Operator product expansion algebra

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based on joint work with M. Fröb, J. Holland and Ch. Kopper

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History

“[...] At this time an idea occurred to me which at first I considered to be mainly of aesthetic value but which turned out to be so fertile that its elaborations and applications determined the direction of my work for many years. [...] My conclusion was that the theory must give us for each region of space-time an algebra corresponding to the set of all observables or operations pertaining to the region. This correspondence between space-time regions and algebras is the content of the theory; nothing more nor less. Relativistic causality demands that the algebras of two regions which lie space-like to each other should commute. In the case of a field theory the algebra of a region is generated by the fields “smeared out” by test functions with support in the region.” [R. Haag: “Some people and some problems met in half a century...” Eur. Phys. J. H. 35 (2010)]

History

The idea to formulate quantum theory in an "algebraic manner" had been proposed already by I. Segal in 1946 [Segal 1946]. NEW IDEAS:

- 1st idea: Segal did not associate different algebras to different Minkowski regions, i.e. a map $N \mapsto \mathfrak{A}(N)$. Special to the relativistic setting.
- 2nd idea: $\mathfrak{A}(N)$ should be "abstract" algebras. In theory with charges

$$\mathcal{H} = \oplus_q \underbrace{\mathcal{H}_q}_{\text{charge } q \text{ ``superselection sector''}} \tag{0.1}$$

Then on each $\mathcal{H}_q$ the algebra acts in a different representation π_q and total representation of $\mathfrak{A}$ is "diagonal"

$$\pi(\mathfrak{A}) = \begin{pmatrix} \ddots & & & \\ & \pi_q(\mathfrak{A}) & & \\ & & \pi_{q+1}(\mathfrak{A}) & \\ & & & \ddots \end{pmatrix} \tag{0.2}$$

$\Longrightarrow$ redundant description.

History

In 1964 Haag and Kastler publish their influential paper which proposes these two ideas. While the 1st idea is well-motivated, they seemed to have settled on the 2nd idea due to their discovery of a mathematical result in the literature (which Haag attributes to Kastler [see "Some people and some problems..."]). This result [Fell 1960] states, in simple terms, that, given n local observables $\mathcal{O}_1, \ldots, \mathcal{O}_n$, one can approximate (for all $i = 1, \ldots, n$)

$$\mathrm{tr}(\underbrace{\rho_q}_{\text{statistical operator in charge } q \text{ Hilbert space}} \mathcal{O}_i)$$

to arbitrary accuracy ε by some statistical operator in charge-0 Hilbert space

$$\mathrm{tr}(\underbrace{\rho_0^{(\varepsilon)}}_{\text{statistical operator in charge } 0 \text{ Hilbert space}} \mathcal{O}_i)$$

$\Longrightarrow$ finitely many local operations cannot distinguish "representation".

History

In 1964, Wilson proposes his “operator product expansion”:

An alternative is proposed to specific Lagrangian models [...] operator products a the same point have no meaning. [...] a generalization of equal time commutation relations is assumed: Operator products at short distances have expansions at short distances involving local field multiplying singular functions [...] [K. Wilson: “Non-Lagrangian models of current algebra” PR 179 (1969)]

Rather than by conceptual thinking as Haag-Kastler, Wilson is influenced by ideas about “current algebras” [Gell-Mann 1962, Lee, Weinberg & Zumino 1967] that are influential around this time. Later, [Zimmermann 1972] shows that Wilsons proposals are indeed consistent with renormalized perturbation theory.

