Entanglement Measures in Quantum Field Theory

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based on joint work with K Sanders

`arXiv:1702.04924 [quant-ph]`

With a laudatio on the occasion of the 65th birthday of B.S. Kay

U. of York
4 April 2017

Bernard Kay

About 20 years ago I arrived in York to start a PhD with Bernard. He suggested that I could perhaps work on an idea of his concerning the possibility of "local vacuum states" in curved spacetime.

Among the papers I was reading for inspiration was the 177p article

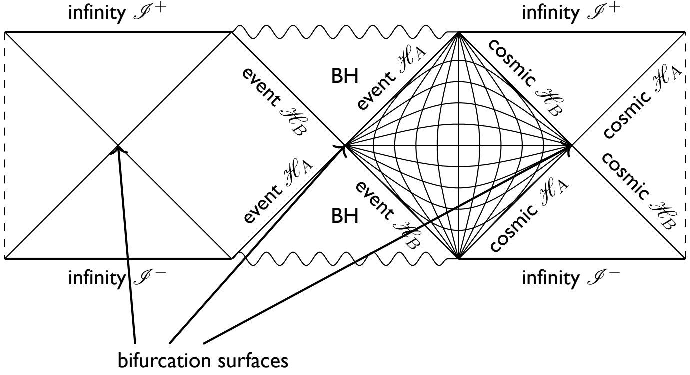
"Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon"

[Kay, Wald 1991]

which I immediately found so interesting that I began to study it in every detail.

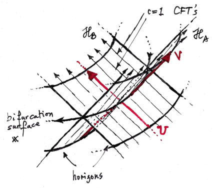
Bifurcate Killing horizons

Such geometries are a generalization of familiar BH spacetimes such as the *extended* Schwarzschild(-deSitter) spacetime, containing as essential geometric feature one (or several) pairs of intersecting horizons:



A key idea of 1991 Kay-Wald paper

The seed of the paper was a calculation of the "restriction" of the 2-point correlation function of a scalar field ϕ to a (Rindler-) horizon:



$$\text{“}\omega(\phi(x_1)\phi(x_2)) \propto \delta(\underbrace{\mathbf{x}_1 - \mathbf{x}_2}_{\text{edge cts.}}) \ln(\underbrace{U_1 - U_2}_{\text{horizon cts.}} - i0)\text{”}$$

See also [Wald, APS Einstein Prize Talk 2017]

Interpretation

Quantum modes of ϕ propagating through horizon organize themselves into those of a "bundle" of ($c = 1$) CFTs on light rays of horizon.

This idea was well ahead of its time in 1991! Its ramifications (BH entropy, holography, thermodynamics, ...) are still being pursued today, and probably tell us something deep about QFT and perhaps even quantum gravity.

P.S.: A reinterpretation of 2-point function formula for light cones also lead to a proposal for a local vacuum state in my PhD thesis.

The main result of the paper (building on 2-point function formula) was:

Main result

Any quantum state ω which is invariant under "boost" symmetry *and* "regular" across horizon necessarily has to be a thermal state at precisely the Hawking-temperature,

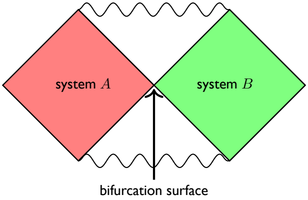
$$T_{\mathrm{Hawking}} = \frac{\kappa}{2\pi} \tag{1}$$

The surface gravity, κ comes in through the relation $U = e^{\kappa u}$ where u is the "boost" parameter. This transformation maps a vacuum state to a thermal state of the CFT. Related to [Bisognano, Wichmann 1972, Unruh 1976, Sewell 1982]

Consequence: a thermal state at a *different* temperature *necessarily* must have a singular behavior of the stress tensor $\omega(T_{ab}) \to \infty$ on the horizons $\mathscr{H}_A$ and $\mathscr{H}_B$, i.e. an observer made out of the quantum field (or coupled to it) will *burn when he/she crosses the horizon* ("firewall").

