The Spectral Action, Heat Coefficients and Loop Quantum Gravity

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joint work with

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The set-up

Consider n-dimensional Riemannian manifolds (M, g), $\dim(M) \geq 3$.
General orthogonal connection for vector fields has the form

$$\nabla_X Y = \nabla^g_X Y + A(X, Y)$$

with ∇^g Levi-Civita connection, A a $(2, 1)$-tensor field.

The torsion $(3, 0)$-tensor

Define torsion $(3, 0)$-tensor field

$$A_{XYZ} = g(A(X, Y), Z)$$

for any $X, Y, Z \in T_p M$

Relation to torsion (2, 1)-tensor $\tilde{A}$

$$\tilde{A}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = A(X, Y) - A(Y, X).$$

Curvatures:

- $\mathrm{Riem}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$
  in general: no Bianchi identity for Riem
- $\mathrm{ric}(X, Y) = \mathrm{tr}_g\big(V \mapsto \mathrm{Riem}(V, X)Y\big)$
  in general: ric not symmetric
- $R = \mathrm{tr}_g(\mathrm{ric})$

For general connection $\nabla_X Y = \nabla^g_X Y + A(X, Y)$:

∇ is orthogonal $\Longleftrightarrow$ $A(X, \cdot)$ is skew-adjoint

The space of all torsion $(3,0)$-tensors:

$$\mathcal{T}(T_pM) = \left\{ A \in \bigotimes^3 T_p^*M \mid A_{XYZ} = -A_{XZY} \quad \forall X, Y, Z \in T_pM \right\}$$

Theorem (E. Cartan, 1925)

One has the following decomposition of $\mathcal{T}(T_pM)$ into irreducible $O(T_pM)$-representations:

$$\mathcal{T}(T_pM) = \mathcal{V}(T_pM) \oplus \mathcal{A}(T_pM) \oplus \mathcal{C}(T_pM).$$

(orthogonal decomposition w.r.t. natural scalar product)

Where we define the vectorial torsion

$$\begin{aligned}\mathcal{V}(T_pM) &= \{A \in \mathcal{T}(T_pM) \mid \exists V \text{ s.t. } \forall X, Y, Z : \\ &\qquad A_{XYZ} = g(X,Y)\cdot g(V,Z) - g(X,Z)\cdot g(V,Y)\},\end{aligned}$$

the totally anti-symmetric torsion

$$\mathcal{A}(T_pM) = \{A \in \mathcal{T}(T_pM) \mid \forall X, Y, Z : A_{XYZ} = -A_{YXZ}\},$$

and the torsion of Cartan-type

$$\begin{aligned}\mathcal{C}(T_pM) &= \{A \in \mathcal{T}(T_pM) \mid \forall X, Y, Z : A_{XYZ} + A_{YZX} + A_{ZXY} = 0 \\ &\qquad \text{and } \sum_{a=1}^{4} A(e_a, e_a, Z) = 0\}\end{aligned}$$

with $e_1, \dots e_n$ orthonormal basis of $T_p(M)$.

Corollary

For any orth. connection $\nabla_X Y = \nabla^g_X Y + A(X, Y)$ exist a unique vector field V, a unique 3-form T and a unique $(3, 0)$-tensor field C with $C_p \in \mathcal{C}(T_p M)$ such that

$$A(X, Y) = g(X, Y)V - g(V, Y)X + T(X, Y, \cdot)^\sharp + C(X, Y, \cdot)^\sharp.$$

Remark

Let ∇ be compatible with the Riemannian metric g.
Then ∇ has same geodesics as ∇^g if and only if $A(X, Y) = T(X, Y, \cdot)^\sharp$ is totally anti-symmetric.

The spinor connection

Now let (M, g) be a spin manifold. Let $e_1, .., e_n$ be an orthonormal frame and $\psi \in \Sigma M$ a spinor field.
The spinor connection induced by ∇ is locally given by

$$\nabla_X \psi = \nabla^g_X \psi + \tfrac{1}{2} \sum_{i<j} A(X, e_i, e_j)\, e_i \cdot e_j \cdot \psi.$$

The Dirac operator D for ∇

$$\begin{aligned} D\psi &= D^g\psi + \tfrac{1}{4} \sum_{i,j,k} A(e_i, e_j, e_k)\, e_i \cdot e_j \cdot e_k \cdot \psi \\ &= D^g\psi + \tfrac{3}{2} T \cdot \psi - \tfrac{n-1}{2} V \cdot \psi, \end{aligned}$$

D^g Dirac operator for ∇^g.