Actually, the Haag-Kastler proposal is also consistent with renormalized perturbation theory [Brunetti & Fredenhagen 1999]

Comparison

Despite obvious differences in motivation, technical setting, etc. there exist several obvious parallels between the OPE proposed by Wilson and the ideas of AQFT proposed by Haag-Kastler

- Both frameworks emphasize algebraic relations between observables (elements of an abstract C^*-algebra here, local point like quantum field there) are independent of the state and the representation. In AQFT-framework, this is because the algebras are to be "abstractly defined". In the OPE, the coefficients do not depend on state.
- Both frameworks emphasize (and exploit) that there is a freedom of choosing the "generators" of the algebraic structure. In OPE: field redefinitions
- Neither framework in principle requires Lagrangian formulation
- Both frameworks emphasize that "equal time" algebraic relations are unsuitable in QFT.
- Relationship between both approaches was clarified by [Bostelmann 2008]

Further developments

Haag-Kastler nets:

- Superselection structure, braid statistics, ... [Doplicher-Haag-Robers 60s-90s, Fredenhagen-Rehren-Schroer 90s, Buchholz-Fredenhagen 1982, Buchholz-Roberts 2015]
- Relationship with sub factor theory [Longo 90s-]
- Classification of conformal QFTs in $d = 2$ [Kawahigashi, Longo, ... 00s-]
- Algebraic viewpoint extremely natural for quantum field theories formulated on curved spacetimes [Kay-Wald 1990, Radzikowski 1998, Brunetti et al. 2003,...] .
- ... (this conference: Lechner, Longo, Reidei)

Operator product expansion:

- In 1970s, various groups [Polyakov 1974, Mack 1977, Gatto et al. 1973, Schroer et al. 1974] realize that the OPE simplifies in CFTs and associativity constraints can be turned into "conformal bootstrap" recently: numerics, see e.g. [Rychkov 2016].
- In 1980s, OPE to study conformal field theories in $d = 2$ [Belavin et al. 1984].
- Borcherds and others propose to formalize their ideas in the framework of Vertex Operator Algebras [Borcherds 1988]

Technical challenges of QFT

Unfortunately, if they mathematically exist, QFTs must be rather complicated presumably in any approach/framework.

"In those years a theoretical physicist of modest talent could harvest results of great interest, today persons of great talent produce results of modest interest" [F. Hund]

BASIC REASONS

- One can show quite generally that $\mathcal{O}(x)$ at a sharp point x is a meaningless object (probability distribution has infinite fluctuations). One must think of $\mathcal{O}(x)$ as operator valued distribution.
- It is not possible to identify in $\mathcal{H}$ subspaces associated with a definite localization in x-space: The set of vectors $\mathcal{O}(x)|0\rangle$ as $\mathcal{O}(x)$ ranges over composite fields spans entire Hilbert space! [Reeh-Schlieder 1968]
- $\mathcal{O}(x)|0\rangle$ contains arbitrarily many particles when there is interaction $\Rightarrow$ situation worse than in non-relativisitic N-body systems

The inherent technical complications implied by these properties have so far strongly impeded progress in establishing the mathematical existence of interesting QFTs in $d = 4$ dimensions.

Formulating QFT via operator product expansion

An intrinsically "generally covariant" formulation of QFT can be given via algebraic methods, e.g. by formulating QFT via **O**perator **P**roduct **E**xpansion [Hollands-Wald 2012]. A quantum field theory consists of:

- A list of quantum fields $\{\mathcal{O}_A\}$, where A is a label (incl. tensor/spinor indices)
- A state Ψ is an expectation value functional characterized by N-point "functions" $\langle \mathcal{O}_{A_1}(x_1) \dots \mathcal{O}_{A_N}(x_N) \rangle_\Psi$. Such a functional should be "positive" $\rightarrow$ probability interpretation!
- N-point functions should satisfy a "micro local spectrum condition"
- The OPE should hold for a wide class of states Ψ

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \rangle_\Psi = \sum_B \underbrace{\mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \rangle_\Psi$$

- The OPE coefficients are independent of Ψ.
- The OPE coefficients should be generally covariant functionals of the metric $g_{\mu\nu}$.
- The OPE should satisfy associativity law.

