Operator product expansion algebra

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based on joint work with M. Fröb, J. Holland and Ch. Kopper

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“[...] At this time an idea occurred to me which at first I considered to be mainly of aesthetic value but which turned out to be so fertile that its elaborations and applications determined the direction of my work for many years. [...] My conclusion was that the theory must give us for each region of space-time an algebra corresponding to the set of all observables or operations pertaining to the region. This correspondence between space-time regions and algebras is the content of the theory; nothing more nor less. Relativistic causality demands that the algebras of two regions which lie space-like to each other should commute. In the case of a field theory the algebra of a region is generated by the fields “smeared out” by test functions with support in the region.” [R. Haag: “Some people and some problems met in half a century...” Eur. Phys. J. H. 35 (2010)]
The idea to formulate quantum theory in an “algebraic manner” had been proposed already by I. Segal in 1946 [Segal 1946]. NEW IDEAS:

1st idea: Segal did not associate different algebras to different Minkowski regions, i.e. a map $N \mapsto \mathcal{A}(N)$. Special to the relativistic setting.

2nd idea: $\mathcal{A}(N)$ should be “abstract” algebras. In theory with charges

$$\mathcal{H} = \bigoplus_q \mathcal{H}_q$$

charge $q$ “superselection sector”

Then on each $\mathcal{H}_q$ the algebra acts in a different representation $\pi_q$ and total representation of $\mathcal{A}$ is “diagonal”

$$\pi(\mathcal{A}) = \begin{pmatrix} \cdots & \pi_q(\mathcal{A}) & \pi_{q+1}(\mathcal{A}) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$\implies$ redundant description.
In 1964 Haag and Kastler publish their influential paper which proposes these two ideas. While the 1st idea is well-motivated, they seemed to have settled on the 2nd idea due to their discovery of a mathematical result in the literature (which Haag attributes to Kastler [see “Some people and some problems...”]). This result [Fell 1960] states, in simple terms, that, given $n$ local observables $O_1, \ldots, O_n$, one can approximate (for all $i = 1, \ldots, n$)

$$\text{tr}(\rho_q O_i)$$

statistical operator in charge $q$ Hilbert space

to arbitrary accuracy $\varepsilon$ by some statistical operator in charge-0 Hilbert space

$$\text{tr}(\rho_0^{(\varepsilon)} O_i)$$

statistical operator in charge 0 Hilbert space

$\Rightarrow$ finitely many local operations cannot distinguish “representation”. 
In 1964, Wilson proposes his “operator product expansion”:

An alternative is proposed to specific Lagrangian models [...] operator products at the same point have no meaning. [...] a generalization of equal time commutation relations is assumed: Operator products at short distances have expansions at short distances involving local field multiplying singular functions [...] [K. Wilson: “Non-Lagrangian models of current algebra” PR 179 (1969)]

Rather than by conceptual thinking as Haag-Kastler, Wilson is influenced by ideas about “current algebras” [Gell-Mann 1962, Lee, Weinberg & Zumino 1967] that are influential around this time. Later, [Zimmermann 1972] shows that Wilsons proposals are indeed consistent with renormalized perturbation theory.

Actually, the Haag-Kastler proposal is also consistent with renormalized perturbation theory [Brunetti & Fredenhagen 1999]
Comparison

Despite obvious differences in motivation, technical setting, etc. there exist several obvious parallels between the OPE proposed by Wilson and the ideas of AQFT proposed by Haag-Kastler

- Both frameworks emphasize algebraic relations between observables (elements of an abstract $C^*$-algebra here, local point like quantum field there) are independent of the state and the representation. In AQFT-framework, this is because the algebras are to be “abstractly defined”. In the OPE, the coefficients do not depend on state.

- Both frameworks emphasize (and exploit) that there is a freedom of choosing the “generators” of the algebraic structure. In OPE: field redefinitions

- Neither framework in principle requires Lagrangian formulation

- Both frameworks emphasize that “equal time” algebraic relations are unsuitable in QFT.

