Feynman Propagators in a Functional Analytic Setting

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February 03, 2018 @ LQP 41

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Introduction

Klein–Gordon operator (on half-densities):

Lorentzian metric

$$K := |g|^{-\frac{1}{4}} (-\mathrm{i}\partial_\mu - A_\mu)\, g^{\mu\nu}\, |g|^{\frac{1}{2}} (-\mathrm{i}\partial_\nu - A_\mu)\, |g|^{-\frac{1}{4}} + Y$$

electromagnetic potential

scalar potential

Questions

1. Is K **essentially self-adjoint** on $C_c^\infty(M)$ with respect to $L^2(M)$?
2. Does the **resolvent limit** $\lim_{\varepsilon \to 0^+} (K - \mathrm{i}\varepsilon)^{-1}$ exist (in some sense)?

Answer for static spacetimes [Dereziński–Siemssen '17]

If (M, g) is standard static and globally hyperbolic, then K is essen-

Motivation – “I never claimed that this is interesting”

- Natural questions for any partial differential operator
- Powerful functional calculus for self-adjoint operators
- Make rigorous sense of heat kernel methods

$$(K - \mathrm{i}0^+)^{-1} = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}s(K-\mathrm{i}0^+)}\,\mathrm{d}s$$

  to define in-out Feynman propagators [Schwinger ’51, DeWitt ’65, DeWitt ’75, Rumpf–Urbantka ’78, Rumpf ’80]
- Lorentzian spectral geometry? (see also [Bär–Strohmaier ’15])

Recent results

Fredholm properties ...

- ... for the wave operator in asymptotically flat spacetimes [Gell-Redman–Haber–Vasy ’16]
- ... for the Klein–Gordon operator in asymptotically flat spacetimes [Gérard–Wrochna ’16]

Self-adjointness and limiting absorption principle ...

- ... for the Klein–Gordon operator in static spacetimes [Dereziński–Siemssen ’17]
- ... for the wave operator in asymptotically flat spacetimes [Vasy ’17]

Methods and Formalism

Hamiltonian formalism

Suppose, for example, that $M = \mathbf{R} \times \Sigma$ with metric

$$g = -\,\mathrm{d}t^2 + g_\Sigma$$

$g_\Sigma(t)$ restricts to Riemannian metric on Σ

Introduce $\quad B(t) := \begin{pmatrix} W(t) & \mathbf{1} \\ L(t) & \overline{W}(t) \end{pmatrix}$

$$W := -A_0 + \frac{\mathrm{i}}{4}|g|^{-1}\partial_t|g|$$

$$L := |g|^{\frac{1}{4}}(-\mathrm{i}\partial_i - A_i)g^{ij}|g|^{\frac{1}{2}}(-\mathrm{i}\partial_j - A_j)|g|^{-\frac{1}{4}} + Y$$

Note that $\quad (\partial_t + \mathrm{i}B)\begin{pmatrix} u \\ (\partial_t + \mathrm{i}W)u \end{pmatrix} = \begin{pmatrix} 0 \\ Ku \end{pmatrix}$

Generator

$\mathcal{H}_{\mathrm{en}}, \mathcal{H}_{\mathrm{en}^*}$ are **Hilbertizable spaces** such that $\mathcal{H}_{\mathrm{en}} \subset \mathcal{H}_{\mathrm{en}^*}$ continuously.

Introduce a bounded Hermitian form, the **charge form**,

$$\mathcal{H}_{\mathrm{en}} \times \mathcal{H}_{\mathrm{en}^*} \ni (u, v) \mapsto (u|Qv) \in \mathbf{C}$$

Suppose that $B(t)$ is **bounded and invertible** from $\mathcal{H}_{\mathrm{en}}$ to $\mathcal{H}_{\mathrm{en}^*}$ such that

$$(u|v)_{\mathrm{en},t} := (B(t)u|Qv) = (u|QB(t)v)$$
$$(u|v)_{\mathrm{en}^*,t} := (B(t)^{-1}u|Qv) = (u|QB(t)^{-1}v)$$

are compatible with $\mathcal{H}_{\mathrm{en}}, \mathcal{H}_{\mathrm{en}^*}$. This yields **Hilbert spaces** $\mathcal{H}_{\mathrm{en},t}, \mathcal{H}_{\mathrm{en}^*,t}$.

