DECAY PROPERTIES OF THE VACUUM AND THERMAL TWO-POINT FUNCTIONS IN CURVED SPACETIMES FOR A MASSIVE INTERACTING SCALAR FIELD

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Outline

- Motivations and settings;
- Flat case;
- Generalization to curved spacetimes:
  1. Globally hyperbolic spacetimes with compact Cauchy surfaces;
  2. Schwarzschild black hole;
  3. Stationary and asymptotically Minkowskian (SAM) spacetimes.

Motivations and setting

The applicability of AQFT to curved spacetimes allows the description of interesting phenomena, such as:

- Hawking radiation[1,2]
- Unruh effect

Setting

- Globally hyperbolic spacetime (M, g)
- Massive scalar field $\phi : M \to \mathbb{R}$, such that $P\phi = 0$ with P the Klein-Gordon operator
- Algebra $\mathcal{A}(M)$ generated by smeared fields $\phi(f)$ $(f \in \mathcal{D}(M))$, encoding locality, causality, CCR
- Quasi-free Hadamard states over $\mathcal{A} \Rightarrow \omega$ defined by a two-point function of Hadamard form

[1] R. Haag, K. Fredenhagen - Commun. Math. Phys. 127, 273-284 (1990)

[2] G. Collini, V. Moretti, N. Pinamonti - Lett. Math. Phys. 104, 217-232 (2013)

Ground[3] and KMS state

Ground state

The state ω is ground if the map $t \mapsto \omega(A\alpha_t(B))$ is such that

$$\int_{-\infty}^{\infty} \widehat{f}(t)\omega(A\alpha_t(B))dt = 0$$

for each $A, B \in \mathcal{A}(M)$, $f \in C_0^\infty(\mathbb{R}^-)$, with $\{\alpha_t\}_{t\in\mathbb{R}}$ strongly continuous one-parameter $*$-isomorphism of $\mathcal{A}$.

KMS state

The state ω is KMS at inverse temperature β if:

- The functions $t \mapsto \omega(A\alpha_t(B))$ and $t \mapsto \omega(\alpha_t(B)A)$ have an analytic extension to the strip $0 < Imz < \beta$ and $-\beta < Imz < 0$ respectively;
- $\omega(A\alpha_t(B)) = \omega(\alpha_{t+i\beta}(B)A) \qquad \forall A, B \in \mathcal{A}(M)$

[3] H. Sahlmann, R. Verch - Passivity and Microlocal Spectrum Condition (2000)

Interacting theory

Interaction

- Self-interaction $\mathcal{H}_I \in \mathcal{A}$ with coupling const. λ (e.g. $\lambda\phi^3$)
- S relative S-matrix

$$S(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{M^n} d^4x_1 ... d^4x_n T\mathcal{H}_I(x_1)...\mathcal{H}_I(x_n)\lambda(x_1)...\lambda(x_n)$$

with T time-ordering operator

- Interacting algebra $\mathcal{A}_\lambda$: generated by $S_\lambda(f) = S(\lambda)^{-1}S(\lambda + f)$

Interacting KMS state

We can define the interacting dynamics $\alpha_I(t)$ in terms of $\alpha(t)$
$\Rightarrow$ Interacting KMS state with respect to $\alpha_I(t)$

No asymptotically free fields $\Rightarrow$ IR divergences

Time-slice axiom

Time-slice axiom (TSA) in the free theory

- $O \subset M$ open s. t. $O \supset \Sigma$ Cauchy surface of M
- $\mathcal{A}(M)$ and $\mathcal{A}(O)$ algebras of observables on M and O resp.

$\mathcal{A}(M)$ is $*$-isomorphic to $\mathcal{A}(O)$ via the map $f \mapsto P\chi^+ E(f)$, where:

- $f \in \mathcal{D}(M)$ generators of $\mathcal{A}$
- χ^+ smooth function, $\chi^+ = 1$ in $J^+(O) \setminus O$, $\chi^+ = 0$ in $J^-(O) \setminus O$
- E causal propagator of P ($E = E^+ - E^-$, $E^\pm$ advanced/retarded fundamental solution of P)

TSA in the interacting theory

For generic coupling constant, TSA is valid also for interacting theories[4]

[4] B. Chilian, K. Fredenhagen - arXiv:0802.1642v3 (2008)

Interacting state

$\omega_\beta^{I,h}(A)$ ($A \in \mathcal{A}$) in terms of the connected correlation functions ω_β^C:

$$\omega_\beta^{I,h}(A) = \sum_{n=0}^{\infty}(-1)^n \int_{\beta S_n} du_1 ... du_n \int_{\Sigma^n} d^3x_1 ... d^3x_n h(x_1)...h(x_n) \times$$
$$\times\, \omega_\beta^C(A \otimes \mathcal{U}_h(u_1, x_1) \otimes ... \otimes \mathcal{U}_h(u_n, x_n))$$

where:

- h spatial cutoff
- $\beta S_n = \{(u_1, ..., u_n) \in \mathbb{R} | 0 < u_1 < ... < u_n < \beta\}$
- $\mathcal{U}_h(u, x) = \int dt \dot{\chi}^-(t) \alpha_{iu}([\mathcal{H}_I(x)]_{h\chi})$
- χ^-, χ s.t. $\chi = 1 - \chi^+ - \chi^-$

Heuristic adiabatic limit: $h \to 1$ on the time-slice

Lindner-Fredenhagen work[5]

Connected correlation functions:

$$F_n^{vac}(u_1, x_1; ...; u_n, x_n) = \omega_{vac}^{C}(A_0 \otimes \alpha_{iu_1,x_1}(A_1) \otimes ... \otimes \alpha_{iu_n,x_n}(A_n)) = \\ = \sum_G \frac{1}{Sym(G)} F_{n,G}^{vac}(u_1, x_1; ...; u_n, x_n)$$

$$F_{n,G}^{vac}(u_1, z_1; ...; u_n, z_n) = \int dXdY \prod_l D_+^{vac}(x_l - y_l)\Psi(X, Y)$$

where:

- $D_+^{vac}(x-y) = \frac{1}{2\pi} \int \frac{d^3p}{2\omega_p} e^{-i(\omega_p x^0 - px)}$
- $\Psi(X,Y) = \prod_{l \in E(G)} \frac{\delta^2}{\delta\phi_{s(l)}(x_l)\delta\phi_{r(l)}y_l}(A_0 \otimes ... \otimes \alpha_{iu_n,z_n} A_n)|_{\phi_0 \otimes ... \otimes \phi_n = 0}$
- $X = (x_1, ..., x_n)$, $Y = (y_1, .., .y_n)$

[5] K. Fredenhagen, F. Lindner - arXiv:1306.6519v5 (2014)

Lindner-Fredenhagen result

Ground (KMS) state

Let ω_{vac} (ω_β) be the ground (KMS with inverse temperature $0 < \beta < \infty$) state of the free Klein-Gordon field with mass $m > 0$. Then:

$$|F_n(u_1, x_1; ...; u_n, x_n)| \leq ce^{-mr_e/\sqrt{n}}$$

for $r_e > 2R$, with $r_e = \sqrt{\sum_{i=1}^n u_i^2 + |x_i|^2}$ ($r_e = \sqrt{\sum_{i=1}^n |x_i|^2}$) and $(u_1, ..., u_n) \in \beta S_n^\infty = \{(u_1, ..., u_n)|0 < u_1 < ... < u_n\}$ ($(u_1, ..., u_n) \in \beta S_n$). In particular $F_{n,G}^{vac} \in L^1(\beta S_n \times \Sigma)$.

Adiabatic limit

The limit $\lim_{h\to 1} \omega_{vac}^{I,h}(A)$ ($\lim_{h\to 1} \omega_\beta^{I,h}(A)$), with $A \in \mathcal{A}_\lambda(\Sigma)$, exists and defines a ground (KMS) state on $\mathcal{A}_\lambda(O)$.

Spacetimes with compact Cauchy surfaces

What we know

- KMS with respect to $\alpha_I(t)$
- Time-slice axiom
- $F^{vac}_{n,G} \in L^1(\beta S_n \times \Sigma)$

Adiabatic limit for the ground (KMS) state

Let ω_{vac} (ω_β) be a ground (KMS) state on the algebra $\mathcal{A}(O)$ with $(O, g|_O)$ time-slice of the spacetime (M, g). Then the adiabatic limit

$$\lim_{h \to 1} \omega^{I,h}_{vac(\beta)}(A) = \omega^{I}_{vac(\beta)}(A)$$

exists and defines a ground (KMS) state on $\mathcal{A}_\lambda(O)$, which induces a ground (KMS) state on $\mathcal{A}_\lambda(M)$ via pull-back (TSA).

Schwarzschild black hole

Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with M mass of the black hole.

Klein-Gordon equation

Massive scalar field $\phi : M \to \mathbb{R}$ with mass m:

$$(\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu + m^2 \sqrt{-g}) \Phi(x) = 0$$

We are interested in the static region: $r > 2M$

Solutions to KG equation

$$\Phi(x) = \sum_{l,n} Y_l^n(\vartheta, \varphi) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \phi^l(r, \omega)$$

where $\phi^l(r, \omega)$ satisfies:

$$\left[\frac{\partial}{\partial r} r(r - 2M) \frac{\partial}{\partial r} + l(l+1) + m^2 r^2 - \frac{\omega^2 r^3}{r - 2M} \right] \phi^l(r, \omega) = 0$$

