What to expect from logarithmic conformal field theory

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Outline

1. Rational vs logarithmic conformal field theory

2. Examples

Some features of rational conformal field theories

- Space of states is a module over 2 commuting copies of a rational vertex operator algebra.
- Singularities of correlation functions are at worst rational poles. *E.g.*

$$\frac{1}{|z-w|^{\frac{1}{4}}}$$

- Character of space of states is a modular invariant.

Rational vertex operator algebras

- Module theory is semisimple.
  $\to$ Virasoro L_0 operator is diagonalisable.
- Finitely many inequivalent simple modules.
- Span of module characters carries a representation of the modular group SL $(2, \mathbb{Z})$. [Zhu]
- Module category is a *modular tensor category*. [Verlinde, Moore-Seiberg, Huang]

Fusion product derived from correlation functions	$=$	Verlinde product derived from SL $(2, \mathbb{Z})$ action on characters

Features of logarithmic conformal field theories

- Space of states is a module over 2 commuting copies of a logarithmic vertex operator algebra.
- Singularities of correlation functions can be logarithmic. *E.g.*

$$\log |z - w|$$

- Character of space of states is still a modular invariant.

Logarithmic vertex operator algebras

- Module theory is not semisimple.
  $\rightarrow$ Virasoro L_0 operator can have Jordan blocks.
- Finitely many inequivalent simple modules. May or may not fail.
- Span of torus 1-point functions carries a representation of the modular group SL$(2, \mathbb{Z})$. [Miyamoto]
- Module category is not a *modular tensor category*.
  Verlinde formula must fail as characters can't distinguish all modules and modular action does not close on characters.

Examples of logarithmic conformal field theories

The symplectic fermions are generated by two field ξ_1, ξ_2:

$$\xi_1(z)\xi_2(w) \sim \frac{1}{(z-w)^2} \sim -\xi_2(z)\xi_1(w), \quad \xi_1(z)\xi_1(w) \sim 0 \sim \xi_2(z)\xi_2(w).$$

The even subalgebra is called “the $c=-2$ triplet”.

Defines a logarithmic vertex operator algebra/conformal field theory.
[Gaberdiel-Kausch]

$c = -2$ triplet module theory

4 simple modules:

$$S_0, \quad S_1, \quad S_{\frac{-1}{8}}, \quad S_{\frac{3}{8}}$$

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Indices denote conformal highest weight.

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Vertex operator algebra/vacuum module.

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Cannot be extended to form reducible yet indecomposable modules.

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Simple modules are not closed under fusion

S_0 fusion unit

$S_1\colon S_0 \leftrightarrow S_1,\ S_{\frac{-1}{8}} \leftrightarrow S_{\frac{3}{8}}$

$S_{\frac{-1}{8}} \times S_{\frac{-1}{8}} = S_{\frac{3}{8}} \times S_{\frac{3}{8}} = P_0,\ S_{\frac{-1}{8}} \times S_{\frac{3}{8}}$

$S_1\colon P_0 \leftrightarrow P_1$

$S_i \times P_j = 2S_{\frac{-1}{8}} \oplus 2S_{\frac{3}{8}},\ i = \frac{-1}{8}, \frac{3}{8},\ j = 0, 1$

Fusion closes on $S_0,\ S_1,\ S_{\frac{-1}{8}}, S_{\frac{3}{8}},\ P_0,\ P_1$.

The new modules $P_0,\ P_1$ are indecomposable yet reducible, but cannot be further extended.