$B(t)$ is a **self-adjoint** operator on $\mathcal{H}_{\mathrm{en}^*,t}$ with domain $\mathcal{H}_{\mathrm{en}}$.

Scales of Hilbert spaces

We can construct a whole **scale of Hilbert spaces**:

$$\mathcal{H}_{\alpha,t} := |B(t)|^{(1-\alpha)/2}\mathcal{H}_{\mathrm{en},t} = |B(t)|^{(-1-\alpha)/2}\mathcal{H}_{\mathrm{en}^*,t}$$
$$(u|v)_{\alpha,t} := (u||B(t)|^{-1+\alpha}v)_{\mathrm{en},t} = (u||B(t)|^{1+\alpha}v)_{\mathrm{en}^*,t}$$

For $\alpha \in [-1, 1]$ their topology is independent of t.

The central space is $\mathcal{H}_{\mathrm{dyn},t} = \mathcal{H}_{0,t}$. We call it the **dynamical space**, and denote the corresponding Hilbertizable space $\mathcal{H}_{\mathrm{dyn}}$.

The charge form is best understood on the dynamical space:

$$\mathcal{H}_{\mathrm{dyn}} \times \mathcal{H}_{\mathrm{dyn}} \ni (u, v) \mapsto (u|Qv) \in \mathbf{C}$$

Dynamics

Suppose that $t \mapsto B(t)$ is continuous in $B(\mathcal{H}_{\mathrm{en}}, \mathcal{H}_{\mathrm{en}^*})$ and

$$||u||_{\mathrm{en},s} \leq ||u||_{\mathrm{en},t} \exp\left|\int_s^t C(r)\,\mathrm{d}r\right| \quad \text{with} \quad C \in L^1_{\mathrm{loc}}(\mathbf{R}), C \geq 0.$$

We apply the theory of **non-autonomous evolution equations** to find:

Theorem

There is a unique family of bounded operators $\{R(t,s)\}_{s,t\in\mathbf{R}}$ in $\mathcal{H}_{\mathrm{en}^*}$ with the following properties (among several others):

1) $R(t,t) = \mathbf{1}$, $R(t,r)R(r,s) = R(t,s)$
2) $R(t,s)\mathcal{H}_{\mathrm{en}} \subset \mathcal{H}_{\mathrm{en}}$
3) $\mathrm{i}\partial_t R(t,s)u = +B(t)R(t,s)u$ for $u \in \mathcal{H}_{\mathrm{en}}$
4) $\mathrm{i}\partial_s R(t,s)u = -R(t,s)B(s)u$ for $u \in \mathcal{H}_{\mathrm{en}}$

Classical propagators

Given $R(t,s)$, we can write the **kernels for the classical propagators**:

$$E^{\mathrm{PJ}}(t,s) := R(t,s) \qquad \text{(Pauli–Jordan propagator)}$$

$$E^{\vee}(t,s) := \theta(t-s)R(t,s) \qquad \text{(retarded propagator)}$$

$$E^{\wedge}(t,s) := -\theta(s-t)R(t,s) \qquad \text{(advanced propagator)}$$

The corresponding propagators are given by

$$(E^{\bullet}f)(t) = \int_{\mathbf{R}} E^{\bullet}(t,s)f(s)\,\mathrm{d}s$$

They can be understood as **bisolutions** resp. **inverses** between various spaces. For example,

$$(\partial_t + \mathrm{i}B)E^{\mathrm{PJ}}f = 0, \quad f \in L^1_{\mathrm{c}}(\mathbf{R};\mathcal{H}_{\mathrm{en}})$$

$$E^{\mathrm{PJ}}(\partial_t + \mathrm{i}B)f = 0, \quad f \in L^1_{\mathrm{c}}(\mathbf{R};\mathcal{H}_{\mathrm{en}}) \cap AC_{\mathrm{c}}(\mathbf{R};\mathcal{H}_{\mathrm{en}^*})$$

Involutions

We call S an **admissible involution** if $S^2 = \mathbf{1}$ and

$$(u|v)_S = (u|QSv) = (Su|Qv)$$

defines a scalar product compatible with $\mathcal{H}_{\text{dyn}}$.

