Non-Equilibrium Thermodynamics and Conformal Field Theory

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Based on a joint work with S. Hollands

and previous works with Bischoff, Kawahigashi, Rehren and Camassa, Tanimoto, Weiner

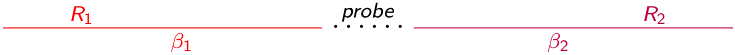
General frame

Non-equilibrium thermodynamics: study physical systems not in thermodynamic equilibrium but basically described by thermal equilibrium variables. Systems, in a sense, near equilibrium.

Non-equilibrium thermodynamics has been effectively studied for decades with important achievements, yet the general theory still missing. The framework is even more incomplete in the quantum case, *non-equilibrium quantum statistical mechanics*.

We aim provide a general, model independent scheme for the above situation in the context of quantum, two dimensional *Conformal Quantum Field Theory*. As we shall see, we provide the general picture for the evolution towards a *non-equilibrium steady state*.

A typical frame described by Non-Equilibrium Thermodynamics:



Two infinite reservoirs R_1, R_2 in equilibrium at their own temperatures $T_1 = \beta_1^{-1}$, $T_2 = \beta_2^{-1}$, and possibly chemical potentials μ_1, μ_2, are set in contact, possibly inserting a probe.

As time evolves, the system should reach a non-equilibrium steady state.

This is the situation we want to analyse. As we shall see the *Operator Algebraic approach to CFT* provides a model independent description, in particular of the asymptotic steady state, and exact computation of the expectation values of the main physical quantities.

Thermal equilibrium states

Gibbs states

Finite system, $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}\, e^{itH}$. Equilibrium state φ at inverse temperature β given by

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H} X)}{\mathrm{Tr}(e^{-\beta H})}$$

KMS states (Haag, Hugenholtz, Winnink)

Infinite volume, $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$, $\mathfrak{B}$ a dense *-subalgebra. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{B}$ $\exists$ $F_{XY} \in A(S_\beta)$ s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$
(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

where $A(S_\beta)$ is the algebra of functions analytic in the strip $S_\beta = \{0 < \Im z < \beta\}$, bounded and continuous on the closure $\bar{S}_\beta$.

Non-equilibrium steady states

Non-equilibrium statistical mechanics:

A *non-equilibrium steady state* **NESS** φ of $\mathfrak{A}$ satisfies property (a) in the KMS condition, for all X, Y in a dense *-subalgebra of $\mathfrak{B}$, but not necessarily property (b).

For any X, Y in $\mathfrak{B}$ the function

$$F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$$

is the boundary value of a function holomorphic in S_β. (Ruelle)

Example: the tensor product of two KMS states at temperatures β_1, β_2 is a NESS with $\beta = \min(\beta_1, \beta_2)$.

Problem: describe the NESS state φ and show that the initial state ψ evolves towards φ

$$\lim_{t\to\infty} \psi \cdot \tau_t = \varphi$$

Möbius covariant nets (Haag-Kastler nets on S^1)

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

Consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I)=B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I),\Omega)$, are

$$U(\delta_I(2\pi t))=\Delta_I^{it},\ \ t\in\mathbb{R}, \qquad \text{dilations}$$
$$U(r_I)=J_I \qquad \text{reflection}$$

  (Fröhlich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)'=\mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I\subset\cup_i I_i\implies\mathcal{A}(I)\subset\vee_i\mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Local conformal nets

$\mathrm{Diff}(S^1) \equiv$ group of orientation-preserving smooth diffeomorphisms of S^1

$\mathrm{Diff}_I(S^1) \equiv \{g \in \mathrm{Diff}(S^1) : g(t) = t \ \forall t \in I'\}$.

A local conformal net $\mathcal{A}$ is a Möbius covariant net s.t.

F. *Conformal covariance*. $\exists$ a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb s.t.

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}_{I'}(S^1), \end{aligned}$$

$\longrightarrow$ unitary representation of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$[L_n, c] = 0$, $L_n^* = L_{-n}$.

