Convergence of the Epstein-Glaser S-matrix in the Sine-Gordon model

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[1] Joint work with Dorothea Bahns.

Outline of the talk

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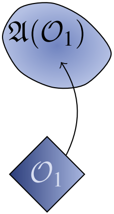

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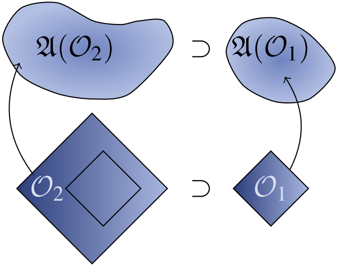

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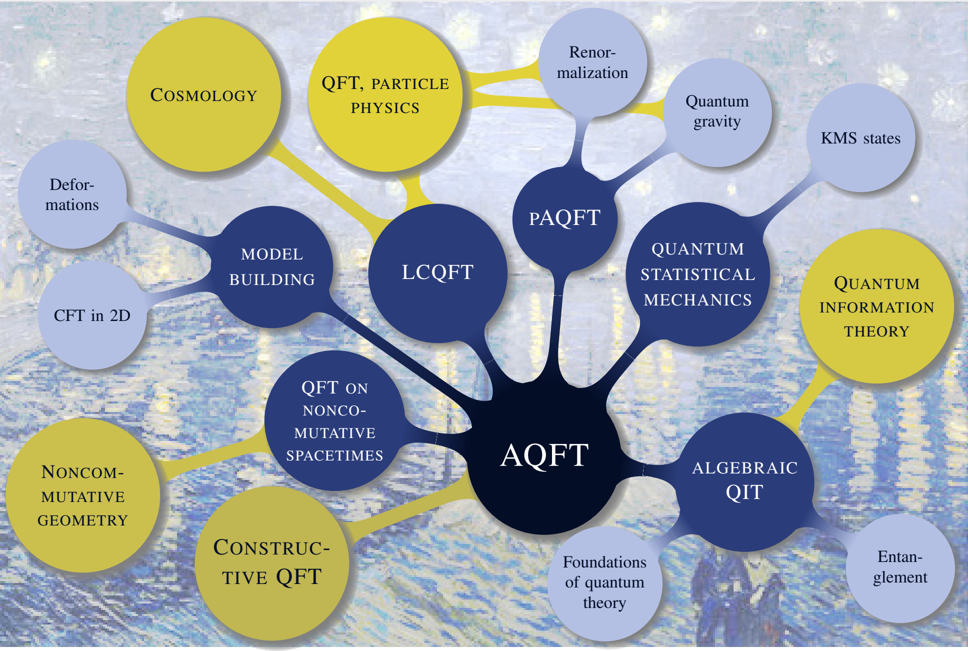
- Mathematicaly, AQFT makes use of functional analysis techniques (operator algebras), but its various generalizations involve many other branches of mathematics.

Different aspects of AQFT and relations to physics



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  - Interaction introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),
  - Generalization to gauge theories using homological algebra ([Hollands 08, Fredenhagen-KR 11]).

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- Dynamics: we use a modification of the Lagrangian formalism. Since the manifold M is non-compact, we need to introduce a cutoff function into the action functional. For the free scalar field
$$S_M(f)(\varphi) = \frac{1}{2}\int(\nabla_\mu\varphi\nabla^\mu\varphi - m^2\varphi^2)(x)f(x)d\mu(x).$$
In general an action is a map $S_M : \mathcal{D}(M) \to \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$, where $\mathcal{D}(M) \equiv \mathcal{C}_c^\infty(M, \mathbb{R})$ are compactly supported smooth functions.

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- Under some technical assumptions on M, P admits retarded and advanced Green's functions Δ^R, Δ^A. They satisfy: $P \circ \Delta^{R/A} = \mathrm{id}_{\mathcal{D}(M)}$, $\Delta^{R/A} \circ (P\big|_{\mathcal{D}(M)}) = \mathrm{id}_{\mathcal{D}(M)}$ and

$$\mathrm{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in J_-(x)\}\,,$$
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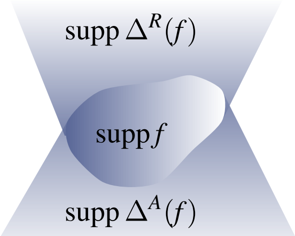

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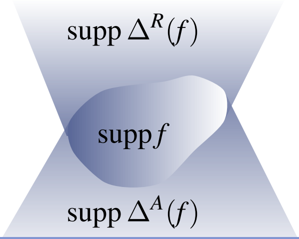

- Their difference is the causal propagator

$$\Delta \doteq \Delta^R - \Delta^A.$$

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$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle ,$$

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- The free QFT is defined as $\mathfrak{A}_0(M) \doteq (\mathcal{F}(M)[[\hbar]], \star, *)$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$ and $\mathcal{F}(M)$ is an appropriate functional space (some WF set conditions on $F^{(n)}(\varphi)$s induced by W).

