Minimal index and dimension for 2-C^*-categories

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Physical motivation: Quantum Information (operator-algebraic setup)

Quantum system: non-commutative *von Neumann algebra* $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$,
(observables = self-adjoint part of $\mathcal{M}$, e.g., projections in $p \in \mathcal{M}$, $p = p^*p$)

Classical part: *center* of $\mathcal{M}$, denoted by $\mathcal{Z}(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}$,
(here assumed to be finite-dimensional, $\mathcal{Z}(\mathcal{M}) \cong \mathbb{C}^n$)

$$\mathcal{M} \cong \bigoplus_{i=1,\ldots,n} \mathcal{M}_i, \quad \mathcal{M}_i := p_i \mathcal{M} p_i, \quad p_i \in \mathcal{Z}(\mathcal{M})$$

canonical decomposition if p_i are minimal, and also $\mathcal{Z}(\mathcal{M}_i) \cong \mathbb{C}$, i.e.,
$\mathcal{M}_i$ is a *factor* ($\rightsquigarrow$ purely quantum part of the system) for every $i = 1, \ldots, n$.

e.g. $$\bigoplus_{i=1,\ldots,n} M_{k_i}(\mathbb{C}) \quad \text{“multi-matrix” algebra,} \quad M_{k_i}(\mathbb{C}) = k_i \times k_i \text{ matrices}$$

(finite-dimensional C^*-algebra, living on $\bigoplus_i \mathbb{C}^{k_i}$, $\rightsquigarrow$ “finite” quantum system)

Aim: develop the mathematical framework for (possibly) “infinite” systems, i.e., bigger and more non-commutative factors $\mathcal{M}_i$)

States: linear maps $\varphi : \mathcal{M} \to \mathbb{C}$, unital $\varphi(\mathbb{1}) = 1$, positive $\varphi(a^*a) \geq 0$, $a \in \mathcal{M}$, normal, faithful.

Channels: (communication, information transfer) among two systems $\mathcal{N}$ and $\mathcal{M}$, linear, unital, normal, completely positive maps $\alpha : \mathcal{N} \to \mathcal{M}$ so

$$\text{for every state } \varphi \text{ on } \mathcal{M}, \quad \alpha^{\#}(\varphi) := \varphi \circ \alpha \quad \text{is a state on } \mathcal{N}$$

e.g., $\alpha =$ *-homomorphism (if injective then $\alpha = \iota : \mathcal{N} \hookrightarrow \mathcal{M}$), conditional expectation (if surjective and $\alpha^2 = \alpha$, then $\alpha = E : \mathcal{N} \to \mathcal{M}$), bimodule ${}_{\mathcal{N}}\mathcal{H}_{\mathcal{M}}$.
(all examples of 1-arrows in suitable 2-categories, or bicategories)

In this setup (*arxiv:1710.00910* [Longo]) gives a mathematical derivation of Landauer's bound: lower bound on the amount of energy (heat) introduced in the system when 1 bit of information is deleted (logically irreversible operation)

$$\text{either} \quad E_\alpha = 0 \quad \text{or} \quad E_\alpha \geq \frac{1}{2} kT \log(2)$$

$k =$ Boltzmann's constant, $T =$ temperature
$\rightsquigarrow$ "solves" the paradox of Maxwell's demon [Bennet]

Mathematical needs: study a **"dimension"** D_α of a channel $\alpha : \mathcal{N} \to \mathcal{M}$

- how to **define** D_α?
- is it **multiplicative**? namely $D_{\beta\circ\alpha} = D_\beta \cdot D_\alpha$ where $\mathcal{N} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{L}$?
  we can also denote $\beta \circ \alpha = \beta \otimes \alpha$.
- is it **additive**? namely $D_{\alpha\oplus\beta} = D_\alpha + D_\beta$ where $\mathcal{N} \xrightarrow{\alpha,\beta} \mathcal{M}$?

In the special case of inclusions of factors $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ (called *"subfactors"*) the dimension is a number, the square root of the **minimal index** (Jones' index)

$$d_\iota = [\mathcal{M} : \mathcal{N}]_0^{1/2}$$

Much more generally, a good notion of dimension is available for objects in "rigid" tensor C^*-categories [Longo-Roberts] provided the tensor unit object I is "simple" (factoriality assumption, indeed if $I = \mathrm{id}_\mathcal{M} : \mathcal{M} \to \mathcal{M}$, $(\mathrm{id}_\mathcal{M}, \mathrm{id}_\mathcal{M}) = \mathcal{Z}(\mathcal{M})$).

