Existence of weak adiabatic limit in almost all models of perturbative QFT

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- The Wightman and Green functions are one of the most important objects in the quantum field theory in *the Minkowski space*.
- The *perturbative* definition of the Green functions in a large class of models was given by Lowenstein (1976) and Breitenloher Maison (1977).
- Using *the Epstein-Glaser* approach both the Wightman and Green functions can be defined. $\rightsquigarrow$ One has to show *the existence of the weak adiabatic limit*.
- The existence of the weak adiabatic limit has been proved so far in purely massive models, the quantum electrodynamics and the massless φ^4 theory.

Main result

The existence of the weak adiabatic limit in a large class of models in the Minkowski space including all models with the interaction vertices of the canonical dimension equal 4.

$\Rightarrow$ The perturbative construction of the Wightman and Green functions.

$\Rightarrow$ The definition of a Poincaré invariant functional on the algebra of the interacting fields.

Plan of the talk

1. Axioms of the time-ordered products.
2. Definition of the Wightman and Green functions in the Epstein-Glaser approach.
3. Known and new results about the existence of the adiabatic limit.
4. Outline of the proof.

Epstein-Glaser approach – the notation

- Scalar, spinor and vector fields.
- Massive or massless fields.
- Only renormalizable models.

The notation:

- The basic generators: $A_1, \ldots, A_{\mathrm{p}}$.
- The generators: $\partial^\alpha A_i$, α - a multi-index
- The algebra of the symbolic fields $=$the free unital commutative algebra generated by $\partial^\alpha A_i$.
- Monomials: $A^r = \prod_{i,\alpha} (\partial^\alpha A_i)^{r(i,\alpha)}$.
- The super-quadri-index: $r : \{1, \ldots, \mathrm{p}\} \times \mathbb{N}^4 \ni (i,\alpha) \mapsto r(i,\alpha) \in \mathbb{N}$.
- Polynomials: $B = \sum_r a_r A^r$, $a_r \in \mathbb{C}$.
- Sub-polynomials: $B^{(s)} = \sum_r \frac{r!}{(r-s)!} a_r A^{r-s}$, s - super-quadri-index.
- Wick polynomials: $:B(x):$.

Epstein-Glaser approach – the time-ordered products

$$\mathrm{T}(B_1,\ldots,B_n)(x_1,\ldots,x_n) \equiv \mathrm{T}(B_1(x_1),\ldots,B_n(x_n)) : \ \mathcal{S}(\mathbb{R}^{4n}) \to L(\mathcal{D}_0)$$

1. $\mathrm{T}(\varnothing) = \mathbb{1}$, $\quad \mathrm{T}(B(x)) = {:}B(x){:}$ and
   $\mathrm{T}(B_1(x_1),\ldots,B_n(x_n),1(x_{n+1})) = \mathrm{T}(B_1(x_1),\ldots,B_n(x_n))$.
2. Symmetry: $\mathrm{T}(B_1(x_1),\ldots,B_n(x_n)) = \mathrm{T}(B_{\pi(1)}(x_{\pi(1)}),\ldots,B_{\pi(n)}(x_{\pi(n)}))$.
3. Translational covariance:
   $$U(a)\,\mathrm{T}(B_1(x_1),\ldots,B_n(x_n))U(a)^{-1} = \mathrm{T}(B_1(x_1+a),\ldots,B_n(x_n+a)).$$
4. Causality: For $x_1,\ldots,x_m \gtrsim x_{m+1},\ldots x_n$ it holds
   $$\mathrm{T}(B_1(x_1),\ldots,B_n(x_n)) = \mathrm{T}(B_1(x_1),\ldots,B_m(x_m))\,\mathrm{T}(B_{m+1}(x_{m+1}),\ldots,B_n(x_n)).$$
5. Wick expansion:
   $$\mathrm{T}(B_1(x_1),\ldots,B_n(x_n)) = \sum_{s_1,\ldots,s_n} (\Omega|\,\mathrm{T}(B_1^{(s_1)}(x_1),\ldots,B_n^{(s_n)}(x_n))\Omega)\ \frac{{:}A^{s_1}(x_1)\ldots A^{s_n}(x_n){:}}{s_1!\ldots s_n!}.$$
6. Bound on the Steinmann's scaling degree:
   $$\mathrm{sd}(\,(\Omega|\,\mathrm{T}(B_1(x_1),\ldots,B_n(x_n),B_{n+1}(0))\Omega)\,) \leqslant \sum_{j=1}^{n+1}(\dim(B_j)+\mathbf{c})$$