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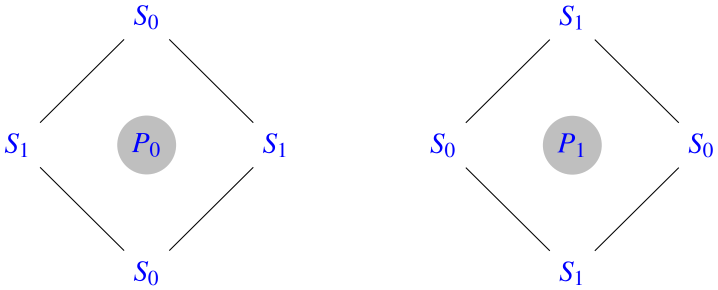
Submodule structure of P_0 and P_1



Module characters: $\mathrm{ch}[M](q) = \mathrm{tr}_M\, q^{L_0 - \frac{c}{24}}$, $q = \mathrm{e}^{2\pi i \tau}$, here $c = -2$.

Characters cannot distinguish indecomposables from the sum of their composition factors: $\mathrm{ch}[P_0] = \mathrm{ch}[P_1] = 2\,\mathrm{ch}[S_0] + 2\,\mathrm{ch}[S_1]$

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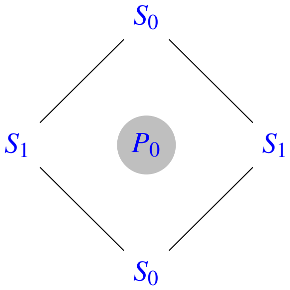

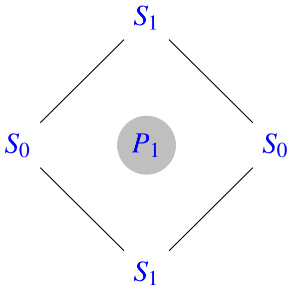

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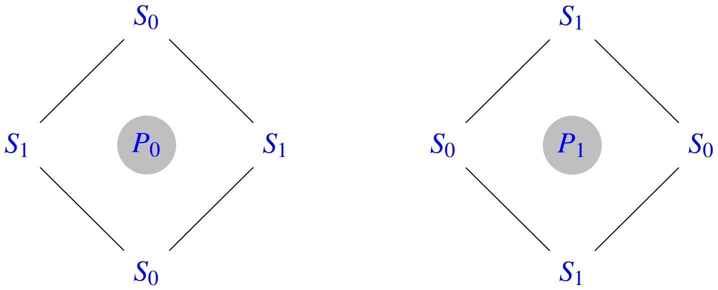

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Action of the modular group on characters

Modular group: $\mathsf{SL}(2,\mathbb{Z}) = \langle \mathsf{S},\mathsf{T} \,|\, \mathsf{S}^2 = (\mathsf{ST})^3,\ \mathsf{S}^4 = 1\rangle$
$\mathsf{T}: \tau \mapsto \tau + 1$, $\mathsf{S}: \tau \mapsto \frac{-1}{\tau}$

S action:

$$\mathrm{ch}[S_0] \mapsto \frac{1}{4}\mathrm{ch}\left[S_{\frac{-1}{8}}\right] - \frac{1}{4}\mathrm{ch}\left[S_{\frac{3}{8}}\right] - \frac{\mathrm{i}\tau}{2}\left(\mathrm{ch}[S_0] - \mathrm{ch}[S_1]\right)$$
$$\mathrm{ch}[S_1] \mapsto \frac{1}{4}\mathrm{ch}\left[S_{\frac{-1}{8}}\right] - \frac{1}{4}\mathrm{ch}\left[S_{\frac{3}{8}}\right] + \frac{\mathrm{i}\tau}{2}\left(\mathrm{ch}[S_0] - \mathrm{ch}[S_1]\right)$$
$$\mathrm{ch}\left[S_{\frac{-1}{8}}\right] \mapsto \mathrm{ch}[S_0] + \mathrm{ch}[S_1] + \frac{1}{2}\mathrm{ch}\left[S_{\frac{-1}{8}}\right] + \frac{1}{2}\mathrm{ch}\left[S_{\frac{3}{8}}\right]$$
$$\mathrm{ch}\left[S_{\frac{3}{8}}\right] \mapsto -\mathrm{ch}[S_0] - \mathrm{ch}[S_1] + \frac{1}{2}\mathrm{ch}\left[S_{\frac{-1}{8}}\right] + \frac{1}{2}\mathrm{ch}\left[S_{\frac{3}{8}}\right]$$

$$\frac{\mathrm{i}\tau}{2}\left(\mathrm{ch}[S_0] - \mathrm{ch}[S_1]\right) \mapsto \frac{1}{2}\mathrm{ch}[S_0] - \frac{1}{2}\mathrm{ch}[S_1]$$