Remark

∇ is compatible with the hermitian metric on the spinor bundle ΣM and compatible with Clifford multiplication.

Theorem (S. Sulanke, T. Friedrich 1979)

The Dirac operator D associated to ∇ is symmetric iff $\operatorname{div}^{\nabla} Z = \operatorname{div}^{\nabla^g} Z$ for any vector field Z.
Equivalent to: vectorial torsion of ∇ is zero.

Lemma

The Dirac operator is independent of the Cartan type component $\mathcal{C}(T_pM)$ of the torsion.

Chamseddine-Connes Spectral Action

The bosonic spectral action for Dirac operator D is the number of eigenvalues of $D^* D$ in the interval $[0, \Lambda^2]$ (with $\Lambda \in \mathbb{R}$):

$$\mathcal{I} = \mathrm{Tr}\, f\left(\frac{D^* D}{\Lambda^2}\right),$$

where Tr is the L^2-trace over the space of spinor fields, and f is cut-off function with support in $[0, +1]$ which is constant near 0.
Spectral action on left- (or right-)handed Spinors:

$$\mathcal{I}_{L/R} = \mathrm{Tr}\, f\left(P_{L/R}\frac{D^* D}{\Lambda^2}\right),$$

where $P_{L/R} = \frac{1}{2}(\mathrm{id} \mp \gamma_5)$.

Asymptotic expansion of Spectral Action

Define $\Delta_{L/R} := P_{L/R} D^* D$.
From the heat trace asymptotics for $t \to 0$

$$\mathrm{Tr}\left(e^{-t\,\Delta_{L/R}}\right) \sim \sum_{n \geq 0} t^{n-2} a_{2n}(\Delta_{L/R})$$

(with Seeley-deWitt coefficients $a_{2n}(\Delta_{L/R})$)
one gets (by $t = \Lambda^{-2}$) an asymptotics for the spectral action

$$\mathcal{I}_{L/R} \sim \Lambda^4 \, f_4 \, a_0(\Delta_{L/R}) + \Lambda^2 \, f_2 \, a_2(\Delta_{L/R}) + \Lambda^0 \, f_0 \, a_4(\Delta_{L/R})$$

as $\Lambda \to \infty$ with f_4, f_2, f_0 moments of cut-off function f.

The square of the Dirac operator

Bochner formula for D

$$\begin{aligned} D^* D\psi &= \Delta\psi + \tfrac{1}{4} R^g\,\psi + \tfrac{3}{2} dT\cdot\psi - \tfrac{3}{4}\|T\|^2\,\psi \\ &\quad + \tfrac{3}{2}\,\mathrm{div}^g(V)\,\psi - \tfrac{9}{2}\,\|V\|^2\,\psi \\ &\quad + 9\Big(T\cdot V\cdot\psi + (V\lrcorner T)\cdot\psi\Big) \end{aligned}$$

where $\Delta = \widetilde{\nabla}^*\widetilde{\nabla}$ is the Laplacian associated to spin connection

$$\widetilde{\nabla}_X\psi = \nabla^g_X\psi + \tfrac{3}{2}(X\lrcorner T)\cdot\psi - \tfrac{3}{2} V\cdot X\cdot\psi - \tfrac{3}{2}\,g(V,X)\,\psi,$$

Spectral Action $\mathcal{I}_L$ up to Λ^2

$$a_2(D^*D) = \frac{1}{16\pi^2}\int_M -\frac{1}{3}R^g + 3\|T\|^2 + 18\|V\|^2 dvol$$

$$a_2(\gamma_5 D^*D) = \frac{1}{16\pi^2}\int_M 6dT + 36\langle T, *V^\flat\rangle_3 dvol$$

$$\mathcal{I}_L = \frac{\Lambda^4 f_4}{2}\int_M dvol + \frac{\Lambda^2 f_2}{32\pi^2}\int_M -\frac{1}{3}R^g + 3\|T\|^2 + 18\|V\|^2 dvol + \frac{\Lambda^2 f_2}{32\pi^2}\int_M 6dT + 36\langle T, *V^\flat\rangle_3\, dvol$$

The Holst density

Let $dim(M) = 4$ and $\theta^1, ..., \theta^4$ be the dual basis of $e_1, ..., e_4$. Define the Holst density

$$\rho_H = \sum_{a,b=1}^{4} g(\mathrm{Riem}(\cdot,\cdot)e_a, e_b) \wedge \theta^a \wedge \theta^b$$

For any orthogonal connection ∇ without Cartan-type torsion one finds

$$\rho_H = 6\, dT + 12\, \langle T, *V^\flat \rangle_3\, dvol$$

where $\langle \cdot, \cdot \rangle_3$ is the scalar product on 3-forms induced by g.