Epstein-Glaser approach – interacting models

The interaction vertices $\mathcal{L}_1, \ldots, \mathcal{L}_{\mathrm{q}}$, the coupling constants $e_1, \ldots, e_{\mathrm{q}}$ and the switching functions $g_1, \ldots, g_{\mathrm{q}} \in \mathcal{S}(\mathbb{R}^4)$, $g := (g_1, \ldots, g_{\mathrm{q}})$.

Interacting fields with IR regularization – Bogliubov's formula

$$S(g;h) = \mathrm{Texp}\left(\mathrm{i}\int \mathrm{d}^4x \sum_{j=1}^{\mathrm{q}} e_j g_j(x)\mathcal{L}_j(x) + \mathrm{i}\int \mathrm{d}^4x\, h(x)B(x)(x)\right) \tag{1}$$

$$B_{\mathrm{ret}}(g;x) := (-\mathrm{i})\frac{\delta}{\delta h(x)} S(g;0)^{-1}S(g;h)\Big|_{h=0} \tag{2}$$

Time-ordered products of interacting fields with IR regularization

$$S(g;h) = \mathrm{Texp}\left(\mathrm{i}\int \mathrm{d}^4x \sum_{j=1}^{\mathrm{q}} e_j g_j(x)\mathcal{L}_j(x) + \mathrm{i}\int \mathrm{d}^4x \sum_{j=1}^{m} h_j(x)B_j(x)(x)\right) \tag{3}$$

$$\mathrm{T}(B_{1,\mathrm{ret}}(g;x_1), \ldots, B_{m,\mathrm{ret}}(g;x_m)) := (-\mathrm{i})^m \frac{\delta}{\delta h_m(x_m)} \ldots \frac{\delta}{\delta h_1(x_1)} S(g;0)^{-1}S(g;h)\Big|_{h=0} \tag{4}$$

Wightman and Green functions with IR regularization

$$(\Omega| B_{1,\mathrm{ret}}(g;x_1) \ldots B_{m,\mathrm{ret}}(g;x_m)\Omega) \tag{5}$$

$$(\Omega|\, \mathrm{T}(B_{1,\mathrm{ret}}(g;x_1), \ldots, B_{m,\mathrm{ret}}(g;x_m))\Omega) \tag{6}$$

The adiabatic limit

1. The Wightman and Green functions are well-defined as formal power series in $e_1, \dots, e_{\mathrm{q}}$ as long as all the switching functions belong to the Schwartz class.

   To make physical predictions one has to take the limit $g_1(x), \dots, g_{\mathrm{q}}(x) \to 1$.

2. For any $g \in \mathcal{S}(\mathbb{R}^N)$ such that $g(0) = 1$ we define a one-parameter family of Schwartz tests functions:
$$g_\epsilon(x) := g(\epsilon x) \quad \text{for} \quad \epsilon > 0. \tag{7}$$
   We have $\lim_{\epsilon \searrow 0} g_\epsilon(x) = 1$ pointwise, $\lim_{\epsilon \searrow 0} \tilde{g}_\epsilon(q) = (2\pi)^N \delta(q)$ in $S'(\mathbb{R}^N)$.