- how about **non-simple unit** case? in particular, minimal index for **non-factor inclusions** $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$?

Idea: $\mathcal{N}$, $\mathcal{M}$ von Neumann algebras (possibly infinite-dimensional), $\mathcal{N} \subset \mathcal{M}$, unital. **Jones' index** $[\mathcal{M} : \mathcal{N}]$ measures the relative size of $\mathcal{M}$ w.r.t. $\mathcal{N}$.

Examples

- inclusion of full matrix algebras (finite type I subfactor)

$$\mathcal{N} \subset \mathcal{M} \cong M_k(\mathbb{C}) \otimes \mathbb{1}_l \subset M_{\tilde{k}}(\mathbb{C}), \quad \tilde{k} = kl$$

  then $[\mathcal{M} : \mathcal{N}] = \tilde{k}^2/k^2 = l^2$, dimension $= l$, and $[\mathcal{M} : \mathcal{N}] \in \{1, 4, 9, \ldots\}$.

- multi-matrix inclusion (not a subfactor, finite-dimensional algebras)

$$\mathcal{N} \subset \mathcal{M} \cong \bigoplus_{j=1,\ldots,n} M_{k_j}(\mathbb{C}) \hookrightarrow \bigoplus_{i=1,\ldots,m} M_{\tilde{k}_i}(\mathbb{C})$$

  then $[\mathcal{M} : \mathcal{N}] = \|\Lambda\|^2$, dimension $= \|\Lambda\|$, where $\Lambda =$ "inclusion matrix", $m \times n$, and $[\mathcal{M} : \mathcal{N}] \in \{4\cos^2(\pi/q), q = 3, 4, 5, \ldots\} \cup [4, +\infty[$.

- $\mathcal{N} \subset \mathcal{M}$ type II_1 subfactor (infinite-dimensional von Neumann algebras, with a trace state $\mathrm{tr} : \mathcal{M} \to \mathbb{C}$, $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, $a, b \in \mathcal{M}$) $\rightsquigarrow$ Jones' index.

More generally [Kosaki]: for arbitrary **factors** $\mathcal{N}$, $\mathcal{M}$ (possibly of type III) the **index** of $\mathcal{N} \subset \mathcal{M}$ is defined w.r.t. normal faithful **conditional expectations** $E : \mathcal{M} \to \mathcal{N}$ (in particular $E(n_1 m n_2) = n_1 E(m) n_2$ for $m \in \mathcal{M}$, $n_1, n_2 \in \mathcal{N}$)

$$\mathrm{Ind}(\mathcal{N} \overset{E}{\subset} \mathcal{M}) \in [1, +\infty].$$

Examples of expectations: for $M_k(\mathbb{C}) \otimes \mathbb{1}_l \subset M_{kl}(\mathbb{C}) \cong M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$, let $E = \mathrm{id}_k \otimes \mathrm{tr}_l$ "partial trace", or any $E = \mathrm{id}_k \otimes \varphi$, where φ state on $M_l(\mathbb{C})$.

Theorem (Longo, Hiai, Havet)

If a subfactor $\mathcal{N} \subset \mathcal{M}$ has finite index, i.e., admits some $E : \mathcal{M} \to \mathcal{N}$ with finite index, then $\exists!$ **minimal** *conditional expectation $E_0 : \mathcal{M} \to \mathcal{N}$, i.e., such that*

$$\mathrm{Ind}(\mathcal{N} \overset{E_0}{\subset} \mathcal{M}) \leq \mathrm{Ind}(\mathcal{N} \overset{E}{\subset} \mathcal{M}) \quad \textit{for every other } E$$

and $[\mathcal{M} : \mathcal{N}]_0 := \mathrm{Ind}(\mathcal{N} \overset{E_0}{\subset} \mathcal{M})$ is called the **minimal index** *of $\mathcal{N} \subset \mathcal{M}$.*