The Holst action

Let M be a 4-dim. manifold with Riemannian metric g and connection ∇ without Cartan-type torsion. The Holst action is

$$
\begin{aligned}
\mathcal{I}_H &= \frac{1}{16\pi G}\int_M \left(-R\,dvol + \frac{1}{\gamma}\,\rho_H\right) \\
&= -\frac{1}{16\pi G}\int_M \left(R^g - 6\,\|V\|^2 - \|T\|^2\right) dvol \\
&\quad + \frac{1}{16\pi G}\int_M \frac{1}{\gamma}\left(6dT + 12\langle T, *V^\flat\rangle_3\right) dvol.
\end{aligned}
$$

Loop Quantum Gravity

$\mathcal{I}_H$ is one of the starting points for LQG.
The parameter γ is called Barbero-Immirzi parameter.

Spectral Action $\mathcal{I}_L$ up to Λ^2

$$a_2(D^*D) = \frac{1}{16\pi^2}\int_M -\frac{1}{3}R(3T,3V)dvol$$

$$a_2(\gamma_5 D^*D) = \frac{1}{48\pi^2}\int_M \rho_H(3T,3V)dvol$$

$$\mathcal{I}_L = \frac{\Lambda^4 f_4}{2}\int_M dvol - \frac{\Lambda^2 f_2}{96\pi^2}\int_M R(3T,3V)dvol + \frac{\Lambda^2 f_2}{96\pi^2}\int_M \rho_H(3T,3V)\, dvol$$

Specific particle model

The ingredients:

- internal Hilbert space: $\mathcal{H}_f$ with connection $\nabla^{\mathcal{H}_f}$
- full Hilbert space: $\mathcal{H}_{SM} = L^2(M, S) \otimes \mathcal{H}_f \ni \psi \otimes \chi$
- twisted connection (with totally anti-symmetric torsion): $\widehat{\nabla}^{SM} = \nabla \otimes \mathrm{id}_{\mathcal{H}_f} + \mathrm{id}_S \otimes \nabla^{\mathcal{H}_f}$
- associated Dirac operator: $D^{\widehat{\nabla}^{SM}}$
- field Φ of endomorphisms of $\mathcal{H}_f$
  (encodes Higgs boson φ, Yukawa couplings, etc.)

The generalised Dirac operator

$$D_\Phi(\psi \otimes \chi) = D^{\widehat{\nabla}^{SM}}(\psi \otimes \chi) + \gamma_5 \psi \otimes \Phi(\chi)$$

Fermion-doubling and the "physical" Dirac operator

The twisted Hilbert space $\mathcal{H}_{SM} = L^2(M,S) \otimes \mathcal{H}_f$ has too many degrees of freedom (fermion doubling problem).

$$\begin{aligned}
\mathcal{H}_{SM} &= (L^2(M,S)^- \oplus L^2(M,S)^+) \otimes (\mathcal{H}_f^\ell \oplus \mathcal{H}_f^r) \\
&= L^2(M,S)^- \otimes \mathcal{H}_f^\ell \oplus L^2(M,S)^+ \otimes \mathcal{H}_f^r \\
&\quad \oplus L^2(M,S)^+ \otimes \mathcal{H}_f^\ell \oplus L^2(M,S)^- \otimes \mathcal{H}_f^r.
\end{aligned}$$

Project with P onto the "physical" Hilbert space

$$\mathcal{H}_{SM}^{phys.} = L^2(M,S)^- \otimes \mathcal{H}_f^\ell \oplus L^2(M,S)^+ \otimes \mathcal{H}_f^r$$

And define

$$\Delta := P D_\Phi^* D_\Phi$$

The $a_2(\Delta)$-term)

$$\begin{aligned} a_2(\Delta) &= -\frac{\mathrm{rk}(\mathcal{H}_{SM})}{96\pi^2}\int_M R(3T,3V)\,dx + \frac{1}{\gamma}\rho_H(3T,3V) \\ &\quad -\frac{1}{8\pi^2}\int_M \mathrm{tr}(\Phi^2)dx \end{aligned}$$

with Barbero-Immirzi parameter $|\gamma| = \left|\frac{\mathrm{rk}(\mathcal{H}_{SM})}{\mathrm{tr}(\gamma_f)}\right| \geq 1$