3. Let $t \in \mathcal{S}'(\mathbb{R}^N)$ and consider the limit
$$\lim_{\epsilon \searrow 0} \int \mathrm{d}^N x \; t(x) g_\epsilon(x) = \lim_{\epsilon \searrow 0} \int \frac{\mathrm{d}^N q}{(2\pi)^N} \; \tilde{t}(q) \tilde{g}_\epsilon(-q) = c. \tag{8}$$
   If the above limit exists and its value is independent of the choice of $g \in \mathcal{S}(\mathbb{R}^N)$ such that $g(0) = 1$ then we say that

   - the adiabatic limit of t exists and equals c and
   - the distribution $\tilde{t}$ has the value c at zero *in the sense of Łojasiewicz*.

Known results about the existence of the adiabatic limit

Epstein Glaser (1973) – the weak adiabatic limit

The existence of the weak adiabatic limit in purely massive theories:

$$\mathrm{W}(B_1(x_1),\ldots,B_m(x_m)) := \lim_{\epsilon\searrow 0}(\Omega|B_{1,\mathrm{ret}}(g_\epsilon;x_1),\ldots,B_{m,\mathrm{ret}}(g_\epsilon;x_m)\Omega) \tag{9}$$

$$\mathrm{G}(B_1(x_1),\ldots,B_m(x_m)) := \lim_{\epsilon\searrow 0}(\Omega|\,\mathrm{T}(B_{1,\mathrm{ret}}(g_\epsilon;x_1),\ldots,B_{m,\mathrm{ret}}(g_\epsilon;x_m))\Omega) \tag{10}$$

The above limits are taken in $S'(\mathbb{R}^{4m})$.

Blanchard Seneor (1975) – the weak adiabatic limit

The existence of the weak adiabatic limit in the quantum electrodynamics and the massless φ^4 theory.

Epstein Glaser (1976) – the strong adiabatic limit

The existence of the S-matrix in purely massive theories:

$$S\Psi := \lim_{\epsilon\searrow 0} S(g_\epsilon)\Psi \quad \text{for all} \quad \Psi\in\mathcal{D}_1. \tag{11}$$

Fredenhagen Lindner (2014)

The existence of expectation values of the products of the interacting fields in thermal states.

Existence of the weak adiabatic limit

Theorem

Assume that the interaction vertices $\mathcal{L}_1, \ldots, \mathcal{L}_{\mathrm{q}}$ of a given model satisfy one of the following conditions:

1. $\mathbf{c} = 0$, $\dim(\mathcal{L}_l) = 4$ for all l,
2. $\mathbf{c} = 1$, $\dim(\mathcal{L}_l) = 3$ and $\mathcal{L}_l$ contains at least one massive field for all l.

$\Rightarrow$ **It is possible to normalize the time-ordered products such that the weak adiabatic limit exists** (the explicit form of the required normalization condition is stated on the next slide).

The bound on the Steinmann's scaling degree implies that

$$\mathrm{sd}\big(\, (\Omega|\, \mathrm{T}\left(\mathcal{L}_{l_1}^{(s_1)}(x_1), \ldots, \mathcal{L}_{l_n}^{(s_n)}(x_n), \mathcal{L}_{l_{n+1}}^{(s_{n+1})}(0)\right)\Omega)\,\big) \leqslant \omega - 4n, \tag{12}$$

where

- $\omega := 4 - \sum_{i=1}^{\mathrm{p}} [\dim(A_i)\, \mathrm{e}(A_i) + \mathrm{d}(A_i)]$ is a function of $s_1, \ldots, s_{n+1}$,
- $\mathrm{e}(A_i) = \#$ the external lines corresponding to A_i,
- $\mathrm{d}(A_i) = \#$ the derivatives acting on the external lines corresponding to A_i,
- $\dim(A_i)$ is the canonical dimension of A_i, $\dim(\varphi){=}\dim(A_\mu){=}1$, $\dim(\psi){=}\dim(\bar{\psi}){=}\frac{3}{2}$.