Example:Free field

For a free scalar field theory in $d = 4$ dimensions with action $\int |\partial\phi|^2$, the basic OPE relation is

$$\phi(x_1)\phi(x_2) = \frac{\lambda}{|x_1 - x_2|^2} \cdot 1 \\ \underbrace{+\phi^2(x_2) + \sum \frac{(x_1 - x_2)^{\mu_1} \dots (x_1 - x_2)^{\mu_N}}{N!} \phi \partial_{\mu_1 \dots \mu_N} \phi(x_2)}_{\text{smooth part}} \tag{0.3}$$

The composite fields such as $\mathcal{O} = \phi^2$ are defined by this equation. Other composite fields $\mathcal{O} = \phi^4, \phi^3 \nabla_\mu \phi, \dots$ similarly occur in OPE of ϕ^2, etc. Everything is constrained by associativity.

So in this theory one has, e.g.

$$\mathcal{C}_{AB}^{C} = \frac{\lambda}{|x_1 - x_2|^2}$$

when $\mathcal{O}_A = \mathcal{O}_B = \phi, \mathcal{O}_C = 1$, etc. In curved spacetime the distances $|x_1 - x_2|$ in the coefficients are replaced by geometric quantities related to the theory of geodesics.

Example: Conformal field theory (CFT)

In conformal field theory $(d = 4)$ on flat spacetime $\mathbb{R}^4$, it is natural to group composite fields into "multiplets" transforming under the conformal group $O(4, 2)$. Each multiplet contains a "primary field" $\mathcal{O}$, together with its "descendants", which are roughly given by $\partial_{\mu_1} \dots \partial_{\mu_N} \mathcal{O}$.
E.g. ϕ^2 is a primary field, $\phi \partial_\mu \phi$ a descendant. The OPE between two primary fields $\mathcal{O}_A, \mathcal{O}_B$ takes the form

$$\mathcal{O}_A(x_1)\mathcal{O}_B(x_2) = \sum_{\text{primary } C} \frac{\lambda_{AB}^C}{|x_1 - x_2|^{\Delta_A + \Delta_B - \Delta_C}} \mathcal{P}(x_1 - x_2, \partial)\mathcal{O}_C(x_2)$$

where $\mathcal{P} = \mathcal{P}_{AB}^C$ is a (pseudo-) differential operator that is determined <u>completely</u> by group theoretical considerations [Schroer & Swieca 1974]. Thus the content of the theory is determined by (i) structure constants λ_{AB}^C and (ii) dimensions Δ_A.
Associativity+OS positivity put very stringent conditions on these data $\to$ conformal bootstrap [Mack 1977, Polyakov 1974, Dolan-Osborne 2000,..., present].

It is natural to ask:

1. How to compute OPE coefficients $\mathcal{C}_{AB\dots}^{C}$ (even in principle) beyond free field or CFTs?
2. In what sense does associativity hold in general?
3. What is the magnitude of the “remainder” in the OPE (=error term)?
4. Can one devise an axiomatic framework for QFT in terms of OPE?

In this talk, I will give some answers to these questions.

Outline

General idea

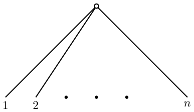
- For free field theories (e.g. free scalar field theory with action $S_{CFT} = \int |\partial\phi|^2$) one can construct OPE directly by "Wick's theorem."
- CFTs in $d = 2$, one can use representation theoretic methods (Virasoro-algebra, W-algebras, current algebras, ...)
- For CFTs in $d > 2$ dimensions, one can use conformal bootstrap including its numerical versions [Polyakov, Mack, Gatto et al., ..., Rychkov et al., ...]
- Some progress has been made for lattice QFTs (numerical) [Monahan et al. 2013,2014]
- For perturbations of free field theories or CFTs (given intuitively by $S_{CFT} + g \int \mathcal{O}$, where $\mathcal{O}$ is some "marginal" or "relevant" operator), one can attempt to derive a differential equation for the OPE coefficients $\mathcal{C}_{AB\ldots C}^{D}$ as a function of the coupling g.
- This type of equation was found (and proved) by [Holland & Holland 2014], generalizing and correcting an earlier attempt by [Guida & Magnoli 1995].