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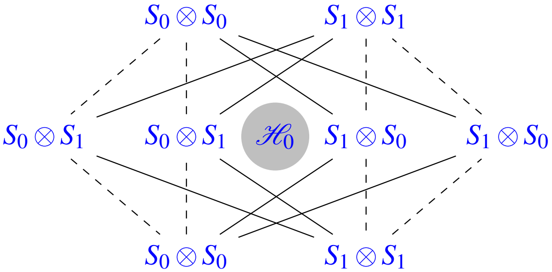
The full 'bulk' conformal field theory

Space of states: $\mathscr{H} = S_{\frac{-1}{8}} \otimes S_{\frac{-1}{8}} \oplus S_{\frac{3}{8}} \otimes S_{\frac{3}{8}} \oplus \mathscr{H}_0$, where



$$\mathrm{ch}\left[\mathscr{H}\right] = \left|\mathrm{ch}\left[S_{\frac{-1}{8}}\right]\right|^2 + \left|\mathrm{ch}\left[S_{\frac{3}{8}}\right]\right|^2 + \mathrm{ch}\left[\mathscr{H}_0\right]$$

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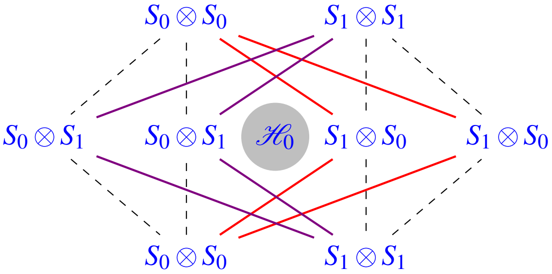

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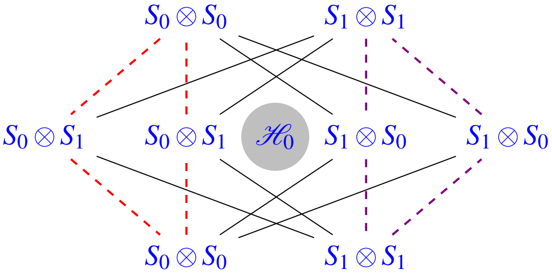

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A logarithmic Verlinde formula?

- Verlinde formula is derived from characters. At best it could predict fusion at the level of characters.
- To salvage a Verlinde formula for logarithmic conformal field theory we need to deal with τ-dependent S transformations.
- We do this by giving up on having only finitely many simple modules.
- If done properly the τ-dependence will be restricted to a ‘subset of modules of measure 0’.

The $\beta\gamma$ ghost logarithmic conformal field theory

Two generating fields β and γ.

$$\gamma(z)\beta(w) \sim \frac{1}{z-w} \sim -\beta(z)\gamma(w), \quad \beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w)$$
$$T(z) = -:\beta(z)\partial\gamma(z):, \quad J(z) = :\beta(z)\gamma(z):.$$

Let $\beta(z) = \sum_n \beta_n z^{-n-1}$ and $\gamma(z) = \sum_n \gamma_n z^{-n}$ then

$$[\gamma_m, \beta_n] = \delta_{m+n,0}\mathbf{1}, \quad [\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n]$$