- Relationship between both approaches was clarified by [Bostelmann 2008]
Further developments

Haag-Kastler nets:

- Superselection structure, braid statistics, ... [Doplicher-Haag-Robers 60s-90s, Fredenhagen-Rehren-Schroer 90s, Buchholz-Fredenhagen 1982, Buchholz-Roberts 2015]

- Relationship with sub factor theory [Longo 90s-]

- Classification of conformal QFTs in $d = 2$ [Kawahigashi, Longo, ... 00s-]


- ... (this conference: Lechner, Longo, Reidei)

Operator product expansion:

- In 1970s, various groups [Polyakov 1974, Mack 1977, Gatto et al. 1973, Schroer et al. 1974] realize that the OPE simplifies in CFTs and associativity constraints can be turned into “conformal bootstrap” recently: numerics, see e.g. [Rychkov 2016].

- In 1980s, OPE to study conformal field theories in $d = 2$ [Belavin et al. 1984].

- Borcherds and others propose to formalize their ideas in the framework of Vertex Operator Algebras [Borcherds 1988]
Technical challenges of QFT

Unfortunately, if they mathematically exist, QFTs must be rather complicated presumably in any approach/framework.

“In those years a theoretical physicist of modest talent could harvest results of great interest, today persons of great talent produce results of modest interest” [F. Hund]

BASIC REASONS

▶ One can show quite generally that $\mathcal{O}(x)$ at a sharp point $x$ is a meaningless object (probability distribution has infinite fluctuations). One must think of $\mathcal{O}(x)$ as operator valued distribution.

▶ It is not possible to identify in $\mathcal{H}$ subspaces associated with a definite localization in $x$-space: The set of vectors $\mathcal{O}(x)|0\rangle$ as $\mathcal{O}(x)$ ranges over composite fields spans entire Hilbert space! [Reeh-Schlieder 1968]

▶ $\mathcal{O}(x)|0\rangle$ contains arbitrarily many particles when there is interaction ⇒ situation worse than in non-relativistic $N$-body systems

The inherent technical complications implied by these properties have so far strongly impeded progress in establishing the mathematical existence of interesting QFTs in $d = 4$ dimensions.
An intrinsically “generally covariant” formulation of QFT can be given via algebraic methods, e.g. by formulating QFT via Operator Product Expansion [Hollands-Wald 2012]. A quantum field theory consists of:

- A list of quantum fields \( \{ \mathcal{O}_A \} \), where \( A \) is a label (incl. tensor/spinor indices)
- A state \( \Psi \) is an expectation value functional characterized by \( N \)-point “functions” \( \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \rangle \Psi \). Such a functional should be “positive” \( \rightarrow \) probability interpretation!
- \( N \)-point functions should satisfy a “micro local spectrum condition”
- The OPE should hold for a wide class of states \( \Psi \)

\[
\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \rangle \Psi = \sum_B C_{A_1 \cdots A_N}^{B}(x_1, \ldots, x_N) \langle \mathcal{O}_B(x_N) \rangle \Psi
\]

- The OPE coefficients are independent of \( \Psi \).
- The OPE coefficients should be generally covariant functionals of the metric \( g_{\mu \nu} \).
- The OPE should satisfy associativity law.
Example: Free field

For a free scalar field theory in $d = 4$ dimensions with action $\int |\partial \phi|^2$, the basic OPE relation is

$$
\phi(x_1) \phi(x_2) = \frac{\lambda}{|x_1 - x_2|^2} \cdot 1
$$

$$
+ \phi^2(x_2) + \sum \frac{(x_1 - x_2)^{\mu_1} \ldots (x_1 - x_2)^{\mu_N}}{N!} \phi \partial_{\mu_1 \ldots \mu_N} \phi(x_2)
$$

(0.3)

The composite fields such as $\mathcal{O} = \phi^2$ are defined by this equation. Other composite fields $\mathcal{O} = \phi^4, \phi^3 \nabla_\mu \phi, \ldots$ similarly occur in OPE of $\phi^2$, etc.

Everything is constrained by associativity.

So in this theory one has, e.g.

$$
C_{AB}^C = \frac{\lambda}{|x_1 - x_2|^2}
$$

when $\mathcal{O}_A = \mathcal{O}_B = \phi$, $\mathcal{O}_C = 1$, etc. In curved spacetime the distances $|x_1 - x_2|$ in the coefficients are replaced by geometric quantities related to the theory of geodesics.
In conformal field theory \((d = 4)\) on flat spacetime \(\mathbb{R}^4\), it is natural to group composite fields into “multiplets” transforming under the conformal group \(O(4, 2)\). Each multiplet contains a “primary field” \(\mathcal{O}\), together with its “descendants”, which are roughly given by \(\partial_{\mu_1} \ldots \partial_{\mu_N} \mathcal{O}\).

