Fock quantization of the Dirac field in cosmology with unitary dynamics

41st LQP Workshop "Foundations and Constructive Aspects of QFT"
Göttingen, February 3th, 2018

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Motivation

- Dirac (fermion) fields describe realistic matter contents in Physics: cosmology, condensed matter (e.g. graphene)…

- Infinite ambiguity in their quantum description:
    - Choice of (Fock) representation of the CAR's.

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- Infinite ambiguity in their quantum description:

    - Choice of (Fock) representation of the CAR's.

- In symmetric backgrounds, one usually imposes invariance of the vacuum under the symmetries of the system.

Motivation

- Dirac (fermion) fields describe realistic matter contents in Physics: cosmology, condensed matter (e.g. graphene)...

- Infinite ambiguity in their quantum description:

  - Choice of (Fock) representation of the CAR's.

- In non-stationary spacetimes, we require as well that the dynamical transformations are implemented unitarily (quantum coherence).

- Characterization of the field degrees of freedom that evolve unitarily: dynamical separation between spacetime and matter d.o.f.!

- **Result:** <u>unique</u> Fock representation of the CAR's (up to unitary equiv.).

General setting & strategy

The Dirac field in curved spacetimes

- Dirac equation on a globally hyperbolic spacetime,

  $S = \{\psi\}$ linear space of solutions.

- Global hyperbolicity $\Longrightarrow$ $S \approx$ set of data on a Cauchy surface.

- Natural inner product $(\psi_1, \psi_2)_S$, conserved under evolution.

- Analogous construction of $\bar{S}$.

Fermion Complex Structures

- Codify the ambiguity in the choice of Fock representation.

- Real linear map J defined on S and on $\bar{S}$, (equiv. on set of data)

$$J^2 = -I, \qquad (J\psi_1, J\psi_2)_S = (\psi_1, \psi_2)_S$$

- Defines a splitting into its $\pm i$ eigenspaces

$$S_J^{\pm} = \frac{1}{2}(S \mp iJS), \qquad \bar{S}_J^{\pm} = \overline{S_J^{\mp}}$$

$S_J^{+} \longrightarrow$ Particle annihilation $\qquad S_J^{-} \longrightarrow$ Antiparticle creation

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$$\left.\begin{array}{l} \bullet\ \text{Completion of } S_J^+ \longrightarrow \text{1-p Hilbert space} \\ \qquad\qquad\qquad\qquad \oplus \\ \bullet\ \text{Completion of } \overline{S_J^-} \longrightarrow \text{1-ap Hilbert space} \end{array}\right\} \Rightarrow \text{Antisym. Fock sp.}$$