Entanglement in QFT

Perhaps essential feature of the setup studied by Kay and Wald: quantum state is strongly **entangled** (in a particular way!) between a "system A" and a "system B" across bifurcation surface:



Entanglement measures

It turns out that this is the case for **every** (regular) state in QFT across **any** pair of disjoint volumes A and B! How to define entanglement and how to measure it (in QFT)? Rest of this talk.

What is entanglement?

Standard setup of quantum theory (except measurement):

- observables: operators a on Hilbert space $\mathcal{H}$
- state: $\omega \leftrightarrow$ statistical operator, $\omega(a) = \mathrm{Tr}(\rho a) =$ expectation value
- pure state: $\rho = |\Omega\rangle\langle\Omega|$. Cannot be written as convex combination of other states, otherwise mixed.
- independent systems A and B: $\mathcal{H}_A \otimes \mathcal{H}_B$, observables for A: $a \otimes 1_B$, observables for B: $1_A \otimes b$

Separable states:

Convex combinations of product states (statistical operators $\rho_A \otimes \rho_B$).

What is entanglement?

Classically: State on bipartite system $\leftrightarrow$ probability density on phase space $\Gamma_A \times \Gamma_B$. Always separable! This motivates:

Entangled states

A state is called "entangled" if it is not separable.

Example: $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ spin-1/2 systems, Bell state $\rho = |\Omega\rangle\langle\Omega|$

$$|\Omega\rangle \propto |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle.$$

is (maximally) entangled.

Example: n dimensions $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^n$:

$$|\Omega\rangle \propto \sum_j |j\rangle \otimes |j\rangle$$

Example: ∞ dimensions:

$$|\Omega\rangle \propto \sum_j c_j |j\rangle \otimes |j\rangle, \quad c_j \to 0$$

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Example: ∞ dimensions:

$$|\Omega\rangle \propto \sum_j e^{-2\pi E_j/\kappa} |j\rangle \otimes |j\rangle$$ (Killing horizons) [Kay, Wald 1991]

When is a state more entangled than another?

More/less entanglement:

We quantify entanglement by listing the set of operations $\omega \mapsto \mathcal{F}^*\omega$ on states which (by definition!) do not increase it. $\rightarrow$ partial ordering of states.

What are these "operations"? **Single** system (channel):

- "Time" evolution: unitary transformation: $\mathcal{F}(a) = UaU^*$
- Ancillae: n copies of system: $\mathcal{F}(a) = 1_{\mathbb{C}^n} \otimes a$
- v. Neumann measurement: $\mathcal{F}(a) = PaP$, where $P : \mathcal{H} \rightarrow \mathcal{H}'$ projection
- Arbitrary combinations $=$ completely positive maps [Stinespring 1955]

Bipartite system:

Separable operations:

Convex combinations of product channels $\mathcal{F}_A \otimes \mathcal{F}_B$

Entanglement measures

This definition is consistent with basic facts [Plenio, Vedral 1998]:

- No separable state can be mapped to entangled state by separable operation
- Every entangled state can be obtained from maximally entangled state (Bell state) by separable operation

An entanglement measure E on bipartite system should satisfy:

Minimum requirements for **any** entanglement measure:

- No increase "on average" under separable operations:

$$\sum_i p_i E(\tfrac{1}{p_i}\mathcal{F}_i^*\omega) \leq E(\omega)$$

  for all states ω (NB: $p_i = \mathcal{F}_i^*\omega(1) =$ probability that i-th separable operation is performed)
- E must vanish iff state separable
- (Perhaps) various other requirements

Examples of entanglement measures

Example: Relative entanglement entropy [Uhlmann 1977, Plenio, Vedral 1998,...]:

$$E_R(\rho) = \inf_{\sigma \text{ separable}} H(\rho, \sigma) \ .$$

Here, $H(\rho, \sigma) = \mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma) =$ Umegaki's relative entropy [Araki 1970s]

Example: Distillable entanglement [Rains 2000]: $E_D(\rho) =$ log of max. number of Bell-pairs extractable via separable operations from N copies of ρ, per copy

Example: v. Neumann entropy $E_N(\rho) = -\mathrm{Tr}(\rho_A \ln \rho_A)$ of reduced state $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\, \rho$ (restriction to A, or similarly B) is not a reasonable entanglement measure except for pure states!

