Characterization of dynamics in algebraic quantum field theory[1]

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[1]Based on joint work with Detlev Buchholz and with Dorothea Bahns and Kasia Rejzner

Introduction

Quantum physics:

can be described in terms of operations $S(F)$ (local S-matrices)

where F is a local functional on the classical configuration space which characterizes a perturbation of the dynamics.

The S-matrices $S(F)$ satisfy a causality relation and a dynamical relation

both related to the classical Lagrangean of the unperturbed system (cf. the talk of Detlev Buchholz).

In quantum mechanics, one so reproduces the well known mathematical structure, without an a priori choice of canonical commutation relations.

The framework remains meaningful in quantum field theory where it delivers for the first time

a construction of a Haag-Kastler net of C*-algebras for quite general classical Lagrangeans.

Previous attempts to the construction of quantum field theories may be reformulated

as searches for a vacuum state on a given quasi local algebra with an automorphic action of the Poincaré group.

Construction of the net for a scalar field

Configuration space $\mathcal{E} = \mathcal{C}^\infty(\mathbb{M})$.
Let $\mathcal{F}$ denote the family of maps

$$F : \mathcal{E} \to \mathbb{R}$$

of the form

$$F[\phi] = \sum \int V_k(\phi) f_k$$

with test functions f_k, and with support (in the sense of functionals)

$$\mathrm{supp} F \doteq \{x | \forall \text{ neighborhoods } N \text{ of } x \ \exists \phi, \phi' \in \mathcal{E},$$

$$\mathrm{supp}(\phi - \phi') \subset N \text{ with } F[\phi] \neq F[\phi']\}$$

hence $\mathrm{supp} F \subset \bigcup \mathrm{supp} f_k$.

Relations:

$$S(F+G+H) = S(F+H)S(H)^{-1}S(G+H) \quad \text{Causality}$$

if supp$F$ later than supp$G$,

$$S(F) = S(F^{\psi} + \delta\mathcal{L}(\psi)) \ \forall \psi \in \mathcal{E}, \ \text{supp}\psi \text{ compact} \quad \text{Dynamics}$$

with $F^{\psi}[\phi] = F[\phi + \psi]$ and $\delta L(\psi)[\phi] = \int L[\phi + \psi] - L[\phi]$ for the classical Lagrangean L.

Remark: For $F = 0$ one obtains the field equation in the form

$$S(\delta L^{\psi}) = S(0) \ .$$

Definition

Let L be a classical Lagrangean. The dynamical algebra associated to L is the unital C*-algebra $\mathfrak{A}_L$ freely generated by unitaries $S(F)$ modulo the relations Causality, Dynamics and the quantum condition

$$S(c) = e^{ic}1 \quad \text{Quantum condition}$$

for constant functionals $F \mapsto c \in \mathbb{R}$.

Local algebras associated to subregions $\mathcal{O} \subset \mathbb{M}$ are closed subalgebras generated by $S(F)$ with $\mathrm{supp} F \subset \mathcal{O}$.

The net $\mathcal{O} \mapsto \mathfrak{A}_L(\mathcal{O})$ satisfies the Haag Kastler axioms of isotony

$$\mathfrak{A}_L(\mathcal{O}_1) \subset \mathfrak{A}_L(\mathcal{O}_2) \text{ if } \mathcal{O}_1 \subset \mathcal{O}_2$$

and of Einstein causality (locality)

$$AB = BA$$

if $A \in \mathfrak{A}_L(\mathcal{O}_1)$, $B \in \mathfrak{A}_L(\mathcal{O}_2)$ and $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated.

Moreover, if the Lagrangean is Poincaré invariant, it is covariant under the automorphic action of the orthochronous Poincaré group induced by

$$\alpha_g(S(F)) = S(gF) \; , gF[\phi] = F[\phi \circ g]$$

Namely, we have

$$\alpha_g(\mathfrak{A}_L(\mathcal{O})) = \mathfrak{A}_L(g\mathcal{O})$$

and

$$\alpha_g \alpha_h = \alpha_{gh}$$

.

Interaction picture

The parametrization of the net in terms of local functionals on the classical configuration space allows an algebraic formulation of the interaction picture

Let $L = L_0 + L_I$ with $L_I(x)[\phi] = V(\phi(x))$.

Proposition

Let $\mathcal{O}$ be a relatively compact subregion, and let $f \equiv 1$ on $J(\mathcal{O})$, suppf compact. Then

$$\alpha_f^R : \begin{cases} \mathfrak{A}_L(\mathcal{O}) & \rightarrow & \mathfrak{A}_{L_0} \\ S(F) & \mapsto & S(L_I(f))^{-1} S(L_I(f) + F) \end{cases}$$

is an injective homomorphism.

(Bogolubov's formula)

Proof.

Causality and Quantum condition are preserved by standard arguments.
Dynamics:

$$\alpha_f(S(F)) = S(L_I(f))^{-1}S(L_I(f) + F)$$
$$= S(L_I(f))^{-1}S(L_I(f)^\psi + F^\psi + \delta L_0^\psi)$$
$$\alpha_f(S(F^\psi + \delta L^\psi)) = S(L_I(f))^{-1}S(L_I(f) + F^\psi + \delta L^\psi)$$

But $L_I(f)^\psi + F^\psi + \delta L_0^\psi = L_I(f) + F^\psi + \delta L^\psi$ for supp$\psi \subset \mathcal{O}$. □

States

As a C*-algebra, $\mathfrak{A}_L$ has many states, e.g.

$$\omega(S) = 0$$

for all products S of S-matrices and their inverses which differ from a multiple of 1. But this state is rather remote from states one is interested in for applications in physics.

Definition

A vacuum state on $\mathfrak{A}_L$ is a state ω which is invariant under Poincaré transformations

$$\omega \circ \alpha_g = \omega$$

such that the functions on the translation subgroup

$$a \mapsto \omega(A\alpha_a(B))$$

are boundary values of an holomorphic function on the forward tube $\mathbb{M} + iV_+$, V_+ forward lightcone.

In an analogous way one can also characterize KMS states.

Perturbative construction of states:

The subalgebra generated by linear functionals F and with the free Lagrangean L_0 is the Weyl algebra of the free field. In the vacuum representation of this algebra one can use causal perturbation theory (SBEG) and obtains formal states, i.e. functionals with values in formal power series with a suitable positivity condition and the usual normalization condition.

Question: Is this representation faithful?

Answer (still unknown) seems to be related to anomalies.

States on a C*-subalgebra can always be extended to states of the full algebra (Hahn-Banach) but these extensions are not unique and might be quite singular. States with good properties are knwon only in special cases :

- Subtheory generated by polynomial functionals up to second order in 4d was already constructed in connection with the external field problem (Bellissard 1975)
- In 2d, the Haag-Kastler net of the sine-Gordon theory was constructed (Bahns-F-Rejzner), generated by functionals of the form

$$F[\phi] = \int f\phi + g\sin\phi + h\cos\phi$$

and it was shown that the dynamical relation holds.

Conclusions and Outlook

- For scalar fields a Haag-Kastler net of C*-algebras can be constructed for a large class of interactions.
- The formalism is locally covariant.
- The problem of constructing quantum field theories is reduced to the investigation of states of a given algebra.
- Ward identities and renormalization group yield additional structure (work in progress).
- A generalization to Fermi fields should be straightforward.
- Can it also be applied to gauge theories or even gravity? Problem: indefinite metric in the standard treatment (Gupta-Bleuler, BRST, BV)