Strategy

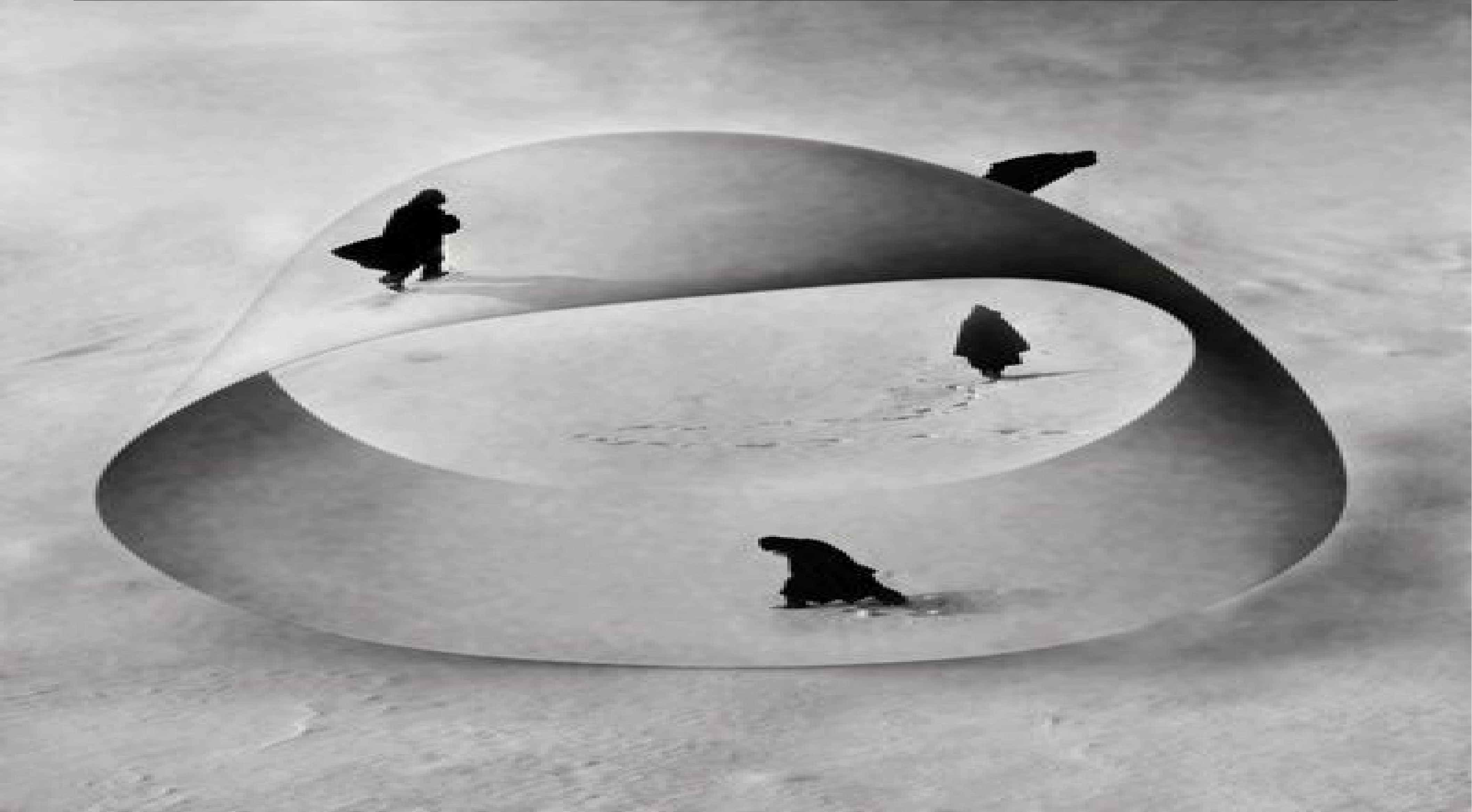
- Characterize those complex structures J that commute with the symmetry transformations of the system under study.

- Consider families of annihilation and creation-like variables defined through linear combinations of the field that can be <u>time-dependent</u>.

- The elements of such families are related by transformations that include the dynamics of the field, but do not trivialize it.

- Determine those families that admit a unitarily implementable dynamics and prove that all of them are, in turn, unitarily equivalent.

Invariant Complex Structures



Cosmological model

- Flat homogeneous and isotropic cosmology, scale factor $\exp[\alpha(\eta)]$, compact spatial sections (isomorphic to three-tori).

- Minimally coupled Dirac field, described by

$$\varphi_A\,,\ \bar{\chi}_{A'} \qquad A=1,2\,;\ A'=1',2', \qquad \text{Grassmann variables}$$

Cosmological model

- Flat homogeneous and isotropic cosmology, scale factor $\exp[\alpha(\eta)]$, compact spatial sections (isomorphic to three-tori).

- Minimally coupled Dirac field.

- After a partial gauge-fixing (time-gauge):

$$\varphi_A(x) = e^{-3\alpha(\eta)/2} \sum_{\vec{k}\in\mathbb{Z}^3} \left[m_{\vec{k}}(\eta) w^{(+)}_{\vec{k}A}(\vec{x}) + \bar{r}_{\vec{k}}(\eta) w^{(-)}_{\vec{k}A}(\vec{x}) \right]$$

$$\bar{\chi}_{A'}(x) = e^{-3\alpha(\eta)/2} \sum_{\vec{k}\in\mathbb{Z}^3} \left[\bar{s}_{\vec{k}}(\eta) \bar{w}^{(+)}_{\vec{k}A'}(\vec{x}) + t_{\vec{k}}(\eta) \bar{w}^{(-)}_{\vec{k}A'}(\vec{x}) \right]$$

$$w^{(+)}_{\vec{k}A} \longrightarrow +\omega_k \qquad \omega_k = O(|\vec{k}|)$$

$$w^{(-)}_{\vec{k}A} \longrightarrow -\omega_k$$

Dirac operator eigenvalues
(asymptotically, indep. of spin structure)

Cosmological models: invariance

$$\varphi_A(x)=e^{-3\alpha(\eta)/2}\sum_{\vec{k}\in\mathbb{Z}^3}\left[m_{\vec{k}}(\eta)\,w^{(+)}_{\vec{k}A}(\vec{x})+\bar{r}_{\vec{k}}(\eta)\,w^{(-)}_{\vec{k}A}(\vec{x})\right]$$