In fact, for pure states one has basic fact [Donald, Horodecki 2002]:

Uniqueness

For pure states, basically all entanglement measures agree with v. Neumann entropy of reduced state.

For mixed states, uniqueness is lost. In QFT, we are always in this situation!

Entanglement measures in QFT

In QFT, systems are tied to spacetime location, e.g. system A

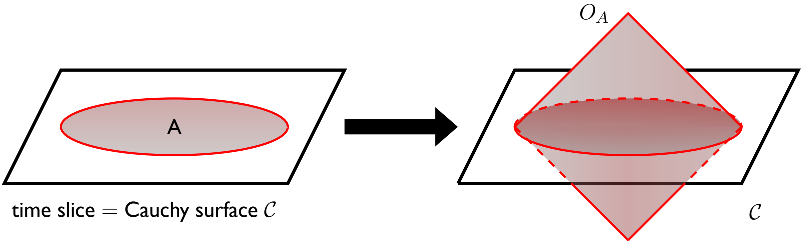

Figure: Causal diamond O_A associated with A.

Set of observables measurable within O_A is an algebra $\mathfrak{A}_A =$ "quantum fields localized at points in O_A". If A and B are regions on time slice (Einstein causality) [Haag, Kastler 1964]

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \ .$$

The algebra of all observables in A and B is called $\mathfrak{A}_A \vee \mathfrak{A}_B =$ v. Neumann algebra generated by $\mathfrak{A}_A$ and $\mathfrak{A}_B$.

Entanglement measures in QFT

Unfortunately [Buchholz, Wichmann 1986, Buchholz, D'Antoni, Longo 1987, Doplicher, Longo 1984, ... Fewster, Verch 2013]:

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \quad \text{does not always imply} \quad \mathfrak{A}_A \vee \mathfrak{A}_B \cong \mathfrak{A}_A \otimes \mathfrak{A}_B \ .$$

This will happen due to boundary effects if A and B touch each other:

Basic conclusion

a) If A and B touch, then there are no (normal) product states, so no separable states, and no basis for discussing entanglement!

b) If A and B do not touch, then there are no pure states (without firewalls)!

Therefore, if we want to discuss entanglement, we **must** leave a safety corridor between A and B, and we **must** accept b).

$\Longrightarrow$ **no unique entanglement measure!**

In the rest of talk, I explain results obtained for relative entanglement entropy E_R for various concrete states/QFTs [Hollands, Sanders 2017, 104pp]

Overview

Results obtained in [Hollands, Sanders 2017]:

1. $1+1$-dimensional integrable models
2. $d+1$-dimensional CFTs
3. Area law
4. Free quantum fields
5. Charged states
6. General bounds for vacuum and thermal states

1) Integrable models

These models (i.e. their algebras $\mathfrak{A}_A$) are constructed using an "inverse scattering" method from their 2-body S-matrix, e.g.

$$S_2(\theta) = \prod_{k=1}^{2N+1} \frac{\sinh\theta - i\sin b_k}{\sinh\theta + i\sin b_k} \ ,$$

by [Schroer, Wiesbrock 2000, Buchholz,Lechner 2004, Lechner 2008, Allazawi,Lechner 2016, Cadamuro,Tanimoto 2016].

$b_i =$ parameters specifying model, e.g. sinh-Gordon model ($N = 0$).

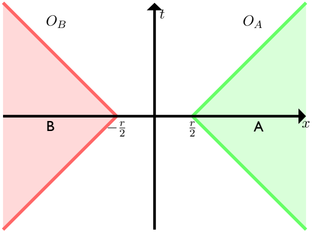

Figure: The regions A, B.