We can find two (lin. ind.) solutions $\phi^l(r, \omega)$ and $\psi^l(r, \omega)$, such that:

$$\phi^l(r, \omega) \overset{r \to \infty}{\sim} \frac{e^{i(qr + M/q(2\omega^2 - m^2) \log r)}}{qri^{l+1}} \qquad \psi^l(r, \omega) \overset{r \to 2M}{\sim} \left(\frac{\mid r - 2M \mid}{2M} \right)^{-2i\omega M}$$

with $q^2 = \omega^2 - m^2$

Ground state two-point function

Boulware ground state[6]

A Green operator has to fulfill the equation

$$\left(-\frac{\partial}{\partial r}r(r-2M)\frac{\partial}{\partial r}+l(l+1)+m^2r^2-\frac{\omega^2r^3}{r-2M}\right)G(r,r',\omega)=\delta(r-r')$$

A solution can be written as:

$$G(r,r',\omega)=\frac{1}{W[\phi,\psi]}\times\begin{cases}\phi(r,\omega)\psi(r',\omega) & r'<r\\ \phi(r',\omega)\psi(r,\omega) & r<r'\end{cases}$$

where $W[\phi,\psi]$ is the Wronskian of ϕ and ψ.

$G(r,r',\omega)$ defines a ground state

[6] D. G. Boulware - Physical Review (1974)

Connected correlation functions

The object we are interested in is the connected correlation function:

$$F^{vac}_{n,G}(u_1, z_1; ...; u_n, z_n) = \int dX dY \prod_l G^{vac}(x_l, y_l)\Psi(X, Y)$$

$\Psi(X, Y)$ is rapidly decreasing (analogous to the flat case, microlocal methods), so $F^{vac}_{n,G}$ is well-defined.

Crucial point: asymptotyc behavior of $G^{vac}(x, y)$

Convergence

Two-point function

- Region $\omega^2 < m^2$

$$\phi(r,\omega) \overset{r\to\infty}{\sim} \frac{a}{i^{l+2}} \frac{e^{-br}}{r} r^{-c}$$

with $a = (m^2 - \omega^2)^{-\frac{1}{2}}$; $b = \sqrt{m^2 - \omega^2}$; $c = M(m^2 - \omega^2)^{-\frac{1}{2}}(2\omega^2 - m^2)$

- Region $\omega > m$

$$\phi(r,\omega) \overset{r\to\infty}{\sim} \frac{e^{i(qr + a\log r)}}{i^{l+1} qr}$$

with $a = (2\omega^2 - m^2)M/q$

(analogous for $\psi(r,\omega)$)
In both cases we have a well-behaved two-point function (and it is valid for an arbitrary ω), so $F^{vac}_{n,G} \in L^1(\beta S_n \times \Sigma)$.

Main result

What we know

- KMS with respect to $\alpha_I(t)$
- Time-slice axiom
- $F^{vac}_{n,G} \in L^1(\beta S_n \times \Sigma)$

Existence of the adiabatic limit

Let ω be a quasi-free Hadamard ground state on the algebra $\mathcal{A}(O)$ with O time-slice in the Schwarzschid spacetime M. Then the adiabatic limit

$$\lim_{h \to 1} \omega^{I,h}_{vac}(A) = \omega^{I}_{vac}(A)$$

exists and defines a ground state on the interacting algebra $\mathcal{A}_\lambda(O)$, which induces a ground state on $\mathcal{A}_\lambda(M)$ via pull-back (TSA).

Thermal state

Integral kernel of the Green operator

$$G(x,x') = i\int_0^\infty d\omega e^{-i\omega(t-t')}\frac{e^{\beta\omega}}{e^{\beta\omega}-1}G_{vac}(r,r',\omega)$$

- exists for the extension: $t \to t + i\tau$, $t' \to t' + i\tau'$
- satisfies the KMS condition

Adiabatic limit

Let ω_β be a quasi-free Hadamard KMS state on the algebra $\mathcal{A}(O)$ with O as before. The adiabatic limit

$$\lim_{h\to 1}\omega_\beta^{I,h}(A) = \omega_\beta^I(A)$$

exists and defines a KMS state on the interacting algebra $\mathcal{A}_\lambda(O)$, which induces a KMS state on $\mathcal{A}_\lambda(M)$ via pull-back (TSA).

Extension to SAM spacetimes?

SAM spacetime

- Stationary $\rightarrow$ KMS condition
- Asymptotically Minkowskian

We basically used asymptotic considerations, so we expect our results to be true for asymptotically Minkowskian spacetimes

Conclusions

What has been done:
Existence of the adiabatic limit for an interacting massive scalar field in:

- Spacetimes with compact Cauchy surfaces (ground and KMS)
- Schwarzschild (ground and KMS)

What has to be done:

- Existence of the adiabatic limit for an interacting massive scalar field in stationary asymptotically Minkowskian spacetimes
- Extension to the massless interacting scalar field[7]
- Extension to spinor fields

[7] N. Drago, T. P. Hack, N. Pinamonti - arXiv:150202705 (2015)