The $a_4(\Delta)$-term

$$
\begin{aligned}
a_4(\Delta) \;=\; & \frac{1}{16\pi^2}\int_M \left(\frac{5}{12}\,\mathrm{tr}\left(\Omega_f^2\right) + \mathrm{tr}\left([\nabla^{\mathcal{H}_f},\Phi]^2\right) + \mathrm{tr}(\Phi^4)\right) dx \\
& -\frac{3\,\mathrm{rk}(\mathcal{H}_{SM})}{64\pi^2}\int_M \frac{1}{30}\|W\|^2\,dx - \left(\|\delta T\|^2 + \|dV^\flat\|^2\right) dx \\
& +\frac{1}{96\pi^2}\int_M \left(R^g - 9\|T\|^2\right)\,\mathrm{tr}(\Phi^2)dx \\
& +\frac{\mathrm{tr}(\gamma_f)}{1152\pi^2}\int_M R(3T,3V)\rho_H(3T,3V) \\
& +\frac{11\,\mathrm{rk}(H_{SM})}{1440}\chi(M) - \frac{\mathrm{tr}(\gamma_f)}{96}p_1(M) + \frac{1}{96\pi^2}\int_M \mathrm{tr}\left(*\,\Omega_f\Omega_f\right)dx
\end{aligned}
$$

The Bosonic Spectral Action

$$
\begin{aligned}
\mathcal{I}_{SM} \;=\; & \frac{\mathrm{rk}(\mathcal{H}_{SM})\,\Lambda^4 f_4}{8\pi^2}\int_M dx - \frac{\mathrm{rk}(\mathcal{H}_{SM})\,\Lambda^2 f_2}{96\pi^2}\int_M R(3T,3V)\,dx \\
& - \frac{\mathrm{rk}(\mathcal{H}_{SM})\,\Lambda^2 f_2}{96\pi^2}\int_M \frac{1}{\gamma}\rho_H(3T,3V) - \frac{\Lambda^2 f_2}{8\pi^2}\int_M \mathrm{tr}(\Phi^2)dx \\
& \frac{f_0}{16\pi^2}\int_M \left(\frac{5}{12}\,\mathrm{tr}\left(\Omega_f^2\right) + \mathrm{tr}\left([\nabla^{\mathcal{H}_f},\Phi]^2\right) + \mathrm{tr}(\Phi^4)\right) dx \\
& - \frac{3\,\mathrm{rk}(\mathcal{H}_{SM})\,f_0}{64\pi^2}\int_M \frac{1}{30}\|W\|^2\,dx - \left(\|\delta T\|^2 + \|dV^\flat\|^2\right) dx \\
& + \frac{f_0}{96\pi^2}\int_M \left(R^g - 9\|T\|^2\right)\,\mathrm{tr}(\Phi^2)dx \\
& + \frac{\mathrm{tr}(\gamma_f)\,f_0}{1152\pi^2}\int_M R(3T,3V)\rho_H(3T,3V) + \text{top. terms}
\end{aligned}
$$

The full Standard Model action

$$\mathcal{I}_{SM} = \mathrm{Tr}\, f\left(\frac{PD_\Phi^* D_\Phi}{\Lambda^2}\right) + \langle \Psi, D_\Phi \Psi \rangle_{\mathcal{J}} \quad \text{with} \quad \Psi \in \mathcal{H}_{SM}^{phys.}$$

Observations

- Vectorial & totally anti-symm. torsion couple to fermions.
- Totally anti-symm. torsion couples to the Higgs field.
- Existence of the derivative terms of T and V.
  $\rightsquigarrow$ Torsion becomes dynamical
- Barbero-Immirzi parameter measures left-right-asymmetry of the particle model: $\mathrm{tr}(\gamma_f) = \mathrm{rk}\,\mathcal{H}_f^r - \mathrm{rk}\,\mathcal{H}_f^\ell$.

Current Projects

- Manifolds with boundaries
  (B. Iochum, C. Levy, D. Vassilevich, A. Connes,...)
- Spectral triples /spectral action in Lorentzian signature
  (R. Verch, A. Rennie, M. Eckstein, N. Franco, C.S.,...)

Some Questions

- Critical points of $\mathcal{I}_{SM}$ with non-zero torsion?
- Dynamical torsion? (see also Tolksdorf)
- Spectral triples for non-symmetric Dirac operators?
- Low Energy value of the Barbero-Immirzi parameter? Renormalisation-group running? (see Reuter et al., Benedetti & Speziale)