From the construction of integrable QFTs to the classification of unitary R-matrices

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partly joint with:
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- The models in the background are **integrable models** on two-dimensional Minkowski space, defined by their 2-body S-matrix ("S-matrix bootstrap").
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- The models in the background are **integrable models** on two-dimensional Minkowski space, defined by their 2-body S-matrix (“S-matrix bootstrap”).
- The mathematical structure of central interest is the **Yang-Baxter equation (YBE)** which is relevant to the factorisation of a $3 \to 3$ scattering process into $2 \to 2$ processes.
- The YBE is also of prominent interest in many other fields: statistical mechanics, subfactors, knot theory, quantum information, braid groups ...
- Will investigate it here with **tools from algebraic QFT**.

S-matrix is main input into S-matrix bootstrap: A continuous map $S : \mathbb{R} \to \mathcal{B}(V \otimes V)$ (with V a finite-dim. Hilbert space labelling particle species) such that

- $S(\theta)^* = S(\theta)^{-1} = S(-\theta)$
- S satisfies the **Yang-Baxter equation** (with spectral parameter):

$$S_1(\theta)S_2(\theta+\theta')S_1(\theta') = S_2(\theta')S_1(\theta+\theta')S_2(\theta)$$

  with $S_1(\theta) := S(\theta) \otimes \mathrm{id}_V$ and $S_2(\theta) := \mathrm{id}_V \otimes S(\theta)$
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▶ To proceed to a **local** QFT, an additional "intertwiner property" of S is required (under control for scalar $\dim V = 1$ theories and certain non-scalar ones).

In the scalar case ($\dim V = 1$) [Bostelmann-L-Morsella '11]:

- One can proceed to a short distance scaling limit if
$$S_\pm := \lim_{\theta \to \pm\infty} S(\theta)$$
exist.
- One finds a massless chiral theory, possibly with a twist between the lightrays encoded in $S(0)$.
- The chiral components are generated by fields localized on half lines. Obstructions to local observables arise from the operators $S_\pm$.

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These structures generalize to the non-scalar setting [Scotford], but now the structure of the matrices $S(0), S_+, S_-$ can be more involved.

- Note: $S(0), S_\pm$ are **R-matrices**, i.e. unitary solutions to the (constant) YBE

$$(R\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes R)(R\otimes \mathrm{id}_V) = (\mathrm{id}_V\otimes R)(R\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes R)$$

  Moreover, $S(0)$ is involutive: $S(0)^2 = 1$.

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- A brute force approach to the YBE is hopeless. In components, YBE is a coupled system of d^6 cubic eqns for d^4 variables.
- Need to embed R-matrices into a richer mathematical context.
- **Plan:** Given any R, define an **endomorphism** λ_R on a von Neumann algebra $\mathcal{M}$ and consider the inclusion

$$\lambda_R(\mathcal{M}) \subset \mathcal{M}.$$

In this way, we can use tools from operator algebras, **subfactors**, and **QFT** (superselection theory).

$V \cong \mathbb{C}^d$: finite-dim. Hilbert space. Define two v.Neumann algebras:

$$\mathcal{N} := \operatorname{End} V \otimes \operatorname{End} V \otimes \ldots \qquad \text{(infinite tensor product)}$$

$$\subset \mathcal{M} := \pi_\omega(\mathcal{O}_V)'' \qquad \text{(generated by Cuntz algebra)}$$

$\mathcal{N}$ is weakly closed w.r.t. **trace** $\tau = \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \ldots$

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$$\lambda_U(x) = \lim_{n\to\infty} U\varphi(U)\cdots\varphi^n(U)\cdot x\cdot\varphi^n(U^*)\cdots\varphi(U^*)U^*.$$

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- In particular, any R-matrix $R \in \operatorname{End} V \otimes \operatorname{End} V \subset \mathcal{O}_V$ defines a **“Yang-Baxter endomorphism”** λ_R of $\mathcal{M}$ (preserving $\mathcal{N}$).
- Choosing $U = F \in \operatorname{End} V \otimes \operatorname{End} V$ as the **flip** gives the **canonical endomorphism** $\varphi := \lambda_F$. On $\mathcal{N}$ it acts as a shift,

$$\varphi(x) = \mathrm{id}_V \otimes x.$$

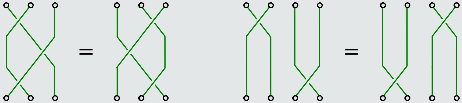
Proposition

Let $R \in \operatorname{End} V \otimes \operatorname{End} V$ be unitary. Then $R \in \mathcal{R}$ iff [Conti/Hong/Szym.'12]

$$R \in \lambda_R^2(\mathcal{M})' \cap \mathcal{M}.$$

In this case, $\pi_R(b_n) := \varphi^{n-1}(R)$ represents the braid group B_∞ in $\mathcal{N}$,

$$b_n b_{n\pm1} b_n = b_{n\pm1} b_n b_{n\pm1}, \qquad b_n b_m = b_m b_n, \quad |n-m| \geq 2.$$



