Non-perturbative Renormalization via Projective Non-Gaussian White Noise Analysis

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Problem

Integration theory on a suitable space of (generalized) fields over a Riemannian manifold M.

Projective measures

- Fields $x : m \in M \mapsto x(m) \in \mathbb{R}$ (just heuristics, technical assumptions would be premature).
- Compatible projections (or regularizing cutoffs)

$$\pi_P : X \to X_P, \quad P \in \mathcal{P}.$$

  Otherwise said, $X \subseteq \operatorname{proj\,lim} X_P$.
- Compatible effective measures μ_P on X_P.

Choice of projections $\pi_P : X \to X_P$

Given by coarse graining procedure:

- Let $\Lambda \subseteq L^\infty(M)$ be a lattice of projections.
- Consider *partitions* $P = \{ p_1, \ldots, p_n \} \subseteq \Lambda$ (i.e. $\sum p_i = 1$ and $p_i p_j = 0$ if $i \neq j$). Family $\mathcal{P}$ of such partitions is directed: $Q \succcurlyeq P$ iff P is contained in the lattice generated by Q.
- Let $X_P = L^\infty(X; P) \cong \mathbb{R}^P$ be the space of P-simple functions. Note that $Q \succcurlyeq P \Leftrightarrow L^\infty(X; P) \subseteq L^\infty(X; Q)$. Define

$$(\pi_{QP} x)_p = \frac{1}{|p|} \sum_{q \leq p} |q| x_q$$

where $|p| = \int_M p(m) \mathrm{d}m$ and $q \leq p \Leftrightarrow qp = q$.

Choice of $\Lambda \subseteq L^\infty(M)$

Less is more: the whole projection lattice of $L^\infty(M)$ encodes just the measure-theoretic structure of M. In cases where the background is fixed, it is natural to take a Λ encoding also the geometry.

- Natural choice: $p \in \Lambda$ associated to bounded region in M with piecewise smooth boundary.
- Partitions $P \in \mathcal{P}$ become smooth cellular structures on M.

Theorem

Let $\{ \nu_\lambda \}_{\lambda \geq 0}$ be a convolution semigroup of probability measures on $\mathbb{R}$. The product measures

$$\mathrm{d}\mu_P(x) = \prod_{p \in P} \mathrm{d}\nu_{|p|}(|p| x_p), \quad x \in X_P$$

are compatible (i.e. $(\pi_{QP})_* \mu_Q = \mu_P$). The corresponding projective measure on $X = \operatorname{proj\,lim} X_P$ is a Lévy white noise field.

Remark

In the Gaussian case $\mathrm{d}\nu_\lambda(x) = \frac{1}{\sqrt{2\pi\lambda}} \mathrm{e}^{-x^2/2\lambda} \mathrm{d}x$ one formally has

$$\mathrm{d}\mu(x) = C \mathrm{e}^{-\frac{1}{2} \int x(m)^2 \mathrm{d}m} \mathrm{d}x = \prod C_m \mathrm{e}^{-\mathcal{L}(x(m)) \mathrm{d}m} \mathrm{d}x(m).$$

In general, divergent running parameters will appear in $\mathcal{L}$.

Definitions

- An *effective observable* is a function $a_P \in L^1(X_P)$.
- A *projective observable* is a collection $a = \{ a_P \}$ of effective observables satisfying the martingale condition

$$\mathbb{E}[a_Q \mid \pi_{QP}] = a_P.$$

- Space of projective observables: $L^1_{\mathrm{eff}}(X) = \operatorname{proj\,lim} L^1(X_P)$ with connecting maps $\mathbb{E}[\cdot \mid \pi_{QP}] : L^1(X_Q) \to L^1(X_P)$. Note: $a \in L^1_{\mathrm{eff}}(X)$ is not necessarily integrable.

Remark

The family $\{ a_P \mu_P \}$ is a projective measure if $a \in L^1_{\mathrm{eff}}(X)$.

Abusing notation, write x_p for the effective evaluation observable $x \in X_P \mapsto x_p \in \mathbb{R}$, where $P \ni p$ is understood from context.

