Spacelike deformations

(e.g. “Maxwell from scalar”)

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Abstract

In Hamiltonian perturbation theory, it is easy to change the mass of a free field.

"Spacelike deformation" allows also to change the helicity of free fields.

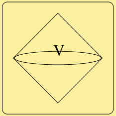
E.g., the Maxwell electric field and the gradient of the massless scalar field are indistinguishable, when restricted to the time-axis.

They differ only away from the time-axis.

(Joint work with V. Morinelli: arXiv:1905.08714)

THE SIMPLE IDEA

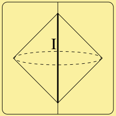
Hamiltonian perturbation theory = **time-zero fields** plus "new" Hamiltonian $\widetilde{H} = H + H_1$, commuting with $\vec{P}$ (plus new boosts). Deforms $A(O_V)$.



(Time-zero fields = "Cauchy data" $\phi(\vec{x})$ and $\pi(\vec{x}) = \dot{\phi}(\vec{x})$.)

Analogously:

Spacelike deformation = fields at $\vec{x} = 0$ (**time-axis fields**) plus new momentum operators $\widetilde{P}_k$, commuting with $P_0 = H$ and the rotations (plus new boosts). Deforms $A(O_I)$.



(Time-axis fields = $\phi(t, \vec{0})$ along with all their spatial derivatives $\nabla_{j_1} \dots \nabla_{j_r} \phi(t, \vec{0})$. They may not be independent: eg, $\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$.)

Idea: Define deformed Poincaré generators on the Hilbert space of the undeformed fields, as functions of the undeformed generators. Take care that they satisfy the Poincaré algebra!

Use deformed translation operators $\widetilde{P}_k$ to define deformed fields away from the time-axis:

$$\widetilde{\phi}(t, \vec{x}) := e^{i\widetilde{P}_k x^k} \phi(t, \vec{0}) e^{-i\widetilde{P}_k x^k}.$$

(Locality is not automatic!)

The deformed field has an expansion in terms of all the time-axis fields including the spatial derivatives.

(For the undeformed generators, this would be just the Taylor expansion.)

For free fields, a one-particle space setting is sufficient:

- **Define** deformed generators $\widetilde{P}_k$ and $\widetilde{M}_{0k}$ on the "undeformed states", labelled by polynomials Q:

$$|Q\rangle_t := Q(-i\vec{\nabla})\phi(t,\vec{0})\Omega \in \mathcal{H}_1$$

- Definition must **respect linear dependences** among such states, due to the equations of motion, eg,

$$|\vec{\nabla}^2 \cdot Q\rangle_t = (\partial_t^2 + m^2)|Q\rangle_t.$$

- Deformed generators must **satisfy the Poincaré algebra** with the undeformed generators P_0 and M_{kl}!

The generators and fields are then lifted to the Fock space by second quantization. **Check locality!**

WARM-UP:
MASS DEFORMATION

We consider distributions in the time t, taking values in the one-particle space $\mathcal{H}_m$ of a **scalar field of mass** m:

$$|Q\rangle_t^m := Q(-i\vec{\nabla})\phi_m(t,\vec{0})\Omega$$

(Q = polynomials).

The time-translations act by

$$P_0|Q\rangle_t^m = -i\partial_t|Q\rangle_t^m,$$

and the spatial rotations by their action on $Q(-i\vec{\nabla})$.
The spatial derivatives act by

$$P_k|Q\rangle_t^m = |-i\nabla_k \cdot Q\rangle_t^m.$$

Mass-shell condition holds:

$$(P_0^2 - \vec{P}^2)|Q\rangle_t^m = -\partial_t^2|Q\rangle_t^m + |\vec{\nabla}^2 \cdot Q\rangle_t^m = m^2|Q\rangle_t^m.$$

The inner product depends on the mass: For Q_ℓ and Q'_ℓ harmonic homogeneous of degree ℓ, and $\hat{Q}(\vec{n}) = Q(\vec{p}/p)$, one has

$$ {}^m_t\langle Q_\ell | Q'_\ell \rangle^m_{t'} = \int_0^\infty \frac{p^2\,dp}{2\sqrt{p^2+m^2}} \cdot p^{2\ell} e^{-i\sqrt{p^2+m^2}(t-t')} \cdot (\hat{Q}_s, \hat{Q}'_{s'})_{S^2} $$

where $(\cdot,\cdot)_{S^2}$ is the L^2 inner product on the sphere.