Let $\mathcal{N} \subset \mathcal{M}$ be a subfactor (infinite factors) with finite index, given $E : \mathcal{M} \to \mathcal{N}$ n.f. conditional expectation, then minimality of E is characterized as follows:

Theorem (Hiai, Longo-Roberts)

$$E = E_0 \quad \Leftrightarrow \quad E_{\restriction \mathcal{N}' \cap \mathcal{M}} = E'_{\restriction \mathcal{N}' \cap \mathcal{M}} \quad \textbf{“sphericality”}$$

where we consider $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{M}' \subset \mathcal{N}'$, the “dual” subfactor, and

$$\begin{aligned} &E : \mathcal{M} \to \mathcal{N}, && E(\mathcal{N}' \cap \mathcal{M}) = \mathcal{N}' \cap \mathcal{N} \cong \mathbb{C} \\ &E' : \mathcal{N}' \to \mathcal{M}', \text{ “dual” expectation}, && E'(\mathcal{N}' \cap \mathcal{M}) = \mathcal{M}' \cap \mathcal{M} \cong \mathbb{C}. \end{aligned}$$

Moreover, E is “left” and E' is “right” in a tensor C^*-categorical (or better 2-C^*-categorical) reformulation.

Notice first that $\mathcal{N}' \cap \mathcal{M} = \{m \in \mathcal{M} : mn = nm, \forall n \in \mathcal{N}\}$ is an *intertwining relation* between $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ and itself, because $\iota(n) = n$, i.e., $\mathcal{N}' \cap \mathcal{M} = (\iota, \iota)$.

Why E is **"left"** and E' is **"right"**?

E, E' correspond to pairs of solutions r, $\bar{r}$ of the **conjugate equations** for $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ (1-arrow in a 2-category), namely there is a "conjugate" 1-arrow $\bar{\iota} : \mathcal{M} \to \mathcal{N}$ and

$$r \in (\mathrm{id}_{\mathcal{N}}, \bar{\iota} \circ \iota), \quad \bar{r} \in (\mathrm{id}_{\mathcal{M}}, \iota \circ \bar{\iota}),$$

intertwining relations in $\mathcal{N}$ and $\mathcal{M}$ respectively, fulfilling the following identities in (ι, ι) and $(\bar{\iota}, \bar{\iota})$ respectively:

$$\bar{r}^* \iota(r) = \mathbb{1}_\iota, \quad r^* \bar{\iota}(\bar{r}) = \mathbb{1}_{\bar{\iota}}.$$

Then

$E(t) = (r^* r)^{-1} \cdot \iota(r^*) \iota\bar{\iota}(t) \iota(r)$ [Longo] indeed $\iota\bar{\iota} = \gamma$ is Longo's canonical endo

$E'(t) = (\bar{r}^* \bar{r})^{-1} \cdot \bar{r}^* t \bar{r}$ [Baillet-Denizeau-Havet, Kawakami-Watatani]

for every $t \in (\iota, \iota)$, actually the fist makes sense for $t \in \mathcal{M}$, the second for $t \in \mathcal{N}'$.

The **dimension** of $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ (subfactor case) is $d = r^*r = \bar{r}^*\bar{r}$ (a number) and $d^2 = [\mathcal{M} : \mathcal{N}]_0$. Moreover:

Theorem (Longo, Kosaki-Longo)

- **normalization:** $d = 1$ *if and only if* $\mathcal{N} = \mathcal{M}$.
- **multiplicativity:** $\mathcal{N} \overset{d_1}{\subset} \mathcal{M} \overset{d_2}{\subset} \mathcal{L}$ *then the dimension of* $\mathcal{N} \subset \mathcal{L}$ *is* $d_1 d_2$, *hence in particular* $E_0^{\mathcal{N}\subset\mathcal{M}} \circ E_0^{\mathcal{M}\subset\mathcal{L}} = E_0^{\mathcal{N}\subset\mathcal{L}}$.
- **additivity:** *for every* $p_1, p_2 \in \mathcal{N}' \cap \mathcal{M}$ *such that* $p_1 + p_2 = \mathbb{1}$, *define* $d_i :=$ *dimension of* $\mathcal{N}_i \subset \mathcal{M}_i$ *where* $\mathcal{N}_i := p_i \mathcal{N} p_i$, $\mathcal{M}_i := p_i \mathcal{M} p_i$, $i = 1, 2$. *Then* $d = d_1 + d_2$.