E.g. \(\phi^2\) is a primary field, \(\phi \partial_{\mu} \phi\) a descendant. The OPE between two primary fields \(\mathcal{O}_A, \mathcal{O}_B\) takes the form

\[
\mathcal{O}_A(x_1)\mathcal{O}_B(x_2) = \sum_{\text{primary } C} \frac{\lambda_{AB}^C}{|x_1 - x_2|^{|\Delta_A + \Delta_B - \Delta_C|}} \mathcal{P}(x_1 - x_2, \partial)\mathcal{O}_C(x_2)
\]

where \(\mathcal{P} = \mathcal{P}_{AB}^C\) is a (pseudo-) differential operator that is determined completely by group theoretical considerations [Schroer & Swieca 1974]. Thus the content of the theory is determined by (i) structure constants \(\lambda_{AB}^C\) and (ii) dimensions \(\Delta_A\).

Associativity+OS positivity put very stringent conditions on these data \(\rightarrow\) conformal bootstrap [Mack 1977, Polyakov 1974, Dolan-Osborne 2000, ..., present].
It is natural to ask:

1. How to compute OPE coefficients $C_{AB}^{C}$... (even in principle) beyond free field or CFTs?
2. In what sense does associativity hold in general?
3. What is the magnitude of the “remainder” in the OPE (=error term)?
4. Can one devise an axiomatic framework for QFT in terms of OPE?

In this talk, I will give some answers to these questions.
Outline

1. How to construct the OPE coefficients
2. OPE factorisation
3. OPE convergence
General idea

- For free field theories (e.g. free scalar field theory with action $S_{CFT} = \int |\partial \phi|^2$) one can construct OPE directly by “Wick’s theorem.”
- CFTs in $d = 2$, one can use representation theoretic methods (Virasoro-algebra, $W$-algebras, current algebras, ...)
- For CFTs in $d > 2$ dimensions, one can use conformal bootstrap including its numerical versions [Polyakov, Mack, Gatto et al., ..., Rychkov et al., ...]
- Some progress has been made for lattice QFTs (numerical) [Monahan et al. 2013,2014]
- For perturbations of free field theories or CFTs (given intuitively by $S_{CFT} + g \int \mathcal{O}$, where $\mathcal{O}$ is some “marginal” or “relevant” operator), one can attempt to derive a differential equation for the OPE coefficients $C_{AB...C}^D$ as a function of the coupling $g$.
- This type of equation was found (and proved) by [Holland & Holland 2014], generalizing and correcting an earlier attempt by [Guida & Magnoli 1995].
To write down the action principle, use graphical notation. I draw an OPE coefficient

\[ C_{A_1 \ldots A_n}^B(x_1, \ldots, x_n) \]

as

\[ \begin{array}{c}
1 \\
2 \\
\ldots \\
n
\end{array} \]

I draw a concatenation of OPE coefficients

\[ C_{A_1 C}^B(x_1, x_n)C_{A_2 \ldots A_n}^C(x_2, \ldots, x_n) \]

as

\[ \begin{array}{c}
1 \\
2 \\
3 \\
\ldots \\
n
\end{array} \]

Attention: None of these diagrams is a “Feynman graph”!
I also write

\[ \int dy \]\n
where

- \( \mathcal{O} \) denotes the “deformation”
- \( \int dy = \text{integral over } \{|y - x_n| < L\} \)
- \( L = \text{length scale that is part of the definition of the theory.} \)
There is a kind of “action principle” for OPE coefficients if we “deform” 
\( S_{CFT} \rightarrow S_{CFT} + g \int O \):

\[
\frac{\partial}{\partial g} = \frac{1}{2} n
\]

**Figure:** Functional equation, left side. The tree represents a coefficient 
\( C^B_{A_1 \ldots A_n} (x_1, \ldots, x_n) \)

\[
\int dy y; O_1 + \cdots + \int dy y; O_n
\]

**Figure:** Functional equation, right side. The composite trees represent 
concatenations of coefficients, e.g. the rightmost tree means 
\( \sum_C C^C_{A_1 \ldots A_n} (x_1, \ldots, x_n) C^B_{O_C} (y, x_n) \)
Action principle

**Theorem (Hollands-JH)**

**To any order in** $g$:

$$
\partial_g C_{A_1 \ldots A_N}^B (x_1, \ldots, x_N) = - \int_{|y-x_N|<L} d^4 y \left[ C_{O,A_1 \ldots A_N}^B (y, x_1, \ldots, x_N) 
\right.

