# Homotopical linear quantum Yang-Mills

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# Outline

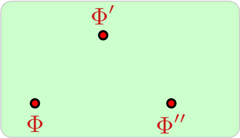
1. Gauge theory and higher structure
2. Linear Yang-Mills
3. Quantization

# Gauge theory and higher structure

# Gauge theory and higher structure

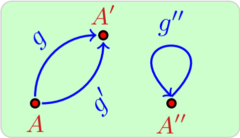
Main idea: gauge fields naturally carry higher structures

"Ordinary" field theory




fields form a set

Gauge theory




gauge fields form a groupoid

ex $\mathbf{B}G^{\mathrm{con}}(U)$ for $U \cong \mathbb{R}^m$: $\begin{cases} \text{Obj:} & A \in \Omega^1(U, \mathfrak{g}) \\ \text{Mor:} & A \xrightarrow{g} A \triangleleft g := g^{-1}Ag + g^{-1}\mathrm{d}g \end{cases}$

Weaker notion than equality: equivalence

Technical challenge: constructions must respect weak equivalences (model category theory or higher category theory)

# Gauge theory and higher structure

So gauge fields naturally carry higher structures...

...and these structures are important:

- Naive gauge orbit space (quotient) does not have all information e.g. for $\mathbf{B}G^{\mathrm{con}}$ it cannot distinguish between structure groups $\mathbb{R}$ and $U(1)$
- Constructing global fields from local fields:

$$\mathbf{B}G^{\mathrm{con}}(M) = \mathrm{holim}\Big( \prod_i \mathbf{B}G^{\mathrm{con}}(U_i) \rightrightarrows \prod_{ij} \mathbf{B}G^{\mathrm{con}}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{ijk} \mathbf{B}G^{\mathrm{con}}(U_{ijk}) \cdots \Big)$$

  homotopy limit allows for gluing up to gauge transformations

- We'll see: BV/BRST structures arise from taking a derived critical locus

# Chain complexes

Linear gauge theory can be described by chain complexes

**def** A chain complex is a sequence of vector spaces with a differential $\mathrm{d}$

$$C_\bullet = \left(\ldots \xleftarrow{d} C_{n-1} \xleftarrow{d} C_n \xleftarrow{d} C_{n+1} \xleftarrow{d} \ldots\right)$$

such that $d^2 = 0$

**def** The homology of a chain complex is

$$H_n(C_\bullet) = \mathsf{ker}\left(C_{n-1} \xleftarrow{d} C_n\right) \ / \ \mathsf{im}\left(C_n \xleftarrow{d} C_{n+1}\right)$$

**def** A quasi-isomorphism of chain complexes is a map that induces an isomorphism on homology

These are our weak equivalences

# Linear Yang-Mills

# Input data

Start with:

- A globally hyperbolic spacetime $M$
- The linear gauge field complex with structure group $\mathbb{R}$

$$\mathfrak{F}(M) = \Big( \overset{(0)}{\Omega^1(M)} \xleftarrow{\;\mathrm{d}\;} \overset{(1)}{\Omega^0(M)} \Big)$$

So gauge fields $A \in \Omega^1(M)$, gauge transformations $\epsilon \in \Omega^0(M)$ :

$$A \to A + \mathrm{d}\epsilon$$

- An action

$$S(A) = \frac{1}{2} \int_M \mathrm{d}A \wedge *\mathrm{d}A$$

yields equation of motion

$$\delta \mathrm{d}A = 0$$

# The solution complex

def Solution complex $\mathfrak{Sol}(M)$: implement the equation of motion

$$\delta_{\mathsf{var}} S = 0$$

in a way that preserves equivalences: derived critical locus

$$\begin{tikzcd} \mathfrak{Sol}(M) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "h"] & \mathfrak{F}(M) \arrow[d, "\delta_{\mathsf{var}} S"] \\ \mathfrak{F}(M) \arrow[r, "0"'] & T^*\mathfrak{F}(M) \end{tikzcd}$$