If $\omega < 0$ then in the inductive construction of the time-ordered products the distribution

$$(\Omega|\, \mathrm{T}\left(\mathcal{L}_{l_1}^{(s_1)}(x_1), \ldots, \mathcal{L}_{l_{n+1}}^{(s_{n+1})}(x_{n+1})\right)\Omega) \tag{13}$$

is determined uniquely by the time-ordered products with at most n arguments.

Existence of the weak adiabatic limit

Normalization condition which implies the existence of the weak adiabatic limit

The weak adiabatic limit exists if for all super-quadri-indices $s_1, \ldots, s_k$ which involve only massless fields the time-ordered products satisfy the condition

$$(\Omega|\, \mathrm{T}(\tilde{\mathcal{L}}_{l_1}^{(s_1)}(q_1), \ldots, \tilde{\mathcal{L}}_{l_k}^{(s_k)}(q_k))\Omega) = (2\pi)^4 \delta(q_1 + \ldots + q_k)\, \Sigma(q_1, \ldots, q_{k-1}), \qquad (14)$$

where $\partial_q^\gamma \Sigma(0) = 0$ for all multi-indices γ such that $|\gamma| < \omega$, i.e. Σ has zero of order ω at zero. The value of the distribution $\partial_q^\gamma \Sigma$ at zero is defined in the sense of Łojasiewicz.

In the case of the above-mentioned class of models it is always possible to define the time-ordered products such that they satisfy the above condition.

Comments:

1. According to the above condition the photon self-energy corrections have zero of order 2 in the sense of Łojasiewicz at vanishing external momentum.
2. The correct mass normalization of all massless fields (= vanishing of the self-energy at vanishing external momentum) is necessary for the existence of the weak adiabatic limit.
3. Since the correct mass normalization is not possible in the massless φ^3 theory, the weak adiabatic limit does not exist in this theory (the massless φ^3 theory does not satisfy the assumption of the theorem from the previous slide).

Compatibility of normalization conditions

The normalization condition given on the previous slide is compatible with all the standard normalization conditions:

1. unitarity,
2. Poincaré covariance,
3. CPT covariance,
4. field equations,
5. Ward identities in the QED.

Almost homogenous scaling in *purely massless* models

Let $A^{r_1}, \ldots, A^{r_k}$ be monomials built out of massless fields. Then

$$(\Omega|\, \mathrm{T}(A^{r_1}(x_1), \ldots, A^{r_k}(x_k))\Omega) \tag{15}$$

scales almost homogeneously with degree

$$D = \sum_{j=1}^{k} \dim(A^{r_j}). \tag{16}$$

Outline of the proof

The Wightman and Green functions are formal power series in the coupling constant e

$$\int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_m \; \mathrm{G}(B_1(x_1),\dots,B_m(x_m))\, f(x_1,\dots,x_m)$$
$$= \sum_{k=0}^{\infty} e^k \;\int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_m \; \mathrm{G}_k(B_1(x_1),\dots,B_m(x_m))\, f(x_1,\dots,x_m)\,. \quad (17)$$

The coefficients of the formal power series are obtained by taking the adiabatic limit

$$\lim_{\epsilon\searrow 0} \int \mathrm{d}^4y_1 \dots \mathrm{d}^4y_k\, \mathrm{d}^4x_1 \dots \mathrm{d}^4x_m\, g_\epsilon(y_1)\dots g_\epsilon(y_k)\, f(x_1,\dots,x_m)$$
$$(\Omega|\, \mathrm{R}(\mathcal{L}(y_1),\dots,\mathcal{L}(y_k); B_1(x_1),\dots,B_m(x_m))\Omega). \quad (18)$$