Representations

A (DHR) *representation* ρ of local conformal net $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \rho_I$, with ρ_I a normal rep. of $\mathcal{A}(I)$ on $B(\mathcal{H})$ s.t.

$$\rho_{\tilde{I}} \restriction \mathcal{A}(I) = \rho_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

ρ is diffeomorphism *covariant*: $\exists$ a projective unitary representation U_ρ of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ such that

$$\rho_{gI}(U(g)xU(g)^*) = U_\rho(g)\rho_I(x)U_\rho(g)^*$$

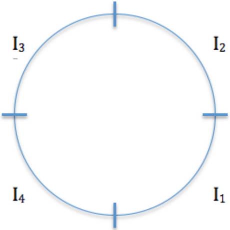
for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \mathrm{Diff}(S^1)$.

Index-statistics relation (L.):

$$d(\rho) = \left[\rho_{I'}\big(\mathcal{A}(I')\big)' : \rho_I\big(\mathcal{A}(I)\big)\right]^{\frac{1}{2}}$$

$$\text{DHR dimension} = \sqrt{\text{Jones index}}$$

Complete rationality



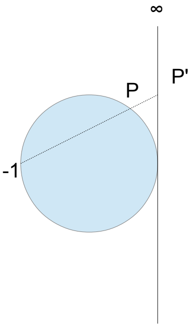
$$\mu_{\mathcal{A}} \equiv \Big[\big(\mathcal{A}(I_1) \vee \mathcal{A}(I_3)\big)' : \big(\mathcal{A}(I_2) \vee \mathcal{A}(I_4)\big) \Big] < \infty$$

$\Longrightarrow$

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

$\mathcal{A}$ *is modular* (Kawahigashi, Müger, L.)

Circle and real line picture



$$z \mapsto i\frac{z-1}{z+1}$$

We shall frequently switch between the two pictures.

KMS and Jones index

Kac-Wakimoto formula (conjecture)

Let $\mathcal{A}$ be a conformal net, ρ representations of $\mathcal{A}$, then

$$\lim_{t \to 0^+} \frac{\mathrm{Tr}(e^{-tL_{0,\rho}})}{\mathrm{Tr}(e^{-tL_0})} = d(\rho)$$

Analog of the Kac-Wakimoto formula (theorem)

ρ a representation of $\mathcal{A}$:

$$(\xi, e^{-2\pi K_\rho}\xi) = d(\rho)$$

where $K\rho$ is the generator of the dilations δ_I and ξ is any vector cyclic for $\rho(\mathcal{A}(I'))$ such that $(\xi, \rho(\cdot)\xi)$ is the vacuum state on $\mathcal{A}(I')$.

$U(1)$-current net

Let $\mathcal{A}$ be the local conformal net on S^1 associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_{\mathbb{R}}(\mathbb{R}),\ \mathrm{supp}\, f \subset I\}''$$

where W is the representation of the Weyl commutation relations

$$W(f)W(g) = e^{-i\int fg'}W(f+g)$$

associated with the vacuum state ω

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty p|\tilde{f}(p)|^2 \mathrm{d}p$$

where $\tilde{f}$ is the Fourier transform of f.

$$W(f) = \exp\Big(-i \int f(x) j(x) dx \Big)$$

$$\big[j(f), j(g) \big] = i \int f g' dx$$

There is a one parameter family $\{\gamma_q, q \in \mathbb{R}\}$ of irreducible sectors and all have index 1 (Buchholz, Mack, Todorov)

$$\gamma_q(W(f)) \equiv e^{i \int F f} W(f), \quad F \in C^\infty, \quad \frac{1}{2\pi} \int F = q \ .$$

q is the called the charge of the sector.

A classification of KMS states (Camassa, Tanimoto, Weiner, L.)

How many KMS states do there exist?

Completely rational case

$\mathcal{A}$ completely rational: only one KMS state (geometrically constructed) $\beta = 2\pi$
exp: net on $\mathbb{R}$ $\mathcal{A} \to$ restriction of $\mathcal{A}$ to $\mathbb{R}^+$

$$\text{exp} \restriction \mathcal{A}(I) = \mathrm{Ad}\, U(\eta)$$

η diffeomorphism, $\eta \restriction I =$ exponential

$$\text{geometric KMS state on } \mathcal{A}(\mathbb{R}) = \text{vacuum state on } \mathcal{A}(\mathbb{R}^+) \ \circ \ \text{exp}$$

$$\varphi_{\mathrm{geo}} = \omega \circ \text{exp}$$

Note: Scaling with dilation, we get the geometric KMS state at any give $\beta > 0$.