Triangular decomposition:

$$\mathfrak{G} = \underbrace{\left(\bigoplus_{n\geq 1} \mathbb{C}\beta_{-n} \oplus \mathbb{C}\gamma_{-n}\right) \oplus \mathbb{C}\gamma_0}_{\mathfrak{n}_-} \oplus \underbrace{\mathbb{C}\mathbf{1}}_{\mathfrak{h}} \oplus \underbrace{\mathbb{C}\beta_0 \oplus \left(\bigoplus_{n\geq 1} \mathbb{C}\beta_n \oplus \mathbb{C}\gamma_n\right)}_{\mathfrak{n}_+}$$

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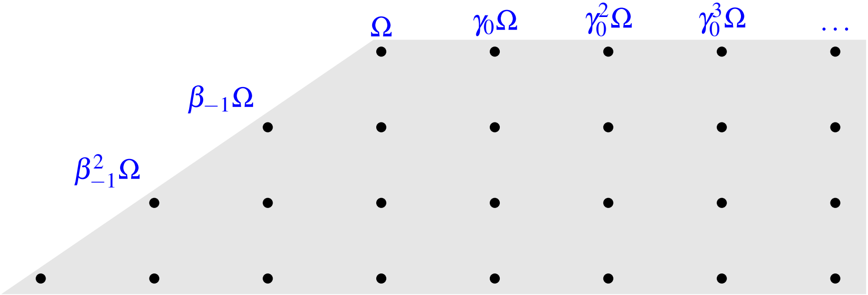
Highest weight modules (category $\mathscr{O}$)

Since $\mathbf{1}$ must act as the identity, there exists only 1 Verma module $\mathscr{V}$ and it is simple.



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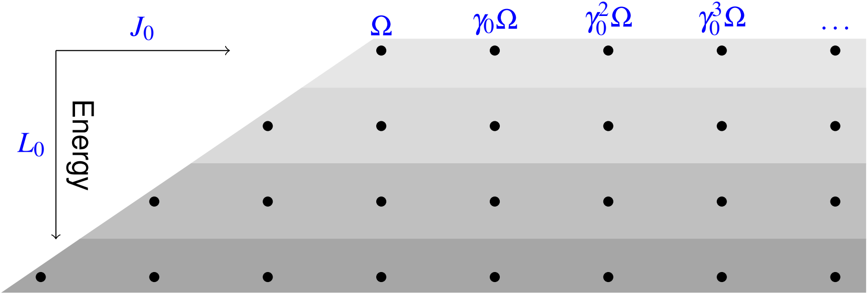

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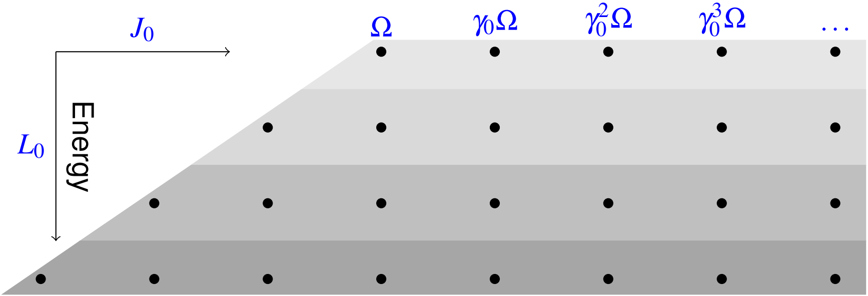

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Going beyond highest weight modules

Enlarge module category by going from triangular decomposition to parabolic decomposition:

$$\mathfrak{G} = \underbrace{\left(\bigoplus_{n\geq 1} \mathbb{C}\beta_{-n} \oplus \mathbb{C}\gamma_{-n}\right)}_{\mathfrak{n}_-^p} \oplus \underbrace{\mathbb{C}\gamma_0 \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\beta_0}_{\mathfrak{h}^p} \oplus \underbrace{\left(\bigoplus_{n\geq 1} \mathbb{C}\beta_{n} \oplus \mathbb{C}\gamma_{n}\right)}_{\mathfrak{n}_+^p}$$

Parabolic Verma modules are modules induced from $\mathfrak{h}^p$ modules, on which $\mathfrak{n}_+^p$ acts trivially, by letting $\mathfrak{n}_-^p$ act freely.