Note that, to each S are associated **two projections**

$$\Pi^{(\pm)} := \frac{1}{2}(\mathbf{1} \pm S)$$

An example of an admissible involution is

$$S(t) = \operatorname{sgn} B(t) = B(t)|B(t)|^{-1}$$

The corresponding projections are $\Pi_t^{(\pm)} = \mathbf{1}_{[0,\infty)}\big(\pm B(t)\big)$

Non-classical propagators

Given $R(t,s)$, an admissible involution S and any $\tau \in \mathbf{R}$, we can write **kernels for non-classical propagators**:

$$E_\tau^{(+)}(t,s) := R(t,\tau)\Pi^{(+)}R(\tau,s) \qquad \text{(positive frequency bisol.)}$$

$$E_\tau^{(-)}(t,s) := -R(t,\tau)\Pi^{(-)}R(\tau,s) \qquad \text{(negative frequency bisol.)}$$

$$E_\tau^{\mathrm{F}}(t,s) := \theta(t-s)E_\tau^{(+)}(t,s) + \theta(s-t)E_\tau^{(-)}(t,s) \qquad \text{(Feynman propagator)}$$

$$E_\tau^{\overline{\mathrm{F}}}(t,s) := \theta(t-s)E_\tau^{(-)}(t,s) + \theta(s-t)E_\tau^{(+)}(t,s) \qquad \text{(anti-Feynman propagator)}$$

Again, they can be understood as **bisolutions** resp. **inverses** between various spaces. Moreover, the following relations hold

$$E_\tau^{\mathrm{F}} = E^\wedge + E_\tau^{(+)} = E^\vee + E_\tau^{(-)}, \qquad E_\tau^{\mathrm{F}} + E_\tau^{\overline{\mathrm{F}}} = E^\vee + E^\wedge, \qquad E_\tau^{(+)} - E_\tau^{(-)} = E^{\mathrm{PJ}},$$

$$E_\tau^{\overline{\mathrm{F}}} = E^\vee - E_\tau^{(+)} = E^\wedge - E_\tau^{(-)}, \qquad E_\tau^{\mathrm{F}} - E_\tau^{\overline{\mathrm{F}}} = E_\tau^{(+)} + E_\tau^{(-)} \qquad E^\vee - E^\wedge = E^{\mathrm{PJ}}.$$

Pairs of projections

Theorem

Let $\varPi_1^{(+)}, \varPi_2^{(+)}$ be **projections** on $\mathcal{H}$. Set $\varPi_\bullet^{(-)} = \mathbf{1} - \varPi_\bullet^{(+)}$. Suppose that

$$\varUpsilon = \mathbf{1} - \left(\varPi_1^{(+)} - \varPi_2^{(+)}\right)^2$$

is **invertible.** Then $\operatorname{Ran} \varPi_1^{(+)}$ and $\operatorname{Ran} \varPi_2^{(-)}$ are **complementary**, i.e.

$$\operatorname{Ran} \varPi_1^{(+)} \cap \operatorname{Ran} \varPi_2^{(-)} = \{0\}, \qquad \operatorname{Ran} \varPi_1^{(+)} + \operatorname{Ran} \varPi_2^{(-)} = \mathcal{H}.$$

Moreover,

$$\varLambda^{(+)} = \varPi_1^{(+)} \varUpsilon^{-1} \varPi_2^{(+)} \qquad \text{(projection onto } \operatorname{Ran} \varPi_1^{(+)} \text{ along } \operatorname{Ran} \varPi_2^{(-)}\text{)}$$

$$\varLambda^{(-)} = \varPi_2^{(-)} \varUpsilon^{-1} \varPi_1^{(-)} \qquad \text{(projection onto } \operatorname{Ran} \varPi_2^{(-)} \text{ along } \operatorname{Ran} \varPi_1^{(+)}\text{)}$$

are **complementary projections**, i.e. $\varLambda^{(+)} + \varLambda^{(-)} = \mathbf{1}$.

Theorem

Let S_1, S_2 be **admissible involutions** (with associated projections $\Pi_\bullet^{(\pm)}$).
Then $\Upsilon = \mathbf{1} - \big(\Pi_1^{(+)} - \Pi_2^{(+)}\big)^2$ is **invertible**.