Comments

About the proof:

Essential use of the *thermal completion* and *Jones index*.

$\mathcal{A}$ net on $\mathbb{R}$, φ KMS state:

In the GNS representation we apply Wiesbrock theorem

$$\mathcal{A}(\mathbb{R}^+) \subset \mathcal{A}(\mathbb{R}) \text{ hsm modular inclusion} \to \text{new net } \mathcal{A}_\varphi$$

Want to prove duality for $\mathcal{A}_\varphi$ in the KMS state, but $\mathcal{A}_\varphi$ satisfies duality up to finite Jones index.

Iteration of the procedure...

Conjecture: $\mathcal{A} \subset \mathcal{B}$ finite-index inclusion of conformal nets, $\varepsilon : \mathcal{B} \to \mathcal{A}$ conditional expectation. If φ is a translation KMS on $\mathcal{A}$ then $\varphi \circ \varepsilon$ is a translation KMS on $\mathcal{B}$.

Non-rational case: $U(1)$-current model

The primary (locally normal) KMS states of the $U(1)$-current net are in one-to-one correspondence with real numbers $q \in \mathbb{R}$; each state φ^q is uniquely determined by

$$\varphi^q\left(W\left(f\right)\right) = e^{iq\int f\,dx} \cdot e^{-\frac{1}{4}\|f\|^2_{S_\beta}}$$

where $\|f\|^2_{S_\beta} = (f, S_\beta f)$ and $\widehat{S_\beta f}(p) := \coth\frac{\beta p}{2}\widehat{f}\,(p)$.

In other words:

Geometric KMS state: $\varphi_{\rm geo} = \varphi^0$

Any primary KMS state:

$$\varphi^q = \varphi_{\rm geo} \circ \gamma_q.$$

where γ_q is a BMT sector.

Virasoro net: $c = 1$

(With $c < 1$ there is only one KMS state: the net is completely rational)

Primary KMS states of the Vir_1 net are in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ is uniquely determined by its value on the stress-energy tensor T:

$$\varphi^{|q|}(T(f)) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f\, dx.$$

The geometric KMS state corresponds to $q = 0$, and the corresponding value of the 'energy density' $\frac{\pi}{12\beta^2} + \frac{q^2}{2}$ is the lowest one in the set of the KMS states.

(We construct these KMS states by composing the geometric state with automorphisms on the larger $U(1)$-current net.)

Virasoro net: $c > 1$

There is a set of primary (locally normal) KMS states of the Vir_c net with $c > 1$ w.r.t. translations in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ can be evaluated on the stress-energy tensor

$$\varphi^{|q|}(T(f)) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f \, dx$$

and the geometric KMS state corresponds to $q = \frac{1}{\beta}\sqrt{\frac{\pi(c-1)}{6}}$ and energy density $\frac{\pi c}{12\beta^2}$.

Are they all? Probably yes...

Rotation KMS states: Recent work with Y. Tanimoto

Chemical potential

$\mathcal{A}$ a local conformal net on $\mathbb{R}$ (or on M) and φ an extremal β-KMS state on $\mathfrak{A}$ w.r.t. the time translation group τ and ρ an irreducible DHR localized endomorphism of $\mathfrak{A} \equiv \overline{\cup_{I\subset\mathbb{R}}\mathcal{A}(I)}$ with finite dimension $d(\rho)$. Assume that ρ is normal, namely it extends to a normal endomorphism of the weak closure $\mathcal{M}$ of $\mathfrak{A}$; automatic e.g. if φ satisfies essential duality $\pi_\varphi\big(\mathfrak{A}(I_\pm)\big)' \cap \mathcal{M} = \pi_\varphi\big((\mathfrak{A}(I_\mp)\big)''$, $I_\pm$ the $\pm$half-line.

U time translation unitary covariance cocycle in $\mathfrak{A}$:

$$\mathrm{Ad}U(t)\cdot\tau_t\cdot\rho = \rho\cdot\tau_t \ , \quad t\in\mathbb{R} \ ,$$

with $U(t+s) = U(t)\tau_t\big(U(s)\big)$ (cocycle relation) (unique by a phase, canonical choice by Möb covariance).

