be chosen to be distinct, as in the usual (DHR) setting.



Velocity superselection and conjugate sectors

But we can also find one conjugate sector which 'compensates' many distinct sectors.



Outlook

1. We exhibited a mathematical structure underlying the infravacuum: the relative normalizer of $R \subset S \subset G$:

$$N_G(R, S) := \{ g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \}.$$

2. We propose the second-conjugate class w.r.t. the infravacuum background as a 'charge class' collecting sectors differing by unobservable 'soft-photon clouds'.

3. Question: How large are the second-conjugate classes?

4. Question: How to incorporate mixed states?

5. Question: How to incorporate positivity of energy?