$$

$$
- \sum_{i=1}^{N} \sum_{[C] \leq [A_i]} C_{C,A_i}^C (y,x_i) C_{A_1 \ldots \tilde{A}_i C \ldots A_N}^B (x_1, \ldots, x_N)

$$

$$
- \sum_{[C] < [B]} C_{A_1 \ldots A_N}^C (x_1, \ldots, x_N) C_{O,C}^B (y, x_N) \right].
$$

- Can compute OPE coefficients to any perturbation order by iteration.
- State independence obvious.
- $L \rightarrow L'$ equivalent to

$$
O_A \rightarrow O'_A = \sum Z_{A}^{B}(g, \tau) \cdot O_B
$$

(1.4)

and $g \rightarrow g' = g(g, \tau)$.  $\Rightarrow$ **RG equations!** ($\tau = \log L/L' = \text{RG “time”}$).
In $\phi^4$ theory ($d = 4$), i.e. $O = -\phi^4$, one can compute OPE coefficients order by order. At each order, one $d^4 y$-integral $\Rightarrow$ at order $g^r$ we have $r$ integrations $\Rightarrow$ similar complexity as “Feynman diagram” method. But: Renormalization “automatic”.

For Gross-Neveu model ($d = 2$), i.e. $O = -(\bar{\psi}\psi)^2$, we have all order bounds on OPE coefficients. Series in $g$ seems to converge [Hollands & Holland, in prep.] $\Rightarrow$ OPE coefficients analytic functions of $g$!

For marginal perturbations of CFTs, simplification of equation to ODE.

For local gauge theories (e.g. YM-theory), there holds a similar action principle, supplemented by an “evolution equation” for the BRST-operator (as a function of $g$) [Fröb 2016]
If I assume to be given a 1-parameter families of CFTs with an exactly marginal operator $\mathcal{O}$ (i.e. $\Delta_{\mathcal{O}} = d$ in $d$ dimensions) parameterized by $g$, then action principle implies an equation of the form

\[
\frac{d}{dg} \lambda = f_{\mathcal{O}}^{\lambda}(\Delta, \lambda) \\
\frac{d}{dg} \Delta = f_{\mathcal{O}}^{\Delta}(\Delta, \lambda)
\]  

(1.5)

where $f_{\mathcal{O}}^{\Delta}$, $f_{\mathcal{O}}^{\lambda}$ are explicit (quadratic) functions that depend on $6j$-symbols of the group $O(4, 2)$ in $d = 4$ (i.e. entirely group theoretic=kinematic). Here $\lambda = \{\lambda^{C}_{AB}(g)\}$ and $\Delta = \{\Delta_{A}(g)\}$ are the CFT data which are now functions of $g$. $\mathcal{O}$ is the (marginal) perturbation of the CFT, which enters the functions. [Hollands, in prep.]
Outline

1 How to construct the OPE coefficients

2 OPE factorisation

3 OPE convergence
### Theorem (Holland-SH)

In \( \phi^4 \)-theory, any arbitrary but fixed loop order:

\[
C^B_{A_1...A_N}(x_1, \ldots, x_N) = \sum_C C^C_{A_1...A_M}(x_1, \ldots, x_M) C^B_{C_{A_{M+1}}...A_N}(x_M, \ldots, x_N)
\]

holds on the domain \( \xi \equiv \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|} < 1 \). (Sum over \( C \) abs. convergent !)

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**Bound on remainder**

**Justification of “action principle”**
The OPE factorises

**Theorem (Holland-SH)**

_in \( \phi^4 \)-theory, any arbitrary but fixed loop order:

\[
C^B_{A_1 \cdots A_N}(x_1, \ldots, x_N) = \sum_C C^C_{A_1 \cdots A_M}(x_1, \ldots, x_M) C^B_{C A_{M+1} \cdots A_N}(x_M, \ldots, x_N)
\]

**holds on the domain**

\[
\xi \equiv \frac{\max_{1 \leq i \leq M} |x_i-x_M|}{\min_{M < j \leq N} |x_j-x_M|} < 1. \text{ (Sum over } C \text{ abs. convergent !)}
\]

*For \( N = 3 \):

\[
\xi = \frac{|x_1-x_2|}{|x_2-x_3|} < 1
\]

\[
\text{for } \xi \ll 1
\]

\[
\text{for } \xi \approx 1
\]
The OPE factorises

**Theorem (Holland-SH)**

In $\phi^4$-theory, any arbitrary but fixed loop order:

$$C^B_{A_1 \ldots A_N}(x_1, \ldots, x_N) = \sum_C C^C_{A_1 \ldots A_M}(x_1, \ldots, x_M) C^B_{C_{A_{M+1}} \ldots A_N}(x_{M}, \ldots, x_N)$$

holds on the domain $\xi \equiv \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|} < 1$. (Sum over $C$ abs. convergent !)