U is equal up to a phase to a Connes Radon-Nikodym cocycle:

$$U(t) = e^{-i2\pi\mu_\rho(\varphi)t} d(\rho)^{-i\beta^{-1}t}\big(D\varphi\cdot\Phi_\rho : D\varphi\big)_{-\beta^{-1}t} \ .$$

$\mu_\rho(\varphi)\in\mathbb{R}$ is the *chemical potential* of φ w.r.t. the charge ρ.

Here Φ_ρ is the left inverse of ρ, $\Phi_\rho \cdot \rho = \mathrm{id}$, so $\varphi \cdot \Phi_\rho$ is a KMS state in the sector ρ.

The geometric β-KMS state φ_0 has zero chemical potential.

By the holomorphic property of the Connes Radon-Nikodym cocycle:

$$e^{2\pi\beta\mu_\rho(\varphi)} = \underset{t \longrightarrow i\beta}{\mathrm{anal.\,cont.}}\, \varphi\big(U(t)\big) / \underset{t \longrightarrow i\beta}{\mathrm{anal.\,cont.}}\, \varphi_0\big(U(t)\big) \ .$$

Example, BMT sectors:

With $\varphi_{\beta,q}$ the β-state associated withe charge q, the chemical potential w.r.t. the charge q is given by

$$\mu_p(\varphi_{\beta,q}) = qp/\pi$$

2-dimensional CFT

$M = \mathbb{R}^2$ Minkowski plane.

$\begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix}$ conserved and traceless stress-energy tensor.

As is well known, $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields,

$$T_L = T_L(t + x), \quad T_R = T_R(t - x).$$

Left and right movers.

Ψ_k family of conformal fields on M: T_{ij} + *relatively local fields*
$\mathcal{O} = I \times J$ double cone, I, J intervals of the chiral lines $t \pm x = 0$

$$\mathcal{A}(\mathcal{O}) = \{e^{i\Psi_k(f)}, \mathrm{supp} f \subset \mathcal{O}\}''$$

then by relative locality

$$\mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J)$$

$\mathcal{A}_L, \mathcal{A}_R$ chiral fields on $t \pm x = 0$ generated by T_L, T_R and other chiral fields

(completely) rational case: $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O})$ finite Jones index

Phase boundaries (Bischoff, Kawahigashi, Rehren, L.)

$M_L \equiv \{(t,x) : x < 0\}$, $M_R \equiv \{(t,x) : x > 0\}$ left and right half Minkowski plane, with a CFT on each half.

Chiral components of the stress-energy tensor:

$$T_+^L(t+x),\, T_-^L(t-x),\, T_+^R(t+x),\, T_-^R(t-x).$$

Energy conservation at the boundary ($T_{01}^L(t,0) = T_{01}^R(t,0)$):

$$T_+^L(t) + T_-^R(t) = T_+^R(t) + T_-^L(t).$$

Transmissive solution:

$$T_+^L(t) = T_+^R(t), \qquad T_-^L(t) = T_-^R(t).$$

A transpartent phase boundary is given by specifying two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on $M_{L/R}$ on the same Hilbert space $\mathcal{H}$;

$$M_L \supset O \mapsto \mathcal{B}^L(O) \ ; \qquad M_R \supset O \mapsto \mathcal{B}^R(O) \ ,$$

$\mathcal{B}^L$ and $\mathcal{B}^R$ both contain a common chiral subnet $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$.

$\mathcal{B}^{L/R}$ extends on the entire M by covariance as the chiral nets $\mathcal{A}_\pm$ on $\mathbb{R}$ contain the Virasoro nets.

By causality:

$$\left[\mathcal{B}^L(O_2), \mathcal{B}^R(O_1)\right] = 0, \quad O_1 \subset M_L,\ O_2 \subset M_R,\ O_1 \subset O_2'$$

By diffeomorphism covariance, $\mathcal{B}^R$ *is thus right local with respect to* $\mathcal{B}^L$

Given a phase boundary, we consider the von Neumann algebras generated by $\mathcal{B}^L(O)$ and $\mathcal{B}^R(O)$:

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O)\ , \quad O \in \mathcal{K}\ .$$

$\mathcal{D}$ is another extension of $\mathcal{A}$, but $\mathcal{D}$ is in general non-local, but relatively local w.r.t. $\mathcal{A}$. $\mathcal{D}(O)$ may have non-trivial center. In the completely rational case, $\mathcal{A}(O) \subset \mathcal{D}(O)$ has finite Jones index, so the center of $\mathcal{D}(O)$ is finite dimensional; by standard arguments, we may cut down the center to $\mathbb{C}$ by a minimal projection of the center, and we may then assume $\mathcal{D}(O)$ to be a factor, as we will do for simplicity in the following.