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- Isometries of the three-torus: composition of translations $T_{\alpha_i}: x_i \longrightarrow x_i+\alpha_i$. On the bi-spinors, direct sum of irreps. of $U(1)\times U(1)\times U(1)$:

$$w^{(\pm)}_{\vec{k}A} \longrightarrow C_{\vec{\alpha}}\, e^{2\pi i\vec{k}\vec{\alpha}/l_0} w^{(\pm)}_{\vec{k}A} \qquad \bar{w}^{(\pm)}_{\vec{k}A} \longrightarrow C_{\vec{\alpha}}\, e^{-2\pi i(\vec{k}+2\vec{\tau})\vec{\alpha}/l_0}\bar{w}^{(+)}_{\vec{k}A}$$

- We also consider the symmetry under the spin rotations generated by the helicity of the field, which is a conserved quantity.

$$w^{(\pm)}_{\vec{k}A}, \bar{w}^{(\pm)}_{\vec{k}A} \longrightarrow \pm 1 \text{ helicity eigenspinors, with } \omega_k\neq 0.$$

Cosmological models: invariance

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- Isometries of the three-torus: direct sum of irreps. of $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$:

$$w^{(\pm)}_{\vec{k}A}\longrightarrow C_{\vec{\alpha}}e^{2\pi i\vec{k}\vec{\alpha}/l_0}w^{(\pm)}_{\vec{k}A}\qquad\qquad \bar{w}^{(\pm)}_{\vec{k}A}\longrightarrow C_{\vec{\alpha}}e^{-2\pi i(\vec{k}+2\vec{\tau})\vec{\alpha}/l_0}\bar{w}^{(+)}_{\vec{k}A}$$

- Spin (helicity) rotations:

$w^{(\pm)}_{\vec{k}A},\bar{w}^{(\pm)}_{\vec{k}A}\longrightarrow\pm 1$ helicity eigenspinors, with $\omega_k\neq 0.$

- Families of invariant annihilation and creation-like: $(x_{\vec{k}},y_{\vec{k}})=(m_{\vec{k}},s_{\vec{k}}),(t_{\vec{k}},r_{\vec{k}})$

$$\boxed{\begin{aligned}a^{(x,y)}_{\vec{k}}(\eta)&=f^{\vec{k}}_1(\eta)x_{\vec{k}}(\eta)+f^{\vec{k}}_2(\eta)\bar{y}_{-\vec{k}-2\vec{\tau}}(\eta)\\ \bar{b}^{(x,y)}_{\vec{k}}(\eta)&=g^{\vec{k}}_1(\eta)x_{\vec{k}}(\eta)+g^{\vec{k}}_2(\eta)\bar{y}_{-\vec{k}-2\vec{\tau}}(\eta)\end{aligned}}\qquad |f^{\vec{k}}_1|^2+|f^{\vec{k}}_2|^2=1,\quad |g^{\vec{k}}_1|^2+|g^{\vec{k}}_2|^2=1,$$

$$f^{\vec{k}}_1\bar{g}^{\vec{k}}_1+f^{\vec{k}}_2\bar{g}^{\vec{k}}_2=0.$$

Unitary dynamics

Fermion dynamics

- First order Dirac equations in the considered cosmology:

$$x_{\vec{k}}{}' = i\,\omega_k x_{\vec{k}} - i m e^{\alpha}\,\bar{y}_{-\vec{k}-2\vec{\tau}}, \qquad y_{\vec{k}}{}' = i\,\omega_k y_{\vec{k}} + i m e^{\alpha}\,\bar{x}_{-\vec{k}-2\vec{\tau}}, \qquad {}' := \frac{d}{d\eta}$$

- Same second order equation for all modes $\{z_{\vec{k}}\} := \{x_{\vec{k}}, y_{\vec{k}}\}$

$$z_{\vec{k}}{}'' = \alpha' z_{\vec{k}}{}' - \left(\omega_k^2 + m^2 e^{2\alpha} + i\,\omega_k\,\alpha'\right) z_{\vec{k}}$$

  - Known asymptotic behavior of its two independent solutions.

$$\Downarrow$$

- The (relevant) asymptotics of the evolution is known

$$x_{\vec{k}}(\eta) = A_k(\eta, \eta_0)\, x_{\vec{k}}(\eta_0) + B_k(\eta, \eta_0)\, \bar{y}_{\vec{k}}(\eta_0),$$

$$\bar{y}_{\vec{k}}(\eta) = \bar{A}_k(\eta, \eta_0)\, \bar{y}_{\vec{k}}(\eta_0) - \bar{B}_k(\eta, \eta_0)\, x_{\vec{k}}(\eta_0).$$