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\text{reg}}(M)[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- We now have an algebraic structure with two products $(\mathcal{F}_{\mathrm{reg}}(M)[[\hbar]], \star, \cdot_{\mathcal{T}})$, where $\star$ is non-commutative, $\cdot_{\mathcal{T}}$ is commutative and they are related by a causal relation:

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- We define the interacting star product as:

$$F \star_{int} G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

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- To extend $\mathcal{T}_n$ to arbitrary local functionals we use e.g. the causal approach of Epstein and Glaser.

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- The 2-point function of the free massless scalar field in 2D coincides with the Hadamard parametrix

$$W(x) = -\frac{1}{4\pi}\ln\left(\frac{-x\cdot x + i\varepsilon t}{\Lambda^2}\right)$$

where $\Lambda > 0$ is the scale parameter.

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- In particular, for $\boldsymbol{x} = \boldsymbol{y}$ and $t > t'$,

$$V_a(t,\boldsymbol{x}) \star V_{a'}(t',\boldsymbol{x}) = e^{aa' i\hbar/2} V_{a'}(t',\boldsymbol{x}) \star V_a(t,\boldsymbol{x}) \,,$$

which is the well-known braiding property for vertex operators.

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- Hence $\star$ and $\star_v$ are equivalent products, and α_v is a "gauge transformation".

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- Similar for the S-matrix:

$$\omega_v(\mathcal{S}(\lambda {:}V{:}_v)) \doteq \alpha_v \left(e_{\mathcal{T}}^{i\lambda {:}V{:}_v/\hbar} \right)(0) = e_{\mathcal{T}_v}^{i\lambda V/\hbar}(0).$$

Here $\cdot_{\mathcal{T}_v}$ is the time-ordered product corresponding to $\star_v$.

Convergence of the S-matrix

Theorem (Bahns, KR 2016)

The formal S-matrix $\alpha_v \circ \mathcal{S}(\lambda :V:_v) = e_{\mathcal{T}_v}^{i\lambda V/\hbar}$ in the Sine-Gordon model with $V = \frac{1}{2}(V_a(f) + V_{-a}(f))$ and $0 < \beta = \hbar a^2/4\pi < 1$, $f \in \mathcal{D}(M)$, converges as a functional on the configuration space in the appropriate topology (related to Hörmander topology on distribution spaces).

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- Direct proof of the convergence of the S-matrix in the Minkowski signature.

Convergence of the S-matrix

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The formal S-matrix $\alpha_v \circ \mathcal{S}(\lambda :V:_v) = e_{\mathcal{T}_v}^{i\lambda V/\hbar}$ in the Sine-Gordon model with $V = \frac{1}{2}(V_a(f) + V_{-a}(f))$ and $0 < \beta = \hbar a^2/4\pi < 1$, $f \in \mathcal{D}(M)$, converges as a functional on the configuration space in the appropriate topology (related to Hörmander topology on distribution spaces).

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- Direct proof of the convergence of the S-matrix in the Minkowski signature.
- No issues with positivity/IR problems, no Wick rotation.
- The abstract formal S-matrix is constructed before a state is chosen.

Some details of the proof I

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- Hence the estimates boil down to estimates of expressions of the form:

$$\prod_{1\leq i<j\leq k} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta} \prod_{1\leq i\leq k, k<j\leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{-\beta} \prod_{k<i<j\leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta} .$$

with the time variable differences $\tau_{ij} = t_i - t_j$ and the space variable differences $\zeta_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$.

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  with the time variable differences $\tau_{ij}=t_i-t_j$ and the space variable differences $\zeta_{ij}=\boldsymbol{x}_i-\boldsymbol{x}_j$.
- After a change of variables, this expression can be rewritten as a determinant (analogous to [Fröhlich 76]):

$$D_{ij}=\begin{cases} w_j^{i-1} & , \quad 1\le i\le l-k \, , \\ 1/(z_{i-l+k}-w_j) & , \quad l-k<i\le l \, . \end{cases}$$

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- The estimates are possible due to the fact that $\beta < 1$ and for the Vandermonde determinants they require one to choose the support of the cutoff function sufficiently small.
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- The estimates are possible due to the fact that $\beta < 1$ and for the Vandermonde determinants they require one to choose the support of the cutoff function sufficiently small.
- The later requirement would not be necessary if we were using a singular state obtained as the limit of the massive theory, instead of a Hadamard state.
- In our future work we expect to be able to drop the condition on the support of the test function, in an appropriate class of Hadamard states.

Constructing a net of von Neumann algebras

- Use the Bogoliubov formula to construct the interacting fields:

$$R_{\lambda V}(F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda :V:)^{-1} \mathcal{S}(\lambda :V: + \mu :F:)\Big|_{\mu=0}$$
$$\equiv -i\hbar \frac{d}{d\mu} \mathcal{S}_{\lambda :V:}(\mu :F:)\Big|_{\mu=0} ,$$

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- Prove covariance.
- Construct the local net using the prescription given in [Fredenhagen, KR 2015].

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- Construct conserved currents for Sine Gordon (i.e. show integrability).
- Apply the same methods to a larger class of integrable models.
- Show equivalence with the $\mathcal{O}(3)$ model and the Thirring model.

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For Bernard and Henning:

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For Dorothea, Chris and Gandalf: Thank you for this wonderful event!

Thank you very much for your attention!