## Proposition

A model for $\mathfrak{Sol}(M)$ is given by:

$$\mathfrak{Sol}(M) = \Big( \overset{(-2)}{\Omega^0(M)} \xleftarrow{\delta} \overset{(-1)}{\Omega^1(M)} \xleftarrow{\delta \mathrm{d}} \overset{(0)}{\Omega^1(M)} \xleftarrow{\mathrm{d}} \overset{(1)}{\Omega^0(M)} \Big)$$

We recover BV/BRST fields: gauge fields $A$ in degree 0

ghost fields $c$ in degree 1

antifields $A^\ddagger$ and $c^\ddagger$ in degrees -1 and -2

# Linear observables

**def** Smooth dual complex :

$$\mathcal{L}(M) \,=\, \Big( \overset{(-1)}{\Omega^0_{\mathrm{c}}(M)} \xleftarrow{\;-\delta\;} \overset{(0)}{\Omega^1_{\mathrm{c}}(M)} \xleftarrow{\;\delta \mathrm{d}\;} \overset{(1)}{\Omega^1_{\mathrm{c}}(M)} \xleftarrow{\;-\mathrm{d}\;} \overset{(2)}{\Omega^0_{\mathrm{c}}(M)} \Big)$$

with integration pairing: $\alpha \in \mathcal{L}(M)_k$, $\beta \in \mathfrak{Sol}(M)_{-k}$

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

**rmk** Because

$$j : \mathcal{L}(M) \hookrightarrow \mathfrak{Sol}(M)[1]$$

one gets a canonical shifted Poisson structure on $\mathcal{L}(M)$ (BV: antibracket)

$$\Upsilon : \mathcal{L}(M) \otimes \mathcal{L}(M) \xrightarrow{\mathrm{id} \otimes j} \mathcal{L}(M) \otimes \mathfrak{Sol}(M)[1] \xrightarrow{\langle\,\cdot\,,\,\cdot\,\rangle} \mathbb{R}[1]$$

**rmk** $M$ is globally hyperbolic: also past compact ($\mathcal{L}_{\mathrm{pc}}$) and future compact ($\mathcal{L}_{\mathrm{fc}}$) support
So $j$ factors:

$$j : \mathcal{L}(M) \hookrightarrow \mathcal{L}_{\mathrm{pc/fc}}(M) \hookrightarrow \mathfrak{Sol}(M)[1]$$

# Retarded/advanced trivializations

## Proposition

Because $M$ is globally hyperbolic, $\mathcal{L}_{\mathrm{pc}}$ (and $\mathcal{L}_{\mathrm{fc}}$) allows for a contracting homotopy:

$$\begin{tikzcd}
\Omega^0_{\mathrm{pc}}(M) \arrow[d, "\mathrm{id}"'] & \Omega^1_{\mathrm{pc}}(M) \arrow[l, "-\delta"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, dashed, "G^+_{\Box}"] & \Omega^1_{\mathrm{pc}}(M) \arrow[l, "\delta\mathrm{d}"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, dashed, "-\delta G^+_{\Box}"] & \Omega^0_{\mathrm{pc}}(M) \arrow[l, "-\mathrm{d}"'] \arrow[d, "\mathrm{id}"'] \\
\Omega^0_{\mathrm{pc}}(M) & \Omega^1_{\mathrm{pc}}(M) \arrow[l, "-\delta"] & \Omega^1_{\mathrm{pc}}(M) \arrow[l, "\delta\mathrm{d}"] & \Omega^0_{\mathrm{pc}}(M) \arrow[l, "-\mathrm{d}"]
\arrow[from=1-1, to=2-2, dashed, "-G^+_{\Box}\mathrm{d}"]
\end{tikzcd}$$

so $\mathrm{id}_{\mathcal{L}_{\mathrm{pc/fc}}} = \partial \mathcal{G}^{\pm}$ and hence $\Upsilon = \partial(\langle\,\cdot\,,\,\cdot\,\rangle \circ \mathrm{id} \otimes \mathcal{G}^{\pm})$