Dynamical transformations

- Fermion dynamics $\longrightarrow$ time-dependent Bogoliubov transformation:

$$a_{\vec{k}}^{(x,y)}(\eta)=\alpha_{\vec{k}}^{f}(\eta,\eta_0)\,a_{\vec{k}}^{(x,y)}(\eta_0)+\beta_{\vec{k}}^{f}(\eta,\eta_0)\,\bar{b}_{\vec{k}}^{(x,y)}(\eta_0)$$

$$\bar{b}_{\vec{k}}^{(x,y)}(\eta)=\alpha_{\vec{k}}^{g}(\eta,\eta_0)\,\bar{b}_{\vec{k}}^{(x,y)}(\eta_0)+\beta_{\vec{k}}^{g}(\eta,\eta_0)\,a_{\vec{k}}^{(x,y)}(\eta_0)$$

with $|\beta_{\vec{k}}^{h}(\eta,\eta_0)|$ given by ($h=f,g$):

$$\Big|\Big[-h_1^{\vec{k}}(h_2^{\vec{k},0}+\Gamma_k h_1^{\vec{k},0})e^{i\int\Lambda_k^1}+\bar{\Gamma}_k h_2^{\vec{k}}h_2^{\vec{k},0}e^{\Delta\alpha}e^{i\int\bar{\Lambda}_k^2}\Big]e^{i\omega_k\Delta\eta}+$$
$$+\Big[h_2^{\vec{k}}(h_1^{\vec{k},0}-\bar{\Gamma}_k h_2^{\vec{k},0})e^{-i\int\bar{\Lambda}_k^1}+\Gamma_k h_1^{\vec{k}}h_1^{\vec{k},0}e^{\Delta\alpha}e^{-i\int\Lambda_k^2}\Big]e^{-i\omega_k\Delta\eta}\Big|$$

$$\Delta\eta=\eta-\eta_0$$
$$\Delta\alpha=\alpha-\alpha_0$$

where $\Gamma_k=\dfrac{m\,e^{\alpha_0}}{2\omega_k+i\alpha'_0}$ and $\Lambda_k^l(\eta)=O(\omega_k^{-1}),\quad l=1,2.$

Conditions for unitary dynamics

- The Bogoliubov transformation is implementable as a unitary operator in the Fock space defined by the initial variables iff the sequences of the beta coefficients are square summable.

- We know the spectral asymptotics of the Dirac operator.

- We disregard as uninteresting any transformation that trivializes the dominant plane wave contribution to the Dirac solutions.

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- The Bogoliubov transformation is implementable as a unitary operator in the Fock space defined by the initial variables iff the sequences of the beta coefficients are square summable.

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$$\Downarrow$$

- This can always be made possible by tuning a specific functional dependence on those oscillations of the f and g functions that define the families of annihilation and creation-like variables.

Unitary dynamics

$$\boxed{\sum_{\vec{k}} |\beta^{f}_{\vec{k}}(\eta,\eta_0)|^2 < \infty, \qquad \sum_{\vec{k}} |\beta^{g}_{\vec{k}}(\eta,\eta_0)|^2 < \infty, \qquad \forall\, \eta.}$$

- This condition requires a specific behavior for $f_1^{\vec{k}}, f_2^{\vec{k}}, g_1^{\vec{k}}, g_2^{\vec{k}}$ both in their dependence on ω_k and on η:

Unitary dynamics

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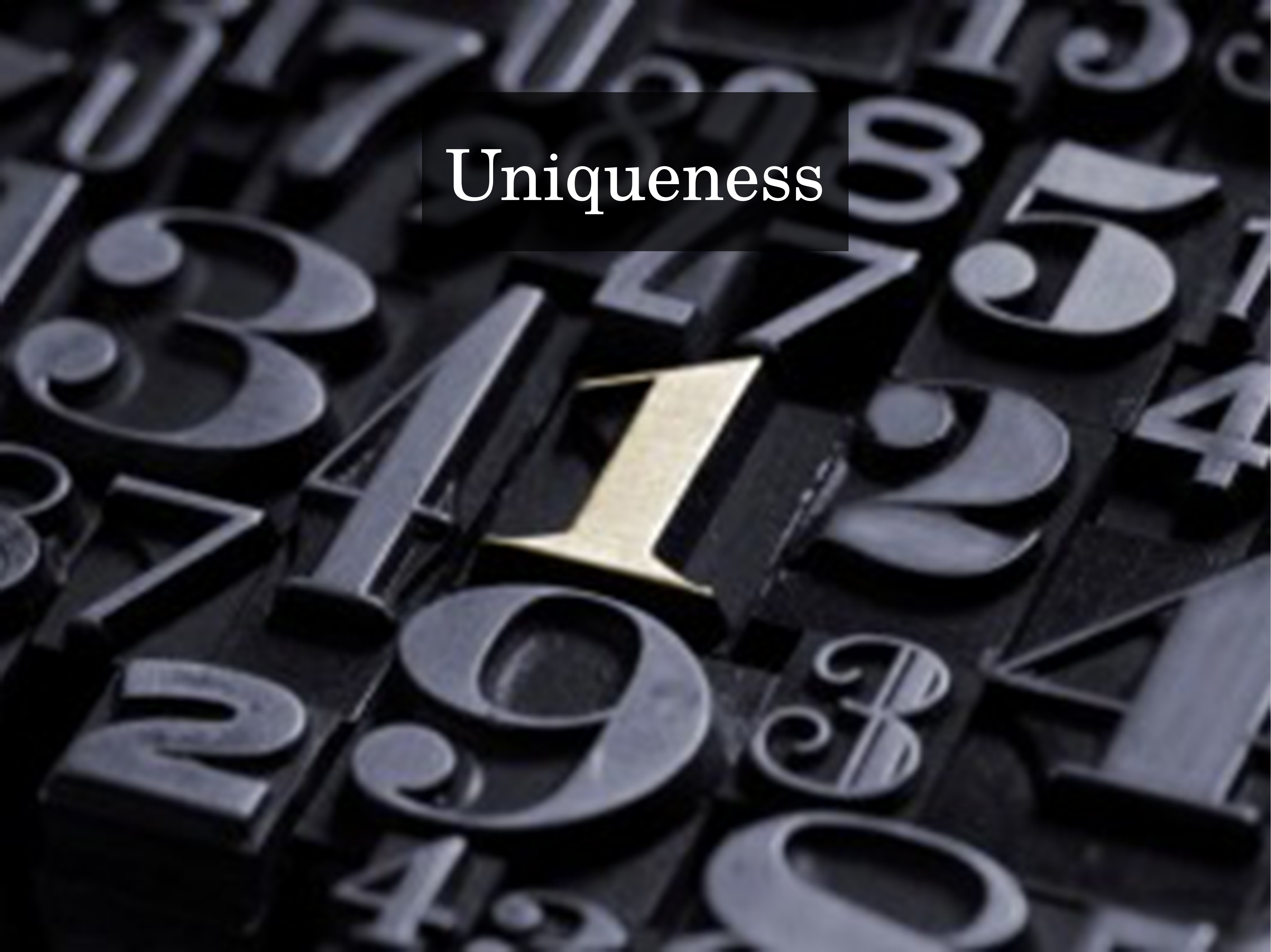
- This condition requires a specific behavior for $f_1^{\vec{k}}, f_2^{\vec{k}}, g_1^{\vec{k}}, g_2^{\vec{k}}$ both in their dependence on ω_k and on η:

$$h_l^{\vec{k}} = (-1)^{l+1}\frac{me^{\alpha}}{2\omega_k} e^{iH_{\tilde{l}}^{\vec{k}}} + o(\omega_k^{-1}), \qquad \{l,\tilde{l}\} := \{1,2\} \text{ as a set, } H_{\tilde{l}}^{\vec{k}} \text{ phase of } h_{\tilde{l}}^{\vec{k}}, \qquad h = f, g.$$

for all $\vec{k} \in \mathbb{Z}_l^3$, with $\mathbb{Z}^3 = \mathbb{Z}_1^3 \cup \mathbb{Z}_2^3$ (up to a finite sublattice and up to two possible complementary infinite sublattices where $h_l^{\vec{k}}$, of order ω_k^{-1} or higher, must be square summable).

- Besides, if we call $\vartheta_{h,l}^{\vec{k}}$ the subdominant terms: $\displaystyle\sum_{\vec{k}\in\mathbb{Z}_l^3} |\vartheta_{h,l}^{\vec{k}}|^2 < \infty$

Uniqueness



Reference quantization

- Reference J_R: Simplest choice of invariant complex structure that admits a unitary quantum dynamics

$$f_1^{\vec{k}} = \frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}} = \sqrt{1 - (f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}} = f_2^{\vec{k}}, \qquad g_2^{\vec{k}} = -f_1^{\vec{k}},$$

both for $(m_{\vec{k}}, s_{\vec{k}})$ and for $(t_{\vec{k}}, r_{\vec{k}})$.

Unitary equivalence

- Reference J_R:

$$f_1^{\vec{k}} = \frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}} = \sqrt{1-(