$\mathfrak{h}^p$ is the Weyl algebra A_1. Its simple modules were classified by Block.

$\mathfrak{h}^p$ weight module classification

Define the eigenvalues of $J = \gamma_0\beta_0$ to be weights.
Weights shifted by $+1$ by γ_0 and by -1 by β_0.

Theorem [Block]

Any simple $\mathfrak{h}^p$ weight module is equivalent to one of the following.

1. Unique highest weight module: $\overline{\mathscr{V}} = \mathbb{C}[\gamma_0]\overline{\Omega}$, $\beta_0\overline{\Omega} = 0$. $\to J\overline{\Omega} = 0$.
2. Unique lowest weight module: $\overline{\mathscr{V}}^* = \mathbb{C}[\beta_0]\overline{\omega}$, $\gamma_0\overline{\omega}$. $\to J\overline{\omega} = \overline{\omega}$.
3. Dense module: For $[\lambda] \in \mathbb{C}/\mathbb{Z}$, $[\lambda] \neq [0]$, let

$$\overline{\mathscr{W}}_\lambda = \mathbb{C}[\beta_0]\beta_0 u_\lambda \oplus u_\lambda \oplus \mathbb{C}[\gamma_0]\gamma_0 u_\lambda$$

be the module generated by a weight vector u_λ, $Ju_\lambda = \lambda u_\lambda$.
$\overline{\mathscr{W}}_\lambda \cong \overline{\mathscr{W}}_\mu$ iff $\lambda - \mu \in \mathbb{Z}$.

For $[\lambda] = 0$, exist two indecomposables characterised by the non-split exact sequences

$$0 \longrightarrow \overline{\mathscr{V}} \longrightarrow \overline{\mathscr{W}}_0^+ \longrightarrow \overline{\mathscr{V}}^* \longrightarrow 0, \quad 0 \longrightarrow \overline{\mathscr{V}}^* \longrightarrow \overline{\mathscr{W}}_0^- \longrightarrow \overline{\mathscr{V}} \longrightarrow 0.$$

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Going beyond highest weight modules

Let $\mathscr{V}$, $\mathscr{V}^*$, $\mathscr{W}_\lambda$, $\mathscr{W}_0^\pm$ be inductions of previous $\mathfrak{h}^p$ modules.

This list is not closed under fusion or action of modular group. One final enlargement needed.

Construct more modules using an algebra automorphism σ called spectral flow: $\sigma(\gamma_n) = \gamma_{n+1}$, $\sigma(\beta_n) = \beta_{n-1}$.

$\rightarrow \sigma(J_0) = J_0 + \mathbf{1}$, $\sigma(L_0) = L_0 - J_0$

Let M be a module. Define spectral flow twist by $\sigma M \cong M$ as vector space. Twisted action $x \cdot_\sigma m = \sigma^{-1}(x) \cdot m$, $x \in \mathfrak{G}$, $m \in M$.

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Let $\mathscr{V}$, $\mathscr{V}^*$, $\mathscr{W}_\lambda$, $\mathscr{W}_0^\pm$ be inductions of previous $\mathfrak{h}^p$ modules.

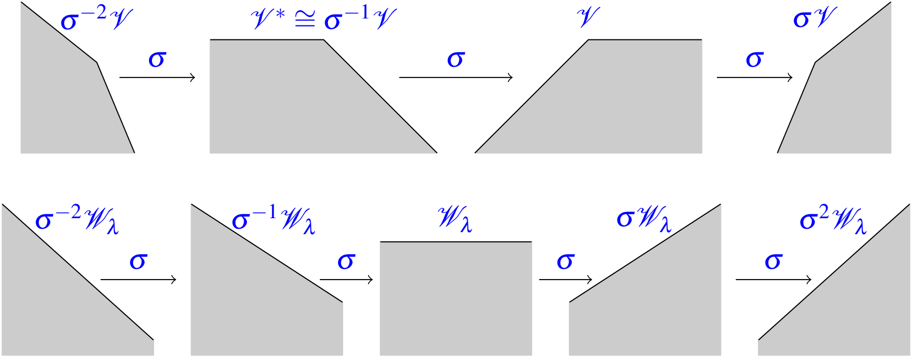
This list is not closed under fusion or action of modular group. One final enlargement needed.