The universal construction

A phase boundary is a transmissive boundary with chiral observables $\mathcal{A}_{2\mathrm{D}} = \mathcal{A}_+ \otimes \mathcal{A}_-$. The phases on both sides of the boundary are given by a pair of Q-systems $A^L = (\Theta^L, W^L, X^L)$ and $A^R = (\Theta^R, W^R, X^R)$ in the sectors of $\mathcal{A}_{2\mathrm{D}}$, describing local 2D extensions $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^L_{2\mathrm{D}}$ and $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^R_{2\mathrm{D}}$.
Now consider the braided product Q-systems (Evans, Pinto)

$$(\Theta = \Theta^L \circ \Theta^R, W = W^L \times W^R, X = (1 \times \epsilon^{\pm}_{\Theta^L,\Theta^R} \times 1) \circ (X^L \times X^R))$$

and the corresponding extensions $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}}$. The original extensions $\mathcal{B}^L_{2\mathrm{D}}$, $\mathcal{B}^R_{2\mathrm{D}}$ are intermediate

$$\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^L_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}} \qquad \mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^R_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}},$$

and the nets $\mathcal{D}^{\pm}_{2\mathrm{D}}$ are generated by $\mathcal{A}_{2\mathrm{D}}$ and two sets of charged fields $\Psi^L_{\sigma\otimes\tau}$ ($\sigma \otimes \tau \prec \Theta^L$) and $\Psi^R_{\sigma\otimes\tau}$ ($\sigma \otimes \tau \prec \Theta^R$), suppressing possible multiplicity indices.

The braided product Q-system determines their commutation relations among each other:

$$\Psi^R_{\sigma\otimes\tau}\Psi^L_{\sigma'\otimes\tau'} = \epsilon^{\pm}_{\sigma'\otimes\tau',\sigma\otimes\tau}\cdot\Psi^L_{\sigma'\otimes\tau'}\Psi^R_{\sigma\otimes\tau}.$$

$\epsilon^{-}_{\sigma'\otimes\tau',\sigma\otimes\tau} = \mathbf{1}$ whenever $\sigma'\otimes\tau'$ is localized to the spacelike left of $\sigma\otimes\tau$. Thus, the choice of $\pm$-braiding ensures that $\mathcal{B}^L$ is left-local w.r.t. $\mathcal{B}^R$, as required by causality. Thus

$$\Theta = (\Theta^L, W^L, X^L)\times^{-}(\Theta^R, W^R, X^R),$$

Universal construction:

The extension $\mathcal{D}$ of $\mathcal{A}$ defined by the above Q-system implements a transmissive boundary condition in the sense. It is universal in the sense that every irreducible boundary condition appears as a representation of $\mathcal{D}$.

Cf. the work of Fröhlich, Fuchs, Runkel, Schweigert (Euclidean setting)

Non-equilubrium in CFT (S. Hollands, R.L.)

Two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on the Minkowski plane M, both containing the same chiral net $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. For the moment $\mathcal{B}^{L/R}$ is completely rational, so the KMS state is unique, later we deal wih chemical potentials.

Before contact. The two systems $\mathcal{B}^L$ and $\mathcal{B}^R$ are, separately, each in a thermal equilibrium state. KMS states $\varphi^{L/R}_{\beta_{L/R}}$ on $\mathfrak{B}^{L/R}$ at inverse temperature $\beta_{L/R}$ w.r.t. τ, possibly with $\beta_L \neq \beta_R$.

$\mathcal{B}^L$ and $\mathcal{B}^R$ live independently in their own half plane M_L and M_R and their own Hilbert space. The composite system on $M_L \cup M_R$ is given by

$$M_L \supset O \mapsto \mathcal{B}^L(O)\,, \qquad M_R \supset O \mapsto \mathcal{B}^R(O)$$

with C^*-algebra $\mathfrak{B}^L(M_L) \otimes \mathfrak{B}^R(M_R)$ and the state

$$\varphi = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(M_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(M_R)}\ ;$$

φ is a stationary state, NESS but not KMS.