For $N = 3$: $\xi = \frac{|x_1 - x_2|}{|x_2 - x_3|} < 1$

This shows associativity really holds!

- Bound on remainder
- Justification of “action principle”
Quantitative bound

Theorem

Up to any perturbation order \( r \in \mathbb{N} \) the bound

\[
\left| \text{Remainder in associativity} \right| \\
\leq K_r \xi^{D+1} \max_{N \leq v < N} |x_i - x_n|^{[B]} \times \frac{\prod_{v=1}^{M} \min_{1 \leq w \leq M, w \neq v} |x_v - x_w|^{[A_v]} + \delta \prod_{i=M+1}^{N} \min_{M < j \leq N, i \neq j} |x_i - x_j|^{[A_i]} + \delta}{\prod_{v=1}^{M} \min_{1 \leq w \leq M, w \neq v} |x_v - x_w|^{[A_v]} + \delta}
\]

holds for some \( \delta > 0 \) and where

\[
\xi := \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|}
\]

and where \( K_r \) is a constant which does not depend on \( D \). (Here \( [A] = \text{dim. of op. in free theory} \).)
Outline

1. How to construct the OPE coefficients
2. OPE factorisation
3. OPE convergence
Wilson proposed his expansion as an asymptotic expansion for short distances.

In CFTs, Mack showed convergence (in $d = 4$) for finite distances [Mack 1977]; for a more formal argument see also [Pappadopulo et al. 2012].

There is a difference between space like separation and light like separation.

For theories without conformal invariance, situation was unclear.
Theorem (Holland-Kopper-SH)

At any perturbation order $r$ and for any $D \in \mathbb{N}$,

$$\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} \mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \right\rangle \right|$$

OPE-Remainder

Spectator fields
Theorem (Holland-Kopper-SH)

At any perturbation order \( r \) and for any \( D \in \mathbb{N} \), there exists a \( K > 0 \) such that

\[
\left| \left< \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} C^B_{A_1 \cdots A_N}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \right> \right| 
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( KM \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \frac{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_{i} \text{dim}[A_i] + 1}{\sup_{\text{dim}[A_i] + 1} \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}}
\]

\[ M = \begin{cases} 
  m & \text{for } m > 0 \\
  \mu & \text{for } m = 0 
\end{cases} \]

mass or renormalization scale
Theorem (Holland-Kopper-SH)

At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that

\[
\left| \left< \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} C^{B}_{A_1 \cdots A_N}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \phi(p_1) \cdots \phi(p_n) \right> \right| \\
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( KM \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}
\]

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

- $|P| = \sup_i |p_i|$: maximal momentum of spectators
Bound on OPE remainder

**Theorem (Holland-Kopper-SH)**

At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that

\[
\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} \mathcal{C}_{A_1 \cdots A_N}^B(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \right\rangle \right| 
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( KM \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \frac{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \text{dim}[A_i] + 1}{\sup_{1 \leq i < j \leq N} |x_i - x_j|} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}
\]

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale
- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu, \varepsilon)$, where $\varepsilon = \min_{I \subset \{1, \ldots, n\}} |\sum_I p_i|$
  $\varepsilon$: distance of $(p_1, \ldots, p_n)$ to "exceptional" configurations
Conclusions from bound on OPE remainder

“OPE remainder” \(\leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i]+1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}\)}
Conclusions from bound on OPE remainder

\[ "OPE remainder" \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i] + 1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)} \]

1. Massive fields \((m > 0)\): Bound is finite for arbitrary \(p_1, \ldots, p_n\)
Conclusions from bound on OPE remainder

“OPE remainder” \(\leq \frac{M^{n-1}}{\sqrt{D!}} \left(\frac{KM \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j|} \right)^{D+1} \sum_i \text{dim}[A_i] + 1 \cdot \sup \left(1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}\)