Construct more modules using an algebra automorphism σ called spectral flow: $\sigma(\gamma_n) = \gamma_{n+1}$, $\sigma(\beta_n) = \beta_{n-1}$.
$\rightarrow \sigma(J_0) = J_0 + \mathbf{1}$, $\sigma(L_0) = L_0 - J_0$

Let M be a module. Define spectral flow twist by $\sigma M \cong M$ as vector space. Twisted action $x \cdot_\sigma m = \sigma^{-1}(x) \cdot m$, $x \in \mathfrak{G}$, $m \in M$.

Spectral flow twists

Spectral flow 'tilts' the energy grading:



The action of the modular group closes on the span of the characters of $\sigma^\ell \mathscr{W}_\lambda$, $\ell \in \mathbb{Z}$, $\lambda \in \mathbb{C}/\mathbb{Z}$.

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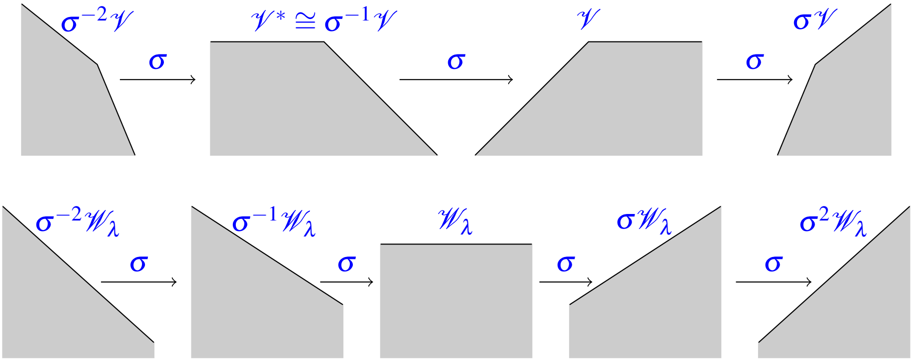

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The rational Verlinde formula

Rational vertex operator algebras admit only a finite number of inequivalent simple modules M_i, $i = 0, \ldots, n$, where M_0 is the vacuum module.

Fusion: $M_i \times M_j = \bigoplus_k N_{i,j}^{k} M_k$, $N_{i,j}^{k} \in \mathbb{N}_0$.

Action of modular group closes on span of module characters:

$$\mathsf{T} : \mathrm{ch}\,[M_i] \mapsto T_i\,\mathrm{ch}\,[M_i]$$
$$\mathsf{S} : \mathrm{ch}\,[M_i] \mapsto S_{i,j}\,\mathrm{ch}\,[M_j]$$

The Verlinde formula relates the fusion structure constants and the S-matrix coefficients.

$$N_{i,j}^{k} = \sum_n \frac{S_{i,n} S_{j,n} \overline{S_{k,n}}}{S_{0,n}}$$

Towards a logarithmic Verlinde formula

The action of the modular group closes on the span of $\sigma^{\ell}\mathscr{W}_{\lambda}$ characters.

$$\mathsf{S} : \mathrm{ch}\left[\sigma^{\ell}\mathscr{W}_{\lambda}\right] \mapsto \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \mathsf{S}\left[\sigma^{\ell}\mathscr{W}_{\lambda} \to \sigma^{m}\mathscr{W}_{\mu}\right] \mathrm{ch}\left[\sigma^{m}\mathscr{W}_{\mu}\right] \mathrm{d}\,\mu$$

$$\mathsf{S}\left[\sigma^{\ell}\mathscr{W}_{\lambda} \to \sigma^{m}\mathscr{W}_{\mu}\right] = (-1)^{\ell+m}\mathsf{e}^{-2\pi\mathsf{i}(\ell\mu+m\lambda)}$$

No τ dependence!