$$ = \frac{1}{2}\int_m^\infty dp_0 \cdot \left[\sqrt{p_0^2 - m^2}\right]^{2\ell+1} e^{-ip_0(t-t')} \cdot (\hat{Q}_\ell, \hat{Q}'_\ell)_{S^2}, $$

Therefore, the operator $U : \mathcal{H}_m \to E(P_0 \geq m)\mathcal{H}_0$

$$ U : |Q_\ell\rangle^m_t \mapsto \left[\sqrt{\frac{P_0^2 - m^2}{P_0^2}}\right]^{\ell+\frac{1}{2}} |Q_\ell\rangle^0_t $$

is unitary, and intertwines the massless and massive P_0 and M_{kl}, and hence the Casimir operators $C = \frac{1}{2}M_{kl}M_{kl}$ of $so(3)$.

Because $C+\frac{1}{4}$ has eigenvalues $(\ell+\frac{1}{2})^2$, U can be written as a function of P_0 and C:

$$U = \left[\sqrt{\frac{P_0^2 - m^2}{P_0^2}}\right]^{\sqrt{C+\frac{1}{4}}} = \exp\left[\frac{1}{4}\sqrt{1+4C}\cdot\log\left(1-\frac{m^2}{P_0^2}\right)\right].$$

This unitary operator is used to "pull back" the translation and boost generators of the massive free field to the projected one-particle space $E(P_0\geq m)\mathcal{H}_0$ of the massless free field:

$$\widetilde{P}_k = UP_k^{(m)}U^*, \quad \widetilde{M}_{0k} = UM_{0k}^{(m)}U^*.$$

Their deviation from the generators $P_k = P_k^{(0)}$, $M_{0k} = M_{0k}^{(0)}$ is the desired deformation.

We have found explicit expressions for $\widetilde{P}_k$ and $\widetilde{M}_{0k}$ in terms of the generators of the massless free field:

$$\widetilde{P}_k = P_k \cdot \sigma(P_0), \qquad \sigma(P_0) \equiv \sqrt{1 - \frac{m^2}{P_0^2}},$$

$$\widetilde{M}_{0k} = \frac{1}{2}\Big(\sigma(P_0)M_{0k} + M_{0k}\sigma(P_0)\Big) + \frac{i}{2}\,[C, P_k] \cdot \sigma'(P_0).$$

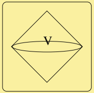
Translating the massless time-axis field with $\widetilde{P}_k$ away from the time axis, yields the massive free field.

In this way, we "obtain" the massive free field by a deformation of the representation of the Poincaré Lie algebra on a subspace of the one-particle space of the massless free field.

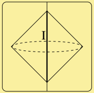
- ✓ Extends to the Fock spaces by second quantization.
- ✓ Generalizes to $m_2 > m_1$ on $E(P_0 \geq m_2)\mathcal{H}_{m_1}$.

Comparison with Hamiltonian perturbation:

- **Hamiltonian:** Local unitary equivalence between time-zero algebras $A_{m_1}(O)$, $A_{m_2}(O)$ (i.e., O doublecones based at $t = 0$): i.e., with unitaries depending on O. [Eckmann-Fröhlich, 1974]
- **Spacelike:** Global unitary equivalence between time-axis algebras $A_{m_1}(O)$, $A_{m_2}(O)$ (i.e., O doublecones spanned by intervals on the time-axis); but only on a subspace of the theory of lower mass.



on $\mathcal{H}_0$ *vs*



on $E_m\mathcal{H}_0$

- Two different unitaries for doublecones at $x = 0$: new insights into **modular theory for massive theories?**

DISCRETE CASE: HELICITY DEFORMATION

Intriguing observation for massless one-particle representations U_h of (integer) helicity:

- Repn U_h of Poincaré extends to V_h of conformal group.
- Restriction to subgp Möb × SO(3) fixing the time-axis:

$$V_h|_{\text{Möb}\times\text{SO}(3)} = \bigoplus_{\ell\geq|h|} V^{\ell+1}\otimes D^\ell.$$

(V^d = one-particle repn of Möb, corresponding to chiral current j_d of scaling dimension d.) E.g., ($h = 0$):

$$\varphi(t,\vec{0}) = j_1(t), \qquad \vec{\nabla}\varphi(t,\vec{0}) = \vec{j}_2(t).$$

- ⇒ **Higher helicity = subrepn of lower helicity** ("less degrees of freedom")
- Used to prove split property for $h \neq 0$ [Longo, Morinelli, Preta, KHR, 2018], after [Buchholz, D'Antoni, Longo, 1990] for $h = 0$.