Action principle

To write down the action principle, use graphical notation. I draw an OPE coefficient

$$\mathcal{C}^{B}_{A_1\dots A_n}(x_1, \dots, x_n)$$

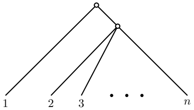
as



I draw a concatenation of OPE coefficients

$$\mathcal{C}^{B}_{A_1 C}(x_1, x_n)\mathcal{C}^{C}_{A_2\dots A_n}(x_2, \dots, x_n)$$

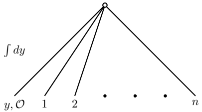
as



Attention: None of these diagrams is a “Feynman graph”!

Action principle

I also write



where

- $\mathcal{O}$ denotes the "deformation"
- $\int dy =$ integral over $\{|y - x_n| < L\}$.
- $L =$ length scale that is part of the definition of the theory.

Action principle

There is a kind of "action principle" for OPE coefficients if we "deform" $S_{CFT} \to S_{CFT} + g \int \mathcal{O}$:

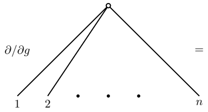


Figure: Functional equation, left side. The tree represents a coefficient $\mathcal{C}^{B}_{A_1 \dots A_n}(x_1, \dots, x_n)$

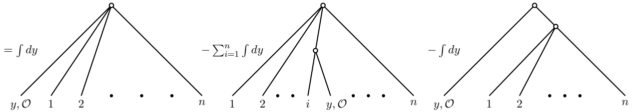


Figure: Functional equation, right side. The composite trees represent concatenations of coefficients, e.g. the rightmost tree means $\sum_C \mathcal{C}^{C}_{A_1 \dots A_n}(x_1, \dots, x_n) \mathcal{C}^{B}_{\mathcal{O}C}(y, x_n)$

Action principle

Theorem (Hollands-JH)

To any order in g:

$$\partial_g \mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N) = -\int_{|y-x_N|<L} \mathrm{d}^4 y \Big[\mathcal{C}^B_{\mathcal{O} A_1 \dots A_N}(y, x_1, \dots, x_N) - \sum_{i=1}^{N} \sum_{[C] \le [A_i]} \mathcal{C}^C_{\mathcal{O} A_i}(y, x_i)\, \mathcal{C}^B_{A_1 \dots \widehat{A_i}\, C \dots A_N}(x_1, \dots, x_N) - \sum_{[C] < [B]} \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{C}^B_{\mathcal{O} C}(y, x_N) \Big].$$

- Can compute OPE coefficients to any perturbation order by iteration.
- State independence obvious.
- $L \to L'$ equivalent to

$$\mathcal{O}_A \to \mathcal{O}'_A = \sum Z^B_A(g, \tau) \cdot \mathcal{O}_B \qquad (1.4)$$

and $g \to g' = g(g, \tau)$. $\Rightarrow$ RG equations! ($\tau = \log L/L'$ = RG "time").

Examples

- In ϕ^4 theory ($d = 4$), i.e. $\mathcal{O} = -\phi^4$, one can compute OPE coefficients order by order. At each order, one d^4y-integral $\Rightarrow$ at order g^r we have r integrations $\Rightarrow$ similar complexity as "Feynman diagram" method. But: Renormalization "automatic".
- For Gross-Neveu model ($d = 2$), i.e. $\mathcal{O} = -(\bar{\psi}\psi)^2$, we have all order bounds on OPE coefficients. Series in g seems to converge [Hollands & Holland, in prep.] $\Rightarrow$ OPE coefficients <u>analytic functions of g</u>!
- For marginal perturbations of CFTs, simplification of equation to ODE.
- For local gauge theories (e.g. YM-theory), there holds a similar action principle, supplemented by an "evolution equation" for the BRST-operator (as a function of g) [Fröb 2016]