What about $\sigma^{\ell}\mathscr{V}$?

Towards a logarithmic Verlinde formula

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Towards a logarithmic Verlinde formula

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What about $\sigma^\ell \mathscr{V}$?

Towards a logarithmic Verlinde formula

Recall, $0 \longrightarrow \sigma^{-1}\mathscr{V} \longrightarrow \mathscr{W}_0^- \longrightarrow \mathscr{V} \longrightarrow 0$.

We can resolve $\mathscr{V}$ in terms of the $\sigma^\ell \mathscr{W}_0^-$ by splicing exact sequences.

$$\cdots \longrightarrow \sigma^{-2}\mathscr{W}_0^- \longrightarrow \sigma^{-1}\mathscr{W}_0^- \longrightarrow \mathscr{W}_0^- \longrightarrow \mathscr{V} \longrightarrow 0$$

The character of $\mathscr{V}$ can then be computed using the Euler-Poincaré principle.

$$\operatorname{ch}\left[\sigma^\ell \mathscr{V}\right] = \sum_{m=0}^{\infty} (-1)^m \operatorname{ch}\left[\sigma^{\ell-m}\mathscr{W}_0\right].$$

S-transformation:

$$\mathsf{S}\left[\sigma^\ell \mathscr{V} \to \sigma^m \mathscr{W}_\mu\right] = (-1)^{\ell+m+1} \frac{\mathsf{e}^{-2\pi\mathsf{i}(\ell+1/2)\mu}}{\mathsf{e}^{\pi\mathsf{i}\mu} - \mathsf{e}^{-\pi\mathsf{i}\mu}}$$

Towards a logarithmic Verlinde formula

We can now conjecture a Verlinde formula for the characters of fusion products.

$$\mathrm{ch}\left[M \times N\right] = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} N_{M,N}^{m,\mu}\, \mathrm{ch}\left[\sigma^m \mathscr{W}_\mu\right] \mathrm{d}\,\mu.$$

The natural generalisation of the rational Verlinde formula is:

$$N_{M,N}^{m,\mu} = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\mathsf{S}\left[M \to \sigma^n \mathscr{W}_\nu\right] \mathsf{S}\left[N \to \sigma^n \mathscr{W}_\nu\right] \overline{\mathsf{S}\left[\sigma^m \mathscr{W}_\mu \to \sigma^n \mathscr{W}_\nu\right]}}{\mathsf{S}\left[\mathscr{V} \to \sigma^n \mathscr{W}_\nu\right]} \mathrm{d}\,\nu$$

Towards a logarithmic Verlinde formula

Fusion predicted by Verlinde formula:

$$\mathrm{ch}\left[\sigma^{\ell}\mathscr{V}\times\sigma^{m}\mathscr{V}\right]=\mathrm{ch}\left[\sigma^{\ell+m}\mathscr{V}\right],$$
$$\mathrm{ch}\left[\sigma^{\ell}\mathscr{V}\times\sigma^{m}\mathscr{W}_{\mu}\right]=\mathrm{ch}\left[\sigma^{\ell+m}\mathscr{W}_{\mu}\right],$$
$$\mathrm{ch}\left[\sigma^{\ell}\mathscr{W}_{\lambda}\times\sigma^{m}\mathscr{W}_{\mu}\right]=\mathrm{ch}\left[\sigma^{\ell+m}\mathscr{W}_{\lambda+\mu}\right]+\mathrm{ch}\left[\sigma^{\ell+m-1}\mathscr{W}_{\lambda+\mu}\right].$$

Fusion products uniquely determined by characters unless $\lambda+\mu\in\mathbb{Z}$.