Results

For vacuum state $\rho_0 = |0\rangle\langle 0|$ and mass $m > 0$:

$$E_R(\rho_0) \lesssim C\, e^{-mr\cos k} \ .$$

for $mr \gg 1$. The constant depends on the scattering matrix, $k > 0, \alpha$.

The proof partly relies on estimates of [Lechner 2008, Allazawi,Lechner 2016]

Conjecturally (i.e. modulo one unproven estimate)

$$E_R(\rho_0) \lesssim C' \,|\ln(mr)|^{\alpha} \ ,$$

for $mr \ll 1$ for constants C', α.

2) CFTs in $3+1$ dimensions

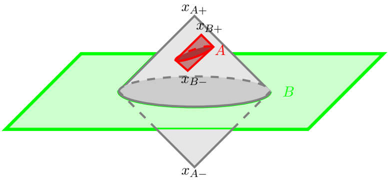

Figure: Nested causal diamonds.

Define conformally invariant cross-ratios u, v by

$$u = \frac{(x_{B+} - x_{B-})^2 (x_{A+} - x_{A-})^2}{(x_{A-} - x_{B-})^2 (x_{A+} - x_{B+})^2} > 0$$

(v similarly) and set

$$\theta = \cosh^{-1}\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{u}}\right) , \quad \tau = \cosh^{-1}\left(\frac{1}{\sqrt{v}} + \frac{1}{\sqrt{u}}\right) .$$

Results

For vacuum state $\rho_0 = |0\rangle\langle 0|$ in any $3+1$ dimensional CFT with local operators $\{\mathcal{O}\}$ of dimensions $d_\mathcal{O}$ and spins $s_\mathcal{O}, s'_\mathcal{O}$:

$$E_R(\rho_0) \leq \ln \sum_{\mathcal{O}} e^{-\tau d_\mathcal{O}} \frac{\sinh \frac{1}{2}(s_\mathcal{O}+1)\theta \ \sinh \frac{1}{2}(s'_\mathcal{O}+1)\theta}{\sinh^2(\frac{1}{2}\theta)} \ .$$

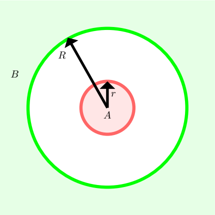

Figure: The regions A and B.

For concentric diamonds with radii $R \gg r$ this gives

$$E_R(\rho_0) \lesssim N_\mathcal{O} \left(\frac{r}{R}\right)^{d_\mathcal{O}} \ ,$$

where $\mathcal{O} =$ operator with the smallest dimension $d_\mathcal{O}$ and $N_\mathcal{O} =$ its multiplicity.

Tools: Hislop-Longo theorem [Brunetti, Guido, Longo 1994], Tomita-Takesaki theory

3) Area law in asymptotically free QFTs

A and B regions separated by a thin corridor of diameter $\varepsilon > 0$ in $d+1$ dimensional Minkowski spacetime, vacuum $\rho_0 = |0\rangle\langle 0|$.

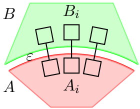

Figure: The the systems A, B

Result ("area law")

Asymptotically, as $\varepsilon \to 0$

$$E_R(\rho_0) \gtrsim \begin{cases} D_2 \cdot |\partial A|/\varepsilon^{d-1} & d > 1, \\ D_2 \cdot \ln \frac{\min(|A|,|B|)}{\varepsilon} & d = 1, \end{cases}$$

where D_2 = distillable entropy E_D of an elementary "Cbit" pair

Tools: Strong super additivity of E_D, bounds [Donald, Horodecki 2002], also [Verch, Werner 2005, Wolf, Werner 2001,HHorodecki 1999]

4) Free massive QFTs

A and B regions in a static time slice in ultra-static spacetime, $\mathrm{d}s^2 = -\mathrm{d}t^2 + h$(space); lowest energy state: $\rho_0 = |0\rangle\langle 0|$. Geodesic distance: r

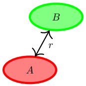

Figure: The the systems A, B

Results (decay + area law)

Dirac field: As $r \to 0$

$$E_R(\rho_0) \lesssim C|\ln(mr)| \sum_{j \geq d-1} r^{-j} \int_{\partial A} a_j$$

where a_j curvature invariants of ∂A. Lowest order $\Longrightarrow$ area law.