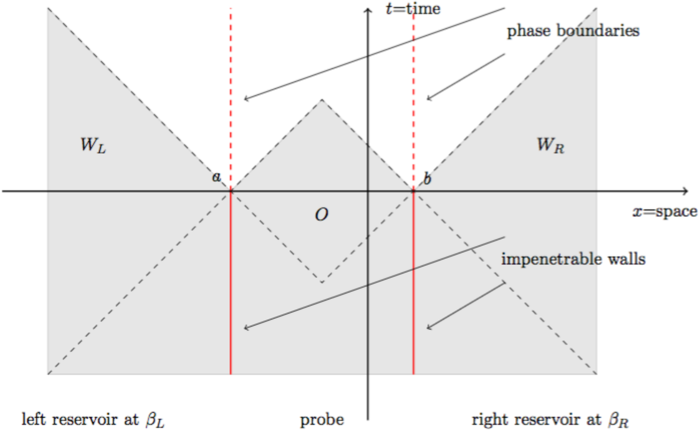


Figure 1: Spacetime diagram of our setup. The initial state ψ is set up in the shaded region before the system is in causal contact with the phase boundaries. In the shaded regions to the left/right of the probe, we have a thermal equilibrium state at inverse temperatures β_L/β_R. In the diamond shaped shaded region O, we have an essentially arbitrary probe state.

After contact.

At time $t = 0$ we put the two systems $\mathcal{B}^L$ on M_L and $\mathcal{B}^R$ on M_R in contact through a totally transmissible phase boundary and the time-axis the defect line. We are in the phase boundary case, with $\mathcal{B}^L$ and $\mathcal{B}^R$ now nets on M acting on a common Hilbert space $\mathcal{H}$.
With $O_1 \subset M_L$, $O_2 \subset M_R$ double cones, the von Neumann algebras $\mathcal{B}^L(O_1)$ and $\mathcal{B}^R(O_2)$ commute if O_1 and O_2 are spacelike separated, so $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ commute.
We want to describe the state ψ of the global system after time $t = 0$. As above, we set

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O)$$

The origin $\mathbf{0}$ is the only $t = 0$ point of the defect line; the observables localized in the causal complement $W_L \cup W_R$ of the $\mathbf{0}$ thus do not feel the effect of the contact, so ψ should be a natural state on $\mathfrak{D}$ that satisfies

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)}, \quad \psi|_{\mathfrak{B}^R(W_R)} = \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)} \ .$$

In particular, ψ is to be a *local thermal equilibrium state* on $W_{L/R}$ in the sense of Buchholz.

Since $\mathfrak{B}^L(M_L)$ and $\mathfrak{B}^R(M_R)$ are not independent, the existence of such state ψ is not obvious. Clearly the C^*-algebra on $\mathcal{H}$ generated by $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ is naturally isomorphic to $\mathfrak{B}^L(W_L) \otimes \mathfrak{B}^R(W_R)$ ($\mathfrak{B}^L(W_L)''$ and $\mathfrak{B}^R(W_R)''$ are commuting factors) and the restriction of ψ to it is the product state $\varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)}$.

Construction of the doubly scaling automorphism:
Let $\mathcal{C}$ be a conformal net on $\mathbb{R}$. Given $\lambda_-, \lambda_+ > 0$, there exists an automorphism α of the C^*-algebra $\mathfrak{C}(\mathbb{R} \smallsetminus \{0\})$ or $\mathfrak{D}(\check{M})$ such that

$$\alpha|_{\mathfrak{C}(-\infty,0)} = \delta_{\lambda_-} \ , \quad \alpha|_{\mathfrak{C}(0,\infty)} = \delta_{\lambda_+} \ ,$$

Then we construct an automorphism on the C^*-algebra $\mathfrak{D}(x \pm t \neq 0)$

$$\alpha|_{\mathfrak{D}(W_L)} = \delta_{\lambda_L} \ , \quad \alpha|_{\mathfrak{D}(W_R)} = \delta_{\lambda_R} \ .$$

where δ_λ is the λ-dilation automorphism of $\mathfrak{A}_\pm(\mathbb{R})$.