1. Massive fields \((m > 0)\): Bound is finite for arbitrary \(p_1, \ldots, p_n\)

2. Massless fields: Bound is finite only for non-exceptional \(p_1, \ldots, p_n\)
Conclusions from bound on OPE remainder

\[
\text{“OPE remainder” } \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i] + 1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}
\]

1. Massive fields \((m > 0)\): Bound is finite for arbitrary \(p_1, \ldots, p_n\)
2. Massless fields: Bound is finite only for non-exceptional \(p_1, \ldots, p_n\)
3. Bound vanishes as \(D \to \infty\)

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Conclusions from bound on OPE remainder

“OPE remainder” \( \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i] + 1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)} \)

1. Massive fields \((m > 0)\): Bound is finite for arbitrary \(p_1, \ldots, p_n\)
2. Massless fields: Bound is finite only for non-exceptional \(p_1, \ldots, p_n\)
3. Bound vanishes as \(D \to \infty \Rightarrow \text{OPE converges at any finite distances!}\)

- \(\|P\|\) is large (“energy scale” of spectators)
- Maximal distance of points \(x_i\) from reference point \(x_N\) is large
- Ratio of max. and min. distances is large, e.g. for \(N = 3\)

Slow convergence

Fast convergence
Conclusions from bound on OPE remainder

“OPE remainder” \( \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i] + 1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)} \)

1. Massive fields \((m > 0)\): Bound is finite for arbitrary \(p_1, \ldots, p_n\)
2. Massless fields: Bound is finite only for non-exceptional \(p_1, \ldots, p_n\)
3. Bound vanishes as \(D \to \infty \Rightarrow \) OPE converges at any finite distances!
4. Convergence is slow if...
   - \(|P|\) is large (“energy scale” of spectators)
Conclusions from bound on OPE remainder

“OPE remainder” \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{KM \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \text{dim}[A_i] + 1} \right)^{D+1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}

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Conclusions from bound on OPE remainder

“OPE remainder” \leq \frac{M^{n-1}}{\sqrt{D!}} \cdot \left( KM \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \cdot \frac{\min_{1 \leq i < j \leq N} |x_i - x_j|}{\sum_i \dim[A_i] + 1} \cdot \sup \left( 1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}

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4. Convergence is slow if...
   - \(|P|\) is large (“energy scale” of spectators)
   - maximal distance of points \(x_i\) from reference point \(x_N\) is large
   - ratio of max. and min. distances is large, e.g. for \(N = 3\)

![Slow convergence](image1)

Slow convergence

![Fast convergence](image2)

Fast convergence
Consider now smeared spectator fields \( \varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x. \)

**Theorem (Holland-Kopper-SH)**

At any perturbation order \( r \) and for any \( D \in \mathbb{N} \),

\[
\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} C^B_{A_1 \cdots A_N}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \varphi(f_1) \cdots \varphi(f_n) \right\rangle \right|
\]

is finite for any \( f_i \in \mathcal{S}(\mathbb{R}^4) \) (Schwartz space).

Let \( \| \hat{f}_i(p) \|_s = 0 \) for \( |p| > j \),

\[
\text{Bound vanishes as } D \to 1
\]

OPE converges at any finite distances!
Consider now smeared spectator fields \( \varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x \).

**Theorem (Holland-Kopper-SH)**

At any perturbation order \( r \) and for any \( D \in \mathbb{N} \), there exists a \( K > 0 \) such that

\[
\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} c_{A_1 \ldots A_N}^{B}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \varphi(f_1) \cdots \varphi(f_n) \right\rangle \right| \\
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( K M \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \min_{1 \leq i < j \leq N} |x_i - x_j|^{\sum_i \text{dim}[A_i] + 1} \sum_{s_1 + \ldots + s_N = 0} \prod_{i=1}^n \frac{\|\hat{f}_i\|^{s_i}}{M^{s_i}}
\]

\( M: \) mass for \( m > 0 \) or renormalization scale \( \mu \) for massless fields 

\( \|\hat{f}\|_s := \sup_{p \in \mathbb{R}^4} \left| (p^2 + M^2)^s \hat{f}(p) \right| \) (Schwartz norm)
Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x$.