Suggests new idea: **Higher-helicity fields as spacelike deformation of scalar fields**: Fix subgroup Möb $\times$ SO(3), deform spacelike translations P_k and boosts M_{0k} and special conformal transformations K_k on the subspace $E_{\ell\geq h}\mathcal{H}_0$.

- If $h \neq 0$, need $U_h \oplus U_{-h}$ to have local fields. This doubles the restriction to Möb $\times$ SO(3).
- Maxwell or higher helicity $\pm h$ lives on the subspace $\ell \geq h$ of two scalar fields = one complex field.
  E.g., Maxwell starts with two conformal currents $\vec{j}_2(t)$ of dimension 2: $\vec{E}(t,\vec{0})$, $\vec{B}(t,\vec{0})$.
- Identify $\vec{\nabla}(\varphi_1 \pm i\varphi_2)|_{\vec{x}=0}\Omega$ with $(\vec{E} \pm i\vec{B})|_{\vec{x}=0}\Omega$.
- Different linear dependences among derivatives:

$$\vec{\nabla}\underbrace{\vec{E}}_{\vec{j}_2} = 0, \quad \vec{\nabla}\times\underbrace{\vec{E}}_{\vec{j}_2} = -\partial_t\underbrace{B}_{\vec{j}_2{}'} \qquad \text{vs} \qquad \vec{\nabla}\underbrace{(\vec{\nabla}\varphi)}_{\vec{j}_2} = \partial_t^2\underbrace{\varphi}_{j_1}.$$

Deformation:

- Identification of $V_h \oplus V_{-h}|_{\text{Möb}\times SO(3)}$ identifies the generators P_0, D, K_0 of $\mathfrak{möb}$ and M_{kl} of $\mathfrak{so}(3)$ of the complex scalar and higher spin fields – the latter on the subspace $E_{\ell\geq h}\mathcal{H}_0$ of the former.
- Ansatz for $h > 0$:

$$\widetilde{P}_k = \sum_{\ell\geq h} a_\ell \cdot (E_{\ell+1}P_kE_\ell + E_\ell P_k E_{\ell+1}) + \sum_{\ell\geq h} b_\ell \cdot Q \cdot \varepsilon_{kmn}P_0M_{mn}E_\ell,$$

where E_ℓ are the projections to the spin-ℓ subspaces, and Q = charge operator of the scalar field.

Motivation for this ansatz:

In scalar QFT, P_k makes only transitions $\ell \to \ell \pm 1$.
In Maxwell, also $\ell \to \ell$, e.g., in $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$.

Symmetric space decomposition $\mathfrak{cf} = [\mathfrak{m\ddot{o}b} \oplus \mathfrak{so}(3)] \oplus \text{Span}\{P_k, M_{0k}, K_k\}$

$\Rightarrow$

Once $\widetilde{P}_k$ are given, then conformal Lie algebra also defines

$$\widetilde{M}_{0k} = \frac{i}{2}[K_0, \widetilde{P}_k], \quad \widetilde{K}_k = -i[K_0, \widetilde{M}_{0k}].$$

Main Result:

Deformed and undeformed generators together give a representation of the conformal Lie algebra iff $i[\widetilde{P}_k, \widetilde{K}_l] = 2\delta_{kl}D + 2M_{kl}$.

This uniquely fixes the coefficients a_ℓ and b_ℓ.

Explicitly:

$$a_\ell = \frac{\sqrt{(\ell+1)^2 - h^2}}{\ell+1}, \qquad b_\ell = \frac{h}{2\ell(\ell+1)}.$$

Discrete set of solutions with "initial condition" $a_{\ell<h} = 0$.

✓ These solutions imply the higher Maxwell equations "on the vacuum" as relations among one-particle states at $\vec{x} = 0$ and, by exponentiating $\vec{P}$, everywhere in Minkowski space.

- Identify fields at $\vec{x} = 0$ and lift generators to Fock space.
- Define deformed fields at $\vec{x} \neq 0$ by action of deformed generators.

✓ Because massless time-axis fields are local (Huygens property), locality in Minkowski space follows by conformal covariance.

✓ By Reeh-Schlieder, the higher Maxwell equations hold "everywhere" in the Fock space.

✓ Generalizes to $h_2 > h_1$ on $E(\ell \geq h_2)\mathcal{H}_{h_1}$.

Prospects

- ↯ $(m > 0, s > 0)$ from $(m = 0, s = 0)$??? No inclusion of one-particle representation spaces.
- Useful for interacting theories??? Must go beyond one-particle space representation theory.
- Will presumably require local aspects.

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