There exists a natural state $\psi \equiv \psi_{\beta_L,\beta_R}$ on $\mathfrak{D}(x \pm t \neq 0)$ such that $\psi|_{\mathfrak{B}(W_{L/R})}$ is $\varphi^{L/R}_{\beta_L/\beta_R}$.

The state ψ is given by $\psi \equiv \varphi \cdot \alpha_{\lambda_L,\lambda_R}$, where φ is the geometric state on $\mathfrak{D}$ (at inverse temperature 1) and $\alpha = \alpha_{\lambda_L,\lambda_R}$ is the above automorphism with $\lambda_L = \beta_L^{-1}$, $\lambda_R = \beta_R^{-1}$.

It is convenient to extend the state ψ to a state on $\mathfrak{D}$ by the Hahn-Banach theorem. *By inserting a probe ψ the state will be normal.*

The large time limit. Waiting a large time we expect the global system to reach a stationary state, a non equilibrium steady state. The two nets $\mathcal{B}^L$ and $\mathcal{B}^R$ both contain the same net $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. And the chiral net $\mathcal{A}_\pm$ on $\mathbb{R}$ contains the Virasoro net with central charge $c_\pm$. In particular $\mathcal{B}^L$ and $\mathcal{B}^R$ share the same stress energy tensor.

Let $\varphi^+_{\beta_L}$, $\varphi^-_{\beta_R}$ be the geometric KMS states respectively on $\mathfrak{A}_+$ and $\mathfrak{A}_-$ with inverse temperature β_L and β_R; we define

$$\omega \equiv \varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R} \cdot \varepsilon \ ,$$

so ω is the state on $\mathfrak{D}$ obtained by extending $\varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R}$ from $\mathfrak{A}$ to $\mathfrak{D}$ by the conditional natural expectation $\varepsilon : \mathfrak{D} \to \mathfrak{A}$. Clearly ω is a stationary state, indeed:

ω is a NESS on $\mathfrak{D}$ with $\beta = \min\{\beta_L, \beta_R\}$.

We now want to show that the evolution $\psi \cdot \tau_t$ of the initial state ψ of the composite system approaches the non-equilibrium steady state ω as $t \to +\infty$.

Note that:

$$\psi|_{\mathcal{D}(O)} = \omega|_{\mathcal{D}(O)} \text{ if } O \in \mathcal{K}(V_+)$$

We have:

For every $Z \in \mathfrak{D}$ we have:

$$\lim_{t \to +\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \ .$$

Indeed, if $Z \in \mathcal{D}(O)$ with $O \in \mathcal{K}(M)$ and $t > t_O$, we have $\tau_t(Z) \in \mathfrak{D}(V_+)$ as said, so

$$\psi\big(\tau_t(Z)\big) = \omega\big(\tau_t(Z)\big) = \omega(Z) \ , \quad t > t_O \ ,$$

because of the stationarity property of ω.
See the picture.

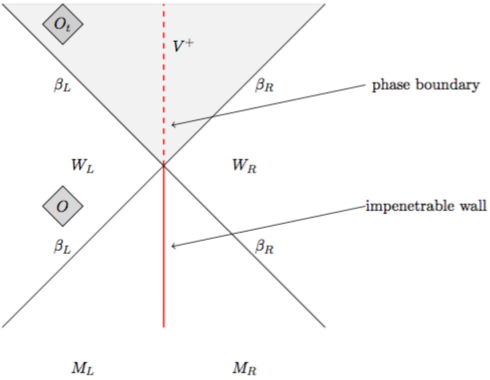


Figure 2: Spacetime diagram of simplified setup. There is just one phase boundary and no probe. Every time-translated diamond will eventually enter the future lightcone V^+.

Case with chemical potential

We suppose here that $\mathcal{A}_\pm$ in the net $\mathcal{C}$ contains is generated by the $U(1)$-current $J^\pm$ (thus $\mathcal{B}^{L/R}$ is non rational with central charge $c = 1$).

Given $q \in \mathbb{R}$, the β-KMS state $\varphi_{\beta,q}$ on $\mathfrak{D}$ with charge q is defined by

$$\varphi_{\beta,q} = \varphi^{+}_{\beta,q} \otimes \varphi^{-}_{\beta,q} \cdot \varepsilon \ ,$$

where $\varphi^{\pm}_{\beta,q}$ is the KMS state on $\mathcal{A}_\pm$ with charge q.