Proposition

Given $P \preccurlyeq Q$ and $q \in Q$, one has

$$\mathbb{E}[x_q \,|\, \pi_{QP}] = x_p$$

where $p \in P$ is uniquely determined by $p \geq q$.

Corollary

Each effective evaluation observable x_p admits an ultraviolet completion (i.e. belongs to a projective observable) consisting only of effective evaluation observables.

Definitions

- A *filter* is a family $\mathfrak{f} \subseteq \Lambda$ which is:
  - Non-trivial: neither $\mathfrak{f} = \varnothing$ nor $\mathfrak{f} = \Lambda$.
  - Downward directed: $p, q \in \mathfrak{f} \Rightarrow pq \in \mathfrak{f}$.
  - Upward saturated: $p \in \mathfrak{f}$ and $q \geq p \Rightarrow q \in \mathfrak{f}$.
- An *ultrafilter* is a filter $\mathfrak{m}$ which is maximal. Equivalently, for each $p \in \Lambda$ either $p \in \mathfrak{m}$ or $\neg p = 1 - p \in \mathfrak{m}$. Write $\mathfrak{M}$ for the space of ultrafilters of Λ.

Proposition

Let $\mathfrak{m} \in \mathfrak{M}$ and $P \in \mathcal{P}$. There is a unique element in $\mathfrak{m} \cap \mathcal{P}$, which will be written $p(\mathfrak{m})$.

Given $\mathfrak{m} \in \mathfrak{M}$, write $x(\mathfrak{m})$ for the projective observable $\{ x_{p(\mathfrak{m})} \}$.

Definition

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n \in \mathfrak{M}$. We define the *Wick product* $x(\mathfrak{m}_1) \diamond \cdots \diamond x(\mathfrak{m}_n) \in L^1_{\mathrm{eff}}(X)$ by

$$\big(x(\mathfrak{m}_1) \diamond \cdots \diamond x(\mathfrak{m}_n)\big)_P = \mathbb{E}\big[x_{q_1} \cdots x_{q_n} \,\big|\, \pi_{QP}\big]$$

where $Q \succcurlyeq P$ is fine enough to admit the existence of *pairwise different* q_i's with $q_i \leq p(\mathfrak{m}_i)$. By independence, one can convince oneself that the actual choice of Q and the q_i's is irrelevant.

Remark

This can also be thought of as a collection of compatible effective products $x_{p_1} \diamond \cdots \diamond x_{p_n} \in L^1(X_P)$.

Examples

- Let μ be Gaussian white noise, i.e. $\mathrm{d}\nu_\lambda(x) = \frac{1}{\sqrt{2\pi\lambda}}\mathrm{e}^{-x^2/2\lambda}\mathrm{d}x$. Then, $x_p^{\diamond n} = H^n_{|p|}(x_p)$ where H^n_λ is the n-th Hermite polynomial with variance $1/\lambda$.
- Let μ be Poisson white noise, i.e. $\hat{\nu}_\lambda(\xi) = \mathrm{e}^{\lambda(\mathrm{e}^{-\mathrm{i}\xi}-1)}$. Then, $x_p^{\diamond n}$ is the falling factorial

$$x_p\big(x_p - |p|^{-1}\big)\cdots\big(x_p - (n-1)|p|^{-1}\big).$$

- Let μ be Γ white noise, i.e. $\hat{\nu}_\lambda(\xi) = (1-\mathrm{i}\xi)^{-\lambda}$. Then,

$$x_p^{\diamond n} = \frac{|p|^n}{|p|(|p|+1)\cdots(|p|+n-1)}x_p^n.$$

Note: multiplicative renormalization!

One can take fixed linear combinations of Wick monomials. But one can also vary the linear combination with the scale! Stochastic integration arises as particular case:

Proposition

Consider a family $\alpha^n = \left\{ \alpha^n_P \right\}_{P \in \mathcal{P}}$ of tensors $\alpha^n_P = (\alpha^n_{p_1 \dots p_n}) \in \mathbb{R}^{P^n}$ satisfying the compatibility condition

$$\alpha^n_{p_1 \dots p_n} = \sum_{q_i \le p_i} \alpha^n_{q_1 \dots q_n}.$$

Then, the effective observables $a_P = \sum_{p_1, \dots, p_n \in P} \alpha^n_{p_1 \dots p_n} x_{p_1} \diamond \dots \diamond x_{p_n}$ define a projective observable $a \in L^1_{\mathrm{eff}}(X)$. Notation:

$$a = \int_{\mathfrak{M}^n} x^{\otimes n} \mathrm{d}\alpha^n = \int_{\mathfrak{M}^n} x(\mathfrak{m}_1) \diamond \dots \diamond x(\mathfrak{m}_n) \mathrm{d}\alpha^n(\mathfrak{m}_1, \dots, \mathfrak{m}_n).$$

A whole Wick calculus can be developed using the $\mathcal{S}$-transform.

