Index theorems for hyperbolic operators and particle creation

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The Gauß-Bonnet Theorem

Theorem (C.F. Gauß, P.O. Bonnet, 1827–1848)

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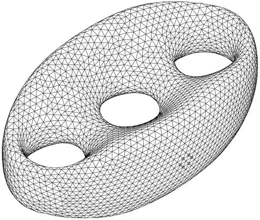
Here $\chi(S)$ is the Euler-number of S:

$$\chi(S) = \#\text{triangles} - \#\text{edges} + \#\text{vertices} = -2g + 2$$



http://commons.wikimedia.org/wiki/File:Tri-brezel.png#/media/File:Tri-brezel.png

The Gauß-Bonnet-Chern Theorem

For curved spaces M of higher dimension n (Riemannian manifolds).

Theorem (C.F. Gauß, P.O. Bonnet, S.-S. Chern, 1945)

$$(2\pi)^{-n/2} \int_M \mathrm{Pf}(\Omega) = \chi(M)$$

Hodge-deRham-Theorie

exterior differential

$$d : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k+1})$$

codifferential:

$$\delta : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k-1})$$

together the form the Euler-operator:

$$d + \delta : C^\infty(M, \Lambda^{\text{even}}) \to C^\infty(M, \Lambda^{\text{odd}})$$

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Therefore:

$$(2\pi)^{-n/2} \int_M \text{Pf}(\Omega) = \text{ind}(d + \delta)$$

Atiyah-Singer Index Theorem

- M closed Riemannian manifold,
- $D : C^\infty(M, E) \to C^\infty(M, F)$ elliptic first order operator

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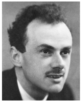
basic example (if a spin structure is given):
Dirac-Operator $D : C^\infty(M,S^+M) \to C^\infty(M,S^-M)$



$$D = \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j}$$

Atiyah-Singer Index Theorem

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Example Dirac-Operator

$$\mathrm{ind}(D) = \int_M \widehat{\mathrm{A}}(\Omega) \wedge \mathrm{ch}(E)$$

E is a twist bundle.

Dimension 4

The Atiyah-Singer integrand in dimension 4 (twisted case):

$$\hat{A} \wedge \mathrm{ch}(E)(x) = \frac{1}{(2\pi)^2} \left(\mathrm{tr}(F^2) + \frac{1}{48} \mathrm{tr}(R^2) \right) .$$

$$\mathrm{tr}(F^2) = \frac{1}{4} \epsilon^{abcd} \mathrm{tr}(F_{ab} F_{cd}),$$

similarly for R.

Spectral Boundary Conditions

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A_0 is an elliptic self-adjoint operator on ∂M.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projection

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APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

Atiyah-Patodi-Singer Index Theorem

Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

imposing APS boundary conditions we have

$$\mathrm{ind}(D) = \int_M \text{alg. expr.}(\Omega) + \int_{\partial M} \text{alg. expr}(\Omega, 2.\ FF) - \frac{h(A_0) + \eta(A_0)}{2}$$

where

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$, and $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

Lorentzian Manifolds

Replace „space “ by „space-times “,
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	model	signature	natural op.	Dirac-op.
	Euclid	positive definite	Laplace $\sum_{j=1}^{n} \partial_j^2$	elliptic
	Minkowski	$(n-1, 1)$	d'Alembert $-\partial_1^2 + \sum_{j=2}^{n} \partial_j^2$	hyperbolic

Closedness?

Problem 1: compact Lorentzian manifolds (without boundary) violate causality conditions
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$\Rightarrow$ not suitable for models in GR

Problem 2: hyperbolic PDE-Theory does not work on such space-times
$\Rightarrow$ no Lorentzian analog of the Atiyah-Singer Theorem.

The Lorentzian Index Theorem

Let M be a compact globally hyperbolic space-time
with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$
Σ_j smooth spacelike Cauchy surfaces
D Dirac-Operator

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Theorem (A. S. , C. Bär, 2015)

With APS-boundary conditions D is a Fredholm operator. Its kernel consists of smooth spinors and one has

$$\mathrm{ind}(D) = \int_M \widehat{\mathrm{A}}(\Omega) + \int_{\partial M} T\widehat{\mathrm{A}}(\Omega, 2.\ FF) - \frac{h(A_1) + h(A_2) + \eta(A_1) - \eta(A_2)}{2}$$

Physical Interpretation: One particle picture

Wave Evolution Operator $Q : C^\infty(\Sigma_1) \to C^\infty(\Sigma_2)$:
For $\varphi \in C^\infty(\Sigma_1)$ solve $D\Phi = 0$ with initial condition $\Phi|_{\Sigma_1} = \varphi$.
Then $Q\varphi = \Phi|_{\Sigma_2}$.
Q extends to a unitary operator $L^2(\Sigma_1) \to L^2(\Sigma_2)$.

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Decompose

$$Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix}$$

wr.t. the splitting

$$L^2(\Sigma_1) = L^2_{[0,\infty)}(\Sigma_1) \oplus L^2_{(-\infty,0)}(\Sigma_1)\,,$$
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Then we have

$$\mathrm{ind}(D) = \dim\ker(Q_{--}) - \dim\ker(Q_{++})$$

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Shale-Stinespring: Q is implementable if Q_{+-} is Hilbert-Schmidt (the case in dim 4).

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The sector of charge N is mapped to sector with charge $N + \mathrm{ind}(D)$.
Normal ordering in Fock space depends on the state (i.e. on the Cauchy surface).

Previous work

- Gibbons ('79), Gibbons and Richer ('80), Lohiya ('83) in explicit models

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- Matsui ('90), Bunke and Hirschmann ('92), $\Sigma_1 = \Sigma_2$
- Delbourgo and Salam ('72) , Dowker ('78), Zahn ('15), non-concervation of axial current

Methods

- evolution operator Q is a Fourier Integral Operator
- compute principal symbol using the calculus by Duistermaat and Hörmander.

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- evolution operator Q is a Fourier Integral Operator
- compute principal symbol using the calculus by Duistermaat and Hörmander.
- define the appropriate function spaces using the theory of hyperbolic PDEs and wavefront sets
- use spectral flow and to transform to usual APS theorem
- Feynman parametrix constructed using a gluing contruction

Thanks for your attention

& Happy Birthday