Action principle in CFT

If I assume to be given a 1-parameter families of CFTs with an exactly marginal operator $\mathcal{O}$ (i.e. $\Delta_{\mathcal{O}} = d$ in d dimensions) parameterized by g, then action principle implies an equation of the form

$$\begin{aligned} \frac{d}{dg}\lambda &= f^{\lambda}_{\mathcal{O}}(\Delta, \lambda) \\ \frac{d}{dg}\Delta &= f^{\Delta}_{\mathcal{O}}(\Delta, \lambda) \end{aligned} \tag{1.5}$$

where $f^{\Delta}_{\mathcal{O}}, f^{\lambda}_{\mathcal{O}}$ are explicit (quadratic) functions that depend on $6j$-symbols of the group $O(4,2)$ in $d = 4$ (i.e. entirely group theoretic=kinematic). Here $\lambda = \{\lambda_{AB}^{C}(g)\}$ and $\Delta = \{\Delta_A(g)\}$ are the CFT data which are now functions of g. $\mathcal{O}$ is the (marginal) perturbation of the CFT, which enters the functions. [Hollands, in prep.]

Outline

The OPE factorises

Theorem (Holland-SH)

In ϕ^4-theory, any arbitrary but fixed loop order:

$$\mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N) = \sum_C \mathcal{C}^{C}_{A_1\ldots A_M}(x_1,\ldots,x_M)\mathcal{C}^{B}_{C A_{M+1}\ldots A_N}(x_M,\ldots,x_N)$$

holds on the domain $\xi \equiv \frac{\max_{1\le i\le M}|x_i - x_M|}{\min_{M<j\le N}|x_j - x_M|} < 1$. *(Sum over C abs. convergent !)*

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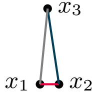
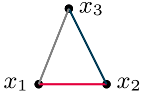
For $N = 3$: $\xi = \frac{|x_1 - x_2|}{|x_2 - x_3|} < 1$



for $\xi \ll 1$



for $\xi \approx 1$

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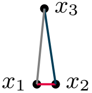

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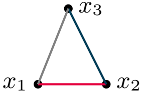

for $\xi \approx 1$

This shows associativity really holds!

- Bound on remainder
- Justification of “action principle”

Quantitative bound

Theorem

Up to any perturbation order $r \in \mathbb{N}$ *the bound*

$$\left| \textit{Remainder in associativity} \right|$$

$$\leq \frac{K_r \xi^{D+1} \max\limits_{N \leq v < N} |x_i - x_n|^{[B]}}{\prod_{v=1}^{M} \min\limits_{1 \leq w \leq M, w \neq v} |x_v - x_w|^{[A_v]+\delta} \prod_{i=M+1}^{N} \min\limits_{M < j \leq N, i \neq j} |x_i - x_j|^{[A_i]+\delta}}$$

holds for some $\delta > 0$ *and where*

$$\xi := \frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|}$$

and where K_r *is a constant which does not depend on* D*. (Here* $[A] =$ *dim. of op. in free theory).*

Outline

Previous results

- Wilson proposed his expansion as an asymptotic expansion for short distances
- In CFTs, Mack showed convergence (in $d = 4$) for finite distances [Mack 1977]; for a more formal argument see also [Pappadopoulo et al. 2012]
- There is a difference between space like separation and light like separation
- For theories without conformal invariance, situation was unclear

Bound on OPE remainder I

Theorem (Holland-Kopper-SH)

At any perturbation order r and for any $D \in \mathbb{N}$,

$$\Bigg| \overbrace{\Bigg\langle \Bigg(\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \sum_{\dim[B] \le D} \mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N)\, \mathcal{O}_B(x_N) \Bigg) \underbrace{\hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n)}_{\text{Spectator fields}} \Bigg\rangle}^{\text{OPE-Remainder}} \Bigg|$$

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$$\leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(K M \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1}}{\min_{1 \leq i < j \leq N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup \left(1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

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$$\leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N}|x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

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- $|P| = \sup_i |p_i|$: maximal momentum of spectators

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- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu,\varepsilon)$, where $\varepsilon = \min_{I\subset\{1,\ldots,n\}} |\sum_I p_i|$
  ε: distance of $(p_1,\ldots,p_n)$ to “exceptional” configurations

Conclusions from bound on OPE remainder

$$\text{"OPE remainder"} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

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Possible Generalisations

- Gauge theories [Fröb 2016]
- Curved manifolds
- Minkowski space
- Non-perturbative constructions
- ...