Proof. First we prove the identity $\Upsilon = \big(\Pi_1^{(-)} + \Pi_2^{(+)}\big)\big(\Pi_2^{(-)} + \Pi_1^{(+)}\big)$. Next we set

$$c = \Pi_1^{(+)}(\mathbf{1} - S_2 S_1)(\mathbf{1} + S_2 S_1)^{-1}\Pi_1^{(-)}$$

and show that $\|c\|_{S_1} < 1$. Wrt. the decomposition $\mathcal{H}_{\text{dyn}} = \operatorname{Ran} \Pi_1^{(+)} \oplus \operatorname{Ran} \Pi_1^{(-)}$ we find

$$\Pi_1^{(-)} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad \Pi_2^{(+)} = \begin{pmatrix} (\mathbf{1} - cc^*)^{-1} & c(\mathbf{1} - c^* c)^{-1} \\ -c^*(\mathbf{1} - cc^*)^{-1} & -c^* c(\mathbf{1} - c^* c)^{-1} \end{pmatrix}.$$

Therefore we can directly compute the inverse:

$$\big(\Pi_1^{(-)} + \Pi_2^{(+)}\big)^{-1} = \begin{pmatrix} \mathbf{1} - 2cc^* & -c \\ c^* & \mathbf{1} \end{pmatrix}.$$

In/out positive/negative frequency bisolutions

Fix two times $t_+, t_- \in \mathbf{R}$ and define (admissible involutions)

$$S_\pm := \operatorname{sgn} B(t_\pm) = B(t_\pm)|B(t_\pm)|^{-1}$$

We define **time-evolved projections**

$$\Pi_\pm^{(+)}(t) := R(t, t_\pm)\Pi_\pm^{(+)}R(t_\pm, t)$$
$$\Pi_\pm^{(-)}(t) := R(t, t_\pm)\Pi_\pm^{(-)}R(t_\pm, t)$$

From this we can define **in/out positive/negative frequency bisolutions**:

$$E_\pm^{(+)}(t, s) := R(t, s)\Pi_\pm^{(+)}(s)$$
$$E_\pm^{(-)}(t, s) := -R(t, t_\pm)\Pi_\pm^{(-)}(s)$$

Often one would take the limit $t_\pm \to \pm\infty$.

In-out Feynman propagator

Since $\Upsilon(t) = \mathbf{1} - \left(\Pi_-^{(+)}(t) - \Pi_+^{(+)}(t)\right)^2$ is invertible, we obtain

$$\Lambda^{(+)}(t) := \Pi_-^{(+)}(t)\Upsilon(t)^{-1}\Pi_+^{(+)}(t)$$
$$\Lambda^{(-)}(t) := \Pi_+^{(-)}(t)\Upsilon(t)^{-1}\Pi_-^{(-)}(t)$$

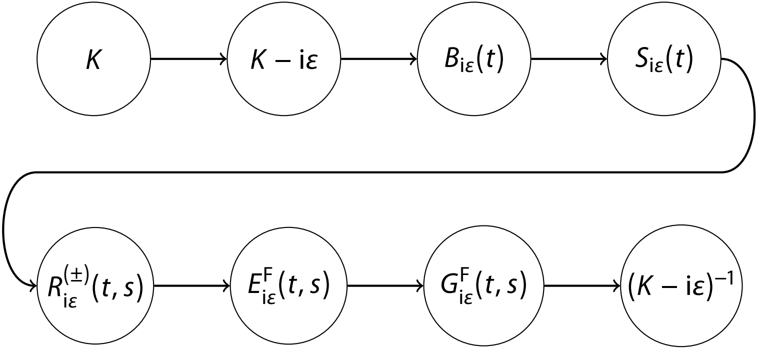
Therefore we can define the **in-out Feynman propagator**

$$E^{\mathrm{F}}(t,s) := \theta(t-s)R(t,s)\Lambda^{(+)}(s) + \theta(s-t)R(t,s)\Lambda^{(-)}(s)$$

NB: This Feynman propagator is generally not associated to a single state!

Taking the limit $t_\pm \to \pm\infty$, we **conjecture** that this kernel yields the same Feynman propagator as obtained from the resolvent limit (if it exists).

Proving the conjecture?



Summary

- The Klein–Gordon operator is essentially self-adjoint in many cases
- The resolvent limit yields a ‘distinguished’ Feynman propagator
- In cases were the Klein–Gordon operator is not essentially self-adjoint, maybe one can construct a distinguished self-adjoint extension