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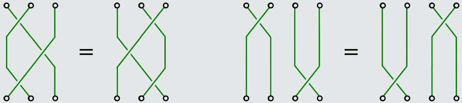

and λ_R **coincides with** φ on the von Neumann algebra $\mathcal{L}_R \subset \mathcal{N}$ generated by the representation.

This structure is strongly reminiscent of **braid group statistics in 2d QFT** [Fredenhagen-Rehren-Schroer '89, Longo '91], generalizing permutation group statistics [DHR '71], and braided subfactors.

$\mathcal{M}$

$$\lambda_R(\mathcal{M}) \subset \mathcal{M}$$

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Simple consequence of $R \in \lambda_R^2(\mathcal{M})' \cap \mathcal{M}$:

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Thus, for non-trivial R

- the inclusion $\lambda_R(\mathcal{M}) \subset \mathcal{M}$ is non-trivial.
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- It could still have trivial relative commutant $\lambda_R(\mathcal{M})' \cap \mathcal{M}$ (then λ_R is called **irreducible**).
- λ_R is not invertible, but has a **left inverse** ϕ_R, related to the **conditional expectation** E_R

$$E_R : \mathcal{M} \to \lambda_R(\mathcal{M}) \quad \text{“projection onto subalgebra”}$$
$$\phi_R := \lambda_R^{-1} \circ E_R : \mathcal{M} \to \mathcal{M}$$

Questions:

- Find all unitary R-matrices (up to an equivalence relation).
- Describe all irreducible endomorphisms.
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- Properties of YB endomorphisms: Index, ergodicity, ...

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All of these are hard problems in general, but partial answers exist.

Irreducibility and reduction

- For (ir)reducibility, one computes the first relative commutant

$$\lambda_R(\mathcal{M})' \cap \mathcal{M} = \{x \in \operatorname{End} V : R^*(x \otimes 1)R = 1 \otimes x\}$$

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This gives a reduction scheme for reducible **involutive** $R = R^*$:

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3. Find

$$R \sim R_p \boxplus R_p^\perp := R_p \oplus R_p^\perp \oplus F$$
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4. Repeat until $R_p, R_p^\perp$ are irreducible.

Here $\sim$ means an **equivalence relation** on $\mathcal{R}$ defined by the "intertwiner property".

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With more work, get:

Theorem

Equivalence classes of involutive R-matrices are in 1:1 correspondence with R-matrices of **normal form**,

$$N = \boxplus_{i=1}^{m} \varepsilon_i \, \mathrm{id}_{d_i}$$

with signs $\{\varepsilon_1, \dots, \varepsilon_m\} \in \{\pm 1\}$ and dimensions $d_1, \dots, d_m \in \mathbb{N}$.

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- "Intertwiner problem" solved for finite-dimensional (purely algebraic) case. Also solved for scalar (pure analytic) case. General mixed case still open.

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- Analogies to DHR analysis of permutation group statistics.
- "Intertwiner problem" solved for finite-dimensional (purely algebraic) case. Also solved for scalar (pure analytic) case. General mixed case still open.
- In QFT models with constant S-matrix, get decomposition into tensor products of free scalar Bose/Fermi theories [Scotford

The Markov property

To understand (certain) non-involutive R-matrices, it is useful to consider the matrix

$$\phi_R(R) \in \operatorname{End} V.$$

Interesting facts:

- $\phi_R(R)$ can be computed explicitly as a partial trace of R, namely

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Have straightforward sufficient condition for the Markov property:

Proposition

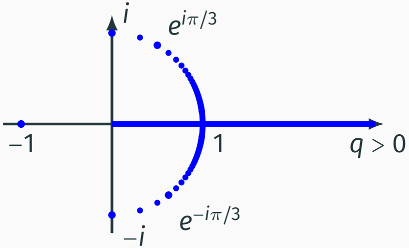
If R has no pair of opposite eigenvalues $q, -q$, then (M) holds.