Definition

The $\mathcal{S}$-transform of $a \in L^1_{\mathrm{eff}}(X)$ is

$$\mathcal{S}a(\xi) = \hat{\mu}(\xi)^{-1}\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi x}a\right], \quad \xi \in \operatorname{inj\,lim} X_P^*.$$

Proposition

If $a, b \in L^1_{\mathrm{eff}}(X)$ are Wick polynomials, then $\mathcal{S}(a \diamond b) = \mathcal{S}(a)\mathcal{S}(b)$.

Definition

Whenever it makes sense, we define

$$f^{\diamond}(a) = \mathcal{S}^{-1}f(\mathcal{S}a), \quad f : \mathbb{R} \to \mathbb{R}.$$

Let μ be Gaussian white noise.

Problem

Find coefficients $\alpha_{p_1 p_2}$ such that $\exp^{\diamond}(-T)\mu$, $T = \int_{\mathfrak{M}^2} x^{\otimes 2} \mathrm{d}\alpha$, is the Gaussian measure $C\mathrm{e}^{-\frac{1}{2}\int (|\nabla x|^2 + x^2)\mathrm{d}m}\mathrm{d}x$.

Solution

Easily done by equating the characteristic functions. One gets

$$\alpha_{p_1 p_2} = \left\langle p_1, \left((-\Delta + 1)^{-1} - 1\right) p_2 \right\rangle .$$

What if μ is Poisson or Γ white noise? Consider the (possibly signed) measure $\exp^{\diamond}(-T)\mu$ where T is as above.

Proposition

$\exp^{\diamond}(-T)\mu$ is a positive measure.

Remarks

- The model is Euclidean invariant by construction.
- Spatial dimension plays no role.

Let μ be Gaussian white noise again. Now consider the measure

$$\exp^{\diamond}(-T - V)\mu, \quad V = \int_{\mathfrak{M}} x(\mathfrak{m})^{\diamond 4} \mathrm{d}\alpha(\mathfrak{m})$$

where $\alpha_p = |p|$ and T is as above.

Proposition

The corresponding characteristic function is, formally,

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi x} \exp^{\diamond}(-T - V)\right] = \mathrm{e}^{-\int \xi(m)^4 \mathrm{d}m} \mathrm{e}^{-\frac{1}{2}\langle \xi, (-\Delta+1)^{-1}\xi\rangle}.$$

Thus, $\exp^{\diamond}(-T - V)\mu$ is not positive (Schoenberg), but it is *reflection positive.*

Finally, let μ be any Lévy noise. Identifying projections $p \in L^\infty(M)$ with their essential supports, define

$$T = \int_{\mathfrak{M}^2} x^{\otimes 2} \mathrm{d}\alpha, \quad \alpha_{p_1 p_1} = \begin{cases} -\operatorname{vol}^{d-1}(p_1 \cap p_2) & p_1 \neq p_2 \\ \operatorname{vol}^{d-1}(\partial p_1) & p_1 = p_2 \end{cases}$$

- The matrix $(\alpha_{p_1 p_2})_{p_1, p_2 \in P}$ is, up to a multiplicative constant, the finite difference Laplacian. T can be understood as a renormalized (up to first order) kinetic energy.
- The effective measure $\mathbb{E}\left[\mathrm{e}^{-T_Q} \,\middle|\, \pi_{QP}\right] \mu_P$ diverges as Q gets finer. Higher-order renormalization is needed.
- The "fully renormalized" version $\exp^\diamond(-T)\mu$ is not positive, but again it is (formally) reflection positive.

Thanks for your attention!

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