The above limit (if exists) is equal to the value at zero in the sense of Łojasiewicz of the following distribution

$$r(q_1,\dots,q_k) := \int \frac{\mathrm{d}^4p_1}{(2\pi)^4} \dots \frac{\mathrm{d}^4p_m}{(2\pi)^4}\, \tilde{f}(p_1,\dots,p_m)$$
$$(\Omega|\, \mathrm{R}(\tilde{\mathcal{L}}(q_1),\dots,\tilde{\mathcal{L}}(q_k); \tilde{B}_1(-p_1),\dots,\tilde{B}_m(-p_m))\Omega). \quad (19)$$

Regularity of a distribution near the origin

Notation $t(q,q') = O^{\mathrm{dist}}(|q|^\delta)$ generalizing notation due to Estrada (1998)

Let $t \in \mathcal{S}'(\mathbb{R}^N \times \mathbb{R}^M)$. For $\delta \in \mathbb{R}$ we write

$$t(q,q') = O^{\mathrm{dist}}(|q|^\delta), \tag{20}$$

iff there exist a neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^N \times \mathbb{R}^M$ and a family of functions $t_\alpha \in C(\mathcal{O})$ indexed by multi-indices α such that

1. $t_\alpha \equiv 0$ for all but finite number of multi-indices α,
2. $|t_\alpha(q,q')| \leqslant \mathrm{const}\,|q|^{\delta+|\alpha|}$ for $(q,q') \in \mathcal{O}$,
3. $t(q,q') = \sum_\alpha \partial_q^\alpha t_\alpha(q,q')$ for $(q,q') \in \mathcal{O}$.

Properties

1. If $t \in C(\mathbb{R}^N \times \mathbb{R}^M)$ such that $t(q,q') = O(|q|^\delta)$, i.e. $|t(q,q')| \leqslant \mathrm{const}\,|q|^\delta$ in some neighborhood of the origin in $\mathbb{R}^N \times \mathbb{R}^M$, then $t(q,q') = O^{\mathrm{dist}}(|q|^\delta)$.
2. If $t \in \mathcal{S}'(\mathbb{R}^N \times \mathbb{R}^M)$ and $t(q,q') = O^{\mathrm{dist}}(|q|^\delta)$ then $t(q,0) = O^{\mathrm{dist}}(|q|^\delta)$.
3. If $t \in \mathcal{S}'(\mathbb{R}^N)$ and $t(q) = c + O^{\mathrm{dist}}(|q|^\delta)$ for some $c \in \mathbb{C}$ and $\delta > 0$ then t has value c at zero in the sense of Łojasiewicz.

Outline of the proof (continued)

Theorem

$$r(q_1,\ldots,q_k) = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\ldots\frac{\mathrm{d}^4 p_m}{(2\pi)^4}\,\tilde{f}(p_1,\ldots,p_m)$$
$$(\Omega|\,\mathrm{R}(\tilde{\mathcal{L}}(q_1),\ldots,\tilde{\mathcal{L}}(q_k);\tilde{B}_1(-p_1),\ldots,\tilde{B}_m(-p_m))\Omega). \quad (21)$$

has value at $q_1 = \ldots = q_m = 0$ in the sense of Łojasiewicz.

$\mathbf{s} = (s_1,\ldots,s_k)$, $\mathbf{r} = (r_1,\ldots,r_m)$ – lists of super-quadri-indices involving only massless fields

$$r^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m) := \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\ldots\frac{\mathrm{d}^4 p_m}{(2\pi)^4}\,\tilde{f}(p_1,\ldots,p_m)$$
$$(\Omega|\,\mathrm{R}(\tilde{\mathcal{L}}^{(s_1)}(q_1),\ldots,\tilde{\mathcal{L}}^{(s_k)}(q_k);\tilde{B}_1^{(r_1)}(q'_1-p_1),\ldots,\tilde{B}_m^{(r_m)}(q'_m-p_m))\Omega) \quad (22)$$