Klein-Gordon field: As $r \to \infty$ decay

$$E_R(\rho_0) \lesssim Ce^{-mr/2}$$

(Dirac: [Islam, to appear])

We expect our methods to yield similar results to hold generally on spacetimes with bifurcate Killing horizon, as studied by Kay and Wald in 1991 paper:

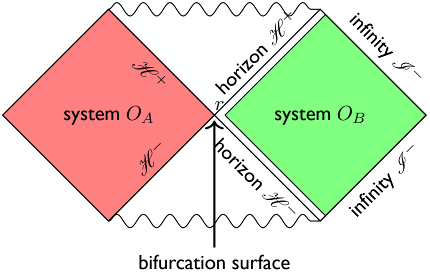

Figure: Spacetime with bifurcate Killing horizon.

5) Charged states

A and B regions, ω any normal state in a QFT in $d+1$ dim. $\chi^*\omega$ state obtained by adding "charges" χ in A or B.

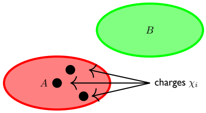

Figure: Adding charges to state in A

Result

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \prod_i \dim(\chi_i)^{2n_i} \ ,$$

n_i: # irreducible charges χ_i type i, and

$$\dim(\chi_i) = \text{quantum dimension} = \sqrt{\text{Jones index}}$$

Tools: Index-statistics theorem [Longo 1990], Jones subfactor theory, Doplicher-Haag-Roberts theory

Examples

Example: $d = 1$, Minimal model type $(p, p+1)$, χ irreducible charge of type (n, m)

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \frac{\sin\left(\frac{\pi(p+1)m}{p}\right)\sin\left(\frac{\pi p n}{p+1}\right)}{\sin\left(\frac{\pi(p+1)}{p}\right)\sin\left(\frac{\pi p}{p+1}\right)} \ .$$

Example: $d > 1$, general QFT, irreducible charge χ with Young tableaux statistics

8	6	5	4	2	1
5	3	2	1		
1					

.

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq 2 \ln 5,945,940$$

6) Decay in general QFTs

A and B regions in a time slice of Minkowski. Distance: r. QFT satisfies nuclearity condition a la Buchholz-Wichmann

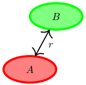

Figure: The the systems A, B

Results (Decay)

Vacuum state in massive theory:

$$E_R(\rho_0) \lesssim C\, e^{-(mr)^k} \; ,$$

for any given $k < 1$ (our C diverges when $k \to 1$)
Thermal state:

$$E_R(\rho_\beta) \lesssim C r^{-\alpha+1} \; ,$$

for $\alpha > 1$ a constant in nuclearity condition. Similar for massless theory.

Let me end the talk coming back to Kay-Wald 1991 paper, which contained many more results.

- An argument that thermal states do not exist in Kerr due to super radiance: major physical prediction
- A technically precise formulation of the notion of "Hadamard state": major role in subsequent further developments of QFT on CST!
- An explanation of the connection with Tomita-Takesaki theory of v. Neumann algebras: TTt important for several results in this talk.

I think that these issues deserve further study, in particular 1)! Perhaps Bernard himself will get involved in this, or perhaps he will carry further many of his other beautiful ideas such as:

- His insightful no-go theorems for quantum field theories on spacetimes with "closed timelike curves" [Kay, Wald, and Radzikowski 1997]
- His pioneering and beautifully simple explanation/calculation of the Casimir effect on a torus, which was an important inspiration for later work on the renormalization problem in perturbative QFT, and the notion of "local and covariance" [Kay 1979]
- His pioneering work on (classical) linear stability of Schwarzschild spacetime [Kay, Wald 1987]

... or perhaps he will do something completely different. At any rate:

Happy Birthday, Bernard!