$G^+_{\Box}$ is the retarded Green operator for the d'Alembertian $\Box = \delta\mathrm{d} + \mathrm{d}\delta$

rmk We call $\mathcal{G}^{\pm}$ retarded and advanced trivializations
They are to be thought of as chain complex analogues of retarded and advanced Green operators

# Unshifted Poisson structures

Using the trivializations $\mathcal{G}^{\pm}$, natural inclusions and the integration pairing we get:

- a causal propagator

$$\mathcal{G} = \mathcal{G}^{+} - \mathcal{G}^{-} : \mathcal{L}(M) \to \mathfrak{Sol}(M)$$

- an unshifted Poisson structure

$$\begin{tikzcd} \mathcal{L}(M) \otimes \mathcal{L}(M) \arrow[rr, "\tau"] \arrow[dr, "\mathrm{id}\otimes\mathcal{G}"'] & & \mathbb{R} \\ & \mathcal{L}(M) \otimes \mathfrak{Sol}(M) \arrow[ur, "\langle \cdot , \cdot \rangle"'] & \end{tikzcd}$$

rmk These structures are uniquely defined up to homotopy:

$$\mathcal{G}^{\pm} \simeq \mathcal{G}^{\pm} + \partial g^{\pm}$$

$$\tau \simeq \tau + \partial \rho$$

rmk Following the same procedure for Klein-Gordon theory, we get the usual Poisson structure

# Quantization

# Quantization

A Poisson structure allows us to canonically quantize:

$$\begin{aligned} \mathfrak{CCR} : \mathbf{PoCh}_{\mathbb{R}} &\longrightarrow \mathbf{dg}^*\mathbf{Alg}_{\mathbb{C}} \\ (V,\tau) &\longmapsto T^{\otimes}_{\mathbb{C}} V \big/ \mathcal{I}_{\mathrm{CCR}(\tau)} \end{aligned}$$

where $\mathcal{I}_{\mathrm{CCR}(\tau)}$ is the ideal generated by canonical commutation relations

$$v \otimes w - (-1)^{|v||w|} w \otimes v - i\tau(v,w)\mathbb{1}$$

## Theorem

$\mathfrak{CCR}$ preserves quasi-isomorphisms and homotopic Poisson structures:

$$\mathfrak{CCR}(V, \tau + \partial\rho) \simeq \mathfrak{CCR}(V,\tau)$$

# Quantization

cor Linear Yang-Mills theory admits a homotopically consistent quantization:

$$\begin{aligned} \big[\widehat{A}(\varphi_1), \widehat{A}(\varphi_2)\big] &= \quad i \int_M \varphi_1 \wedge *G_\Box \varphi_2 \; \mathbb{1} \\ \big[\widehat{A}^\ddagger(\alpha), \widehat{c}(\chi)\big] &= -i \int_M \alpha \wedge *G_\Box \mathrm{d}\chi \; \mathbb{1} \end{aligned}$$

for $\varphi_i \in \mathcal{L}(M)_0$, $\alpha \in \mathcal{L}(M)_1$ and $\chi \in \mathcal{L}(M)_{-1}$

thm The functor

$$\begin{aligned} \mathfrak{A} \,:\, \mathbf{Loc} &\longrightarrow \mathbf{dg}^*\mathbf{Alg}_{\mathbb{C}} \\ M &\longmapsto \mathfrak{CCR}(\mathcal{L}(M), \tau) \end{aligned}$$

is a homotopy algebraic quantum field theory in the sense of [Benini,Schenkel,Woike(2019)]

# Summary

- Gauge theory involves higher structures, that should be respected
- For linear Yang-Mills theory we use chain complexes and find familiar BV/BRST structures
- On globally hyperbolic manifolds we produce an unshifted Poisson structure using retarded and advanced trivializations of the shifted Poisson structure
- With this we can canonically quantize linear Yang-Mills theory in a way that preserves equivalences, producing a first example of a homotopy AQFT