R-matrices with two eigenvalues

Consider R-matrices with two eigenvalues, say -1 and q.

- In this case, R must necessarily be **selfadjoint** or **unitary**. Positivity of the braid group character τ_R defined by R requires

$$q \in \{-1\} \cup \mathbb{R}_+ \cup \{e^{\pm 2\pi i/\ell} \,:\, \ell = 4, 5, 6, \ldots\}.$$



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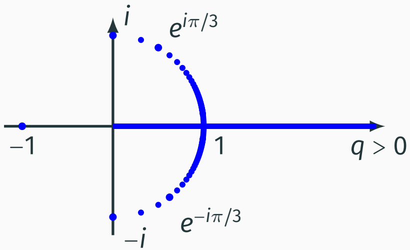

[Wenzl 1988, Fredenhagen/Rehren/Schroer 1989, Ocneanu 1985]

- The case $q = 1$ is the case of involutive R, discussed before.
- Can be extended to $q > 0$ by deformation procedure.

R-matrices with two eigenvalues -1 and $q \neq \pm 1$

- **All such R-matrices have the Markov property.** (No pair of opposite eigenvalues.)
- A Markov trace is uniquely fixed by the value $\tau_R(e_1)$ (with $\pi_R(e_1)$ = eigen projection of R) [Jones '87].

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$$\tau_R(e_1) = \frac{\sin \frac{\pi(k-1)}{\ell}}{2 \cos \frac{\pi}{\ell} \sin \frac{\pi k}{\ell}}, \qquad k \in \{1, \ldots, \ell - 1\}, \;\; q = e^{\pm i\pi/\ell}.$$

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Theorem

- Equivalence classes characterised by d, q, and its multiplicity.
- Unitary R-matrices with eigenvalues -1 and $q \neq \pm 1$ exist if and only if $q = \pm i$ or $q = e^{\pm i\pi/3}$.
- If $q = \pm i$, then $\tau_R(e_1) = \frac{1}{2}$. If $q = e^{\pm i\pi/3}$, then $\tau_R(e_1) \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$.

An almost complete Theorem

The following families (1),(2a),(2b) (and maybe (3)) of equivalence classes of R-matrices occur:

(1) $q = \pm i$, $d_R \in 2\mathbb{N}$, $\tau_R(e_1) = \frac{1}{2}$,

$$R \sim \frac{-1 \pm i}{2} \begin{pmatrix} 1 & & & -1 \\ & 1 & -1 & \\ & 1 & 1 & \\ 1 & & & 1 \end{pmatrix} \boxtimes 1_k$$

(2a) $q = e^{\pm i\pi/3}$, $d_R \in 3\mathbb{N}$, $\tau_R(e_1) = \frac{1}{3}$, with $R \sim (-P + e^{\pm i\pi/3}(1 - P)) \boxtimes 1_k$,

$$P = \frac{1}{3} \begin{pmatrix} 1 & & & & & \bar{q}^2 & & 1 & \\ & 1 & & \bar{q}^2 & & & & & \bar{q}^2 \\ & & 1 & & 1 & & 1 & & \\ & q^2 & & 1 & & & & & 1 \\ & & 1 & & 1 & & 1 & & \\ q^2 & & & & & 1 & & q^2 & \\ & & 1 & & 1 & & 1 & & \\ 1 & & & & & \bar{q}^2 & & 1 & \\ & q^2 & & 1 & & & & & 1 \end{pmatrix}$$

An almost complete Theorem

(...)

(2b) As (2a), but with $P \leftrightarrow (1 - P)$.

(3) There might be another class with $q = e^{\pm i\pi/3}$, $\tau_R(e_1) = \frac{1}{2}$. Then necessarily $d_R \in \{4, 6, 8, \ldots\}$ and index$= 4$ (not Temperley-Lieb).

Cases (1),(2a),(2b) are irreducible, with index 2 and 3, respectively.

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A more general analysis of $\mathcal{R}$ and its endomorphisms is work in progress with R. Conti.

Let’s finish with some **advertisment**

Job announcement

Within the **GAPT** research group at Cardiff University, we have an open permanent position:

Lecturer in Pure Mathematics or Mathematical Physics
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and please share with anybody who might be interested.