Induction hypothesis

There exists $c \in C(\mathbb{R}^{4m})$ such that for all $\epsilon > 0$ it holds

$$r^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m) = c(q'_1,\ldots,q'_m) + O^{\mathrm{dist}}(|q_1,\ldots,q_k|^{\omega'-\varepsilon}), \quad (23)$$

where

$$\omega' = 1 - \sum_{i=1}^{\mathrm{p}}[\dim(A_i)\,\mathrm{e}(A_i) + \mathrm{d}(A_i)]. \quad (24)$$

The distribution $d^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m)$ is defined in analogous way to the distribution $r^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m)$ with the product R replaced by $\mathrm{D} = \mathrm{A} - \mathrm{R}$.

Outline of the proof (continued)

It is possible to represent $d^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m)$ with $k=n$ in terms of

- Wightman two point functions of free fields,
- $r^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m)$ with $k<n$ and
- $(\Omega|\,\mathrm{T}(\tilde{\mathcal{L}}^{(s_1)}(q_1),\ldots,\tilde{\mathcal{L}}^{(s_k)}(q_k)\Omega) = (2\pi)^4\,\delta(q_1+\ldots+q_k)\,\mathbf{\Sigma}^{\mathbf{s}}(q_1,\ldots,q_{k-1})$.

$\mathbf{s}=(s_1,\ldots,s_k)$, $\mathbf{r}=(r_1,\ldots,r_m)$ – lists of super-quadri-indices involving only massless fields

The proof of the inductive step is divided into two parts:

1. Using the above representation and the lemma below we first show that for all $\epsilon>0$
$$d^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m) = O^{\mathrm{dist}}(|q_1,\ldots,q_k|^{\omega'-\varepsilon}). \tag{25}$$

Lemma

It is possible to normalize the time-ordered products such that for all super-quadri-indices $s_1,\ldots,s_k$ which involve only massless fields and all $\epsilon>0$
$$\mathbf{\Sigma}^{\mathbf{s}}(q_1,\ldots,q_{k-1}) = O^{\mathrm{dist}}(|q_1,\ldots,q_{k-1}|^{\omega-\varepsilon}), \tag{26}$$
where
$$\omega = 4-\sum_{i=1}^{\mathrm{p}}[\dim(A_i)\,\mathrm{e}(A_i)+\mathrm{d}(A_i)]. \tag{27}$$

2. Next, using the above result we prove that for all $\epsilon>0$
$$r^{\mathbf{s},\mathbf{r}}(q_1,\ldots,q_k;q'_1,\ldots,q'_m) = c(q'_1,\ldots,q'_m) + O^{\mathrm{dist}}(|q_1,\ldots,q_k|^{\omega'-\varepsilon}). \tag{28}$$

Other results and open problems

Existence of the central splitting solution in the QED

For all polynomials $B_1, \ldots, B_k$ which are sub-polynomials of the interaction vertex the retarded product satisfy the condition

$$(\Omega|\,\mathrm{R}(\tilde{B}_1(q_1), \ldots, \tilde{B}_n(q_n); \tilde{B}_{n+1}(q_{n+1}))\Omega) = (2\pi)^4 \delta(q_1 + \ldots + q_{n+1})\, r(q_1, \ldots, q_n), \quad (29)$$

where $\partial_q^\gamma r(0) = 0$ for all multi-indices γ, such that $|\gamma| \leqslant \omega = \sum_{j=1}^{n+1} \dim(B_j)$.

$\Rightarrow$ It fixes uniquely the time-ordered products of sub-polynomials of the interaction vertex.

$\Rightarrow$ It implies the standard normalization conditions e.g. the Ward identities.

- The existence of the Wightman functions may be used to define a Poincaré invariant functional on the algebra of interacting **fields**.
- This functional is a positive ($\Rightarrow$ it is a state) in the case of models without vector fields.
- Is it possible to define the Poincaré invariant state on the algebra of **observables** in the QED or the non-abelian Yang-Mills theories without matter?

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