Spot checks by direct computation match Verlinde prediction.

Towards a logarithmic Verlinde formula

Fusion predicted by Verlinde formula:

$$\begin{aligned}
\operatorname{ch}\left[\sigma^{\ell}\mathscr{V}\times\sigma^{m}\mathscr{V}\right] &= \operatorname{ch}\left[\sigma^{\ell+m}\mathscr{V}\right],\\
\operatorname{ch}\left[\sigma^{\ell}\mathscr{V}\times\sigma^{m}\mathscr{W}_{\mu}\right] &= \operatorname{ch}\left[\sigma^{\ell+m}\mathscr{W}_{\mu}\right],\\
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\end{aligned}$$

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\end{aligned}$$

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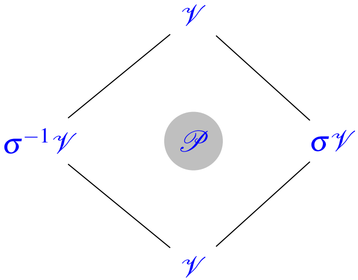
Indecomposable fusion products

By direct computation: $\mathscr{W}_\lambda \times \mathscr{W}_{-\lambda} = \sigma^{-1}\mathscr{P}$, where



Conjecture [Ridout-SW]

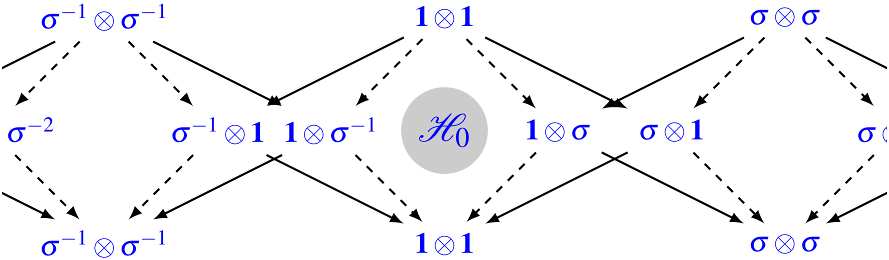
Let $\mathscr{C}$ be the abelian category of $\beta\gamma$ vertex operator algebra modules generated by the closure under extensions of the $\sigma^\ell \mathscr{V}$ and $\sigma^\ell \mathscr{W}_\lambda$, $\ell \in \mathbb{Z}$, $\lambda \in (0,1)$. Then,

- the $\sigma^\ell \mathscr{W}_\lambda$ are simple and projective,
- the $\sigma^\ell \mathscr{P}$ are indecomposable projective covers of $\sigma^\ell \mathscr{V}$,
- the logarithmic Verlinde formula holds.

Modular invariants [Ridout-SW]

As a final exercise we can write down a modular invariant candidate for the space of states.

$$\mathscr{H} = \mathscr{H}_0 \oplus \bigoplus_{\ell\in\mathbb{Z}} \int^{\oplus}_{\mathbb{R}/\mathbb{Z}} \sigma^{\ell}\mathscr{W}_{\lambda} \otimes \sigma^{\ell}\mathscr{W}_{\lambda}\,\mathrm{d}\lambda.$$



$$\mathrm{ch}\left[\mathscr{H}_0\right] = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}\left[\sigma^{\ell}\mathscr{V}\right]\overline{\mathrm{ch}\left[\sigma^{\ell}\mathscr{P}\right]} = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}\left[\sigma^{\ell}\mathscr{P}\right]\overline{\mathrm{ch}\left[\sigma^{\ell}\mathscr{V}\right]}$$

Conclusion

- Logarithmic conformal field theories/vertex operator algebras admit reducible yet indecomposable modules.
- Having finitely many simple modules seems to break the Verlinde formula.
- Verlinde formula can be fixed by allowing infinitely many simple modules.