**Theorem (Holland-Kopper-SH)**

At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that

$$\left|\left\langle \left(\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \right) - \sum_{\dim[B] \leq D} \mathcal{C}^{B}_{A_1 \ldots A_N}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right\rangle \varphi(f_1) \cdots \varphi(f_n) \right| \leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j|} \right)^{D+1} \sum_{s_1 + \ldots + s_N = 0} \prod_{i=1}^{n} \frac{\|\hat{f}_i\|_{s_i}^{s_i}}{M^{s_i}}$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields

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1. Bound is finite for any $f_i \in S(\mathbb{R}^4)$ (Schwartz space)

OPE remainder is a tempered distribution
Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x$.

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At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that

$$
\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} \mathcal{C}_{A_1 \cdots A_N}^{B}(x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \varphi(f_1) \cdots \varphi(f_n) \right\rangle \right|
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( K \frac{M}{\min_{1 \leq i < j \leq N} |x_i - x_j|} \right)^{D+1} \prod_{s_1 + \cdots + s_N = 0} \prod_{i=1}^{n} \frac{\|\hat{f}_i\|_s^{s_i}}{M^{s_i}}
$$

$M$: mass for $m > 0$ or renormalization scale $\mu$ for massless fields

$\|\hat{f}\|_s := \sup_{p \in \mathbb{R}^4} \left| (p^2 + M^2)^s \hat{f}(p) \right|$ (Schwartz norm)

1. Bound is finite for any $f_i \in S(\mathbb{R}^4)$ (Schwartz space)
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2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$:
Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x$.

**Theorem (Holland-Kopper-SH)**

At any perturbation order $r$ and for any $D \in \mathbb{N}$, there exists a $K > 0$ such that

$$
\left| \left\langle \left( \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\text{dim}[B] \leq D} C_{A_1 \ldots A_N}^B (x_1, \ldots, x_N) \mathcal{O}_B(x_N) \right) \varphi(f_1) \cdots \varphi(f_n) \right\rangle \right|
\leq \frac{M^{n-1}}{\sqrt{D!}} \left( \frac{K M \max_{1 \leq i \leq N} |x_i - x_N|}{\min_{1 \leq i < j \leq N} |x_i - x_j|} \right)^{D+1} \sup_{\sum_i \text{dim}[A_i] + 1} \left( 1, \frac{|P|}{M} \right)^{(D+2)(r+5)}
$$

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$\|\hat{f}\|_s := \sup_{p \in \mathbb{R}^4} |(p^2 + M^2)^s \hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in S(\mathbb{R}^4)$ (*Schwartz space*)

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---

**Bound on OPE remainder II**
Consider now smeared spectator fields \( \varphi(f_i) = \int f_i(x) \varphi(x) \, d^4x \).

**Theorem (Holland-Kopper-SH)**

At any perturbation order \( r \) and for any \( D \in \mathbb{N} \), there exists a \( K > 0 \) such that

\[
\left| \left\langle \left( O_{A_1}(x_1) \cdots O_{A_N}(x_N) - \sum_{\dim[B] \leq D} C_{A_1 \cdots A_N}^{B}(x_1, \ldots, x_N) O_B(x_N) \right) \varphi(f_1) \cdots \varphi(f_n) \right\rangle \right|
\leq \frac{M^{n-1}}{\sqrt{D}!} \left( KM \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1} \min_{1 \leq i < j \leq N} |x_i - x_j| \sum_i \dim[A_i] + 1 \sup \left( 1, \frac{|P|}{M} \right)^{(D+2)(r+5)}
\]

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\( \| \hat{f} \|_s := \sup_{p \in \mathbb{R}^4} |(p^2 + M^2)^s \hat{f}(p)| \) (Schwartz norm)

1. **Bound is finite for any** \( f_i \in S(\mathbb{R}^4) \) (**Schwartz space**)
   OPE remainder is a tempered distribution

2. **Let** \( \hat{f}_i(p) = 0 \) **for** \( |p| > |P| \): **Bound vanishes as** \( D \rightarrow \infty \)
   \( \Rightarrow \) OPE converges at any finite distances!
Conclusions & Outlook

1. QFT in CST is best formulated in terms of algebraic relations + states
2. The OPE converges at finite distances in perturbation theory.
3. The OPE factorises (associativity) in perturbation theory.
4. The OPE satisfies an action principle which is also useful for calculations
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Possible Generalisations

- Gauge theories [Fröb 2016]
- Curved manifolds
- Minkowski space
- Non-perturbative constructions
- ...

[8x252]Conclusions & Outlook

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