$\varphi_{\beta,q}$ satisfies the β-KMS condition on $\mathfrak{D}$ w.r.t. to τ.

Similarly as above we have:

Given $\beta_{L/R} > 0$, $q_{L/R} \in \mathbb{R}$, there exists a state ψ on $\mathfrak{D}$ such that

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi_{\beta_L,q_L}|_{\mathfrak{B}^L(W_L)} \ , \qquad \psi|_{\mathfrak{B}^R(W_R)} = \varphi_{\beta_R,q_R}|_{\mathfrak{B}^R(W_R)} \ .$$

and for every $Z \in \mathfrak{D}$ we have:

$$\lim_{t\to+\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \ .$$

We can explicitly compute the expected value of the asymptotic NESS state ω on the stress energy tensor and on the current(chemical potential enters):

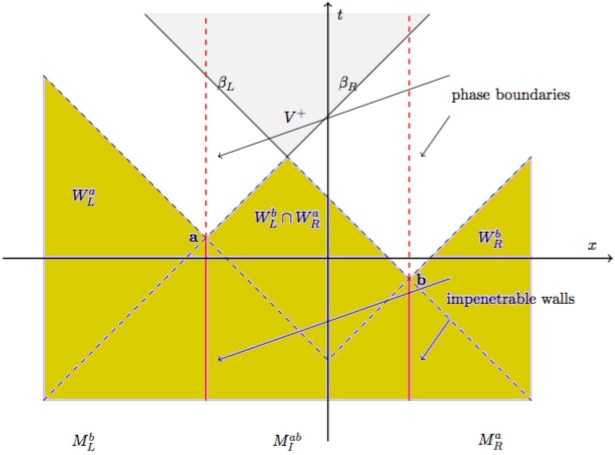


Figure 3: Spacetime diagram of our setup. The initial state ψ is set up in the shaded region before the system is in causal contact with the phase boundaries. In W_L^a resp. W_R^b, we have a thermal equilibrium state at inverse temperatures β_L resp. β_R. In the diamond shaped shaded region $W_L^b \cap W_R^a$, we have an essentially arbitrary probe state. Again, every time-translated causal diamond in this region will eventually enter V^+.

Now $\omega = \varphi^{+}_{\beta_L,q_L} \otimes \varphi^{-}_{\beta_R,q_R} \cdot \varepsilon$ is a steady state is a NESS and ω is determined uniquely by $\beta_{L/R}$ and the charges $q_{L/R}$

$$\varphi^{+}_{\beta_L,q_L}\big(J^{+}(0)\big) = q_L \ , \qquad \varphi^{-}_{\beta_R,q_R}\big(J^{-}(0)\big) = q_R \ .$$

We also have

$$\varphi^{+}_{\beta_L,q_L}\big(T^{+}(0)\big) = \frac{\pi}{12\beta_L^2} + \frac{q_L^2}{2} \ , \qquad \varphi^{-}_{\beta_R,q_R}\big(T^{-}(0)\big) = \frac{\pi}{12\beta_R^2} + \frac{q_R^2}{2} \ .$$

In presence of chemical potentials $\mu_{L/R} = \frac{1}{\pi} q_{L/R}$, the large time limit of the two dimensional current density expectation value (x-component of the current operator J^μ) in the state ψ is, with $J^x(t,x) = J^-(t+x) - J^+(t-x)$

$$\lim_{t\to+\infty} \psi\big(J^x(t,x)\big) = \varphi^{-}_{\beta_L,q_L}\big(J^{-}(0)\big) - \varphi^{+}_{\beta_R,q_R}\big(J^{+}(0)\big) = -\pi(\mu_L - \mu_R) \ ,$$

whereas on the stress energy tensor

$$\lim_{t\to+\infty} \psi\big(T_{tx}(t,x)\big) = \varphi^{+}_{\beta_L,q_L}\big(T^{+}(0)\big) - \varphi^{-}_{\beta_R,q_R}\big(T^{-}(0)\big)$$
$$= \frac{\pi}{12}\big(\beta_L^{-2} - \beta_R^{-2}\big) + \frac{\pi^2}{2}\big(\mu_L^2 - \mu_R^2\big) \ ,$$

(cf. Bernard-Doyon)