News: This is no longer true if $\mathcal{N}$ or $\mathcal{M}$ have a *non-trivial center* (e.g., $\mathcal{N} \subset \mathcal{M}$ multi-matrix inclusion), unless we consider not the "scalar dimension" (whose square is still the minimal index) but the **"dimension matrix"**.

Theorem (Havet, Teruya, Jolissaint)

Let $\mathcal{N} \subset \mathcal{M}$ be a finite index inclusion of von Neumann algebras, assume finite-dimensional centers and "connectedness", i.e., $\mathcal{Z}(\mathcal{N}) \cap \mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbb{1}$. Then $\exists!\ E_0 : \mathcal{M} \to \mathcal{N}$ minimal, i.e.,

$$\| \operatorname{Ind}(\mathcal{N} \overset{E_0}{\subset} \mathcal{M})\| \leq \| \operatorname{Ind}(\mathcal{N} \overset{E}{\subset} \mathcal{M})\| \quad \textit{for every other } E$$

because $\operatorname{Ind}(\mathcal{N} \overset{E}{\subset} \mathcal{M}) \in \mathcal{Z}(\mathcal{M})$ *in general. Moreover,* $\operatorname{Ind}(\mathcal{N} \overset{E_0}{\subset} \mathcal{M}) = c\mathbb{1}$ *and* $c = \| \operatorname{Ind}(\mathcal{N} \overset{E_0}{\subset} \mathcal{M})\|$ *(a number) =: minimal index of* $\mathcal{N} \subset \mathcal{M}$*.*

Questions: How to characterize minimality of E? properties of the minimal index? does it admit a 2-C^*-categorical formulation (hence generalization)? (what does "standard" solution of the conjugate equations mean?)

$$E : \mathcal{M} \to \mathcal{N}, \quad E(\mathcal{N}' \cap \mathcal{M}) = \mathcal{N}' \cap \mathcal{N} = \mathcal{Z}(\mathcal{N})$$
$$E' : \mathcal{N}' \to \mathcal{M}', \quad E'(\mathcal{N}' \cap \mathcal{M}) = \mathcal{M}' \cap \mathcal{M} = \mathcal{Z}(\mathcal{M}),$$

$$E_{\restriction \mathcal{N}' \cap \mathcal{M}} = E'_{\restriction \mathcal{N}' \cap \mathcal{M}} \quad ??$$

Theorem (LG-Longo)

Let $\mathcal{N} \subset \mathcal{M}$, let $p_1, \ldots, p_n$ minimal central projections in $\mathcal{M}$, also called atoms in $\mathcal{Z}(\mathcal{M})$, and $q_1, \ldots, q_m$ atoms in $\mathcal{Z}(\mathcal{N})$. Then

$$E = E_0 \quad (\textit{i.e.}, \ E' = E'_0) \quad \Leftrightarrow \quad \omega_l \circ E_{\restriction \mathcal{N}' \cap \mathcal{M}} = \omega_r \circ E'_{\restriction \mathcal{N}' \cap \mathcal{M}}$$

where ω_l and ω_r are uniquely determined (connectedness) states on $\mathcal{Z}(\mathcal{N})$ and $\mathcal{Z}(\mathcal{M})$ respectively, called "left" and "right" state of $\mathcal{N} \subset \mathcal{M}$.

Let $\omega_s := \omega_l \circ E = \omega_r \circ E'$ on $\mathcal{N}' \cap \mathcal{M}$ and call it **"spherical state"** *of $\mathcal{N} \subset \mathcal{M}$, then ω_s is a tracial and*

$$\omega_s(\cdot)\mathbb{1} = s\text{-}lim\{EE'EE'EE' \ldots\}$$

i.e., the projection $\mathcal{N}' \cap \mathcal{M} \to \mathcal{Z}(\mathcal{N}) \cap \mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbb{1}$.

- do $\omega_{l/r/s}$ depend on $\mathcal{N} \subset \mathcal{M}$ or on $\mathcal{N}$, $\mathcal{M}$ alone?
- can we categorize ω_s? (hence the minimality of E and the dimension)
- is it more data or can we derive it? how to compute the minimal index?

Continued:

Theorem (LG-Longo)

For every $i = 1, \ldots, n$, $j = 1, \ldots, m$, if $p_i q_j \neq 0$, observe that $p_i q_j \in \mathcal{Z}(\mathcal{N}' \cap \mathcal{M})$, set $\mathcal{N}_{ij} := p_i q_j \mathcal{N} p_i q_j$ and $\mathcal{M}_{ij} := p_i q_j \mathcal{M} p_i q_j$, then $\mathcal{N}_{ij} \subset \mathcal{M}_{ij}$ is a subfactor. Set

$$D := (d_{ij})_{i,j} \quad m \times n \text{ matrix, called } \textbf{“dimension matrix”} \text{ of } \mathcal{N} \subset \mathcal{M}$$

where d_{ij} := dimension of $\mathcal{N}_{ij} \subset \mathcal{M}_{ij}$ (quantized as in Jones' theorem), or $d_{ij} := 0$ if $p_i q_j = 0$. Then then minimal index of $\mathcal{N} \subset \mathcal{M}$ equals

$$d^2 = \|D\|^2, \quad d := \|D\| \quad \textbf{“scalar dimension”} \text{ of } \mathcal{N} \subset \mathcal{M}$$

and the (unique, l^2-normalized) Perron-Frobenius eigenvectors

$$D^t D \sqrt{\nu} = d^2 \sqrt{\nu}$$
$$D D^t \sqrt{\mu} = d^2 \sqrt{\mu}$$

and $\nu_j = \omega_l(q_j)$, $\mu_i = \omega_r(p_i)$ are the left/right states of $\mathcal{N} \subset \mathcal{M}$.

Moreover:

- the states $\omega_{l/r/s}$ do depend on the inclusion (even for multi-matrices).
- we can reconstruct E_0 (i.e., the **"standard"** solution of the conjugate eqns. for $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$) out of the minimal expectations in $\mathcal{N}_{ij} \subset \mathcal{M}_{ij}$ and an expectation matrix Λ determined by D and by P-F data:

$$\lambda_{ij} := \frac{d_{ij}}{d} \frac{\sqrt{\mu_i}}{\sqrt{\nu_j}} \quad \text{i.e.} \quad r = \bigoplus_{i,j} \frac{\sqrt[4]{\mu_i}}{\sqrt[4]{\nu_j}} r_{ij}$$

where $r_{ij}, \bar{r}_{ij}$ are the standard solutions for $\iota_{ij} : \mathcal{N}_{ij} \hookrightarrow \mathcal{M}_{ij}$.

- **additivity:** D of $\mathcal{N} \subset \mathcal{M}$ is $D = D_1 + D_2$ if D_1, D_2 correspond to $p_1, p_2 \in \mathcal{N}' \cap \mathcal{M}$, $p_1 + p_2 = \mathbb{1}$. But $d \neq d_1 + d_2$ in general. Indeed $d^2 = d_1^2 + d_2^2$ if $\mathcal{N}$ or $\mathcal{M}$ is a factor and p_1, p_2 are minimal in $\mathcal{Z}(\mathcal{M})$ or $\mathcal{Z}(\mathcal{N})$, i.e., **the index itself may be additive**. More generally

$$d = \sum_{i,j} d_{ij} \sqrt{\nu_j} \sqrt{\mu_i}.$$

- **multiplicativity:** Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{L}$ then D of $\mathcal{N} \subset \mathcal{L}$ is $D = D_2 D_1$ where D_1 and D_2 correspond to the intermediate inclusions, i.e., **the (matrix) dimension is multiplicative**. But $d \neq d_1 d_2$ in general. However $d \leq d_1 d_2$ and equality holds *if* $\nu^{\mathcal{M} \subset \mathcal{L}} = \mu^{\mathcal{N} \subset \mathcal{M}}$, e.g., if $\mathcal{M}$ is a factor. If $\mathcal{N}$ and $\mathcal{L}$ are factors then $d = \cos(\alpha) d_1 d_2$, where $\alpha :=$ angle between vectors D_1 and D_2.
- we have a theory of dimension for rigid 2-C^*-categories with finite-dimensional "centers", how about **infinite-dimensional** ones?
- further applications of "standard" Q-systems to finite index **non-factorial** extensions of **QFTs**? (cf. construction of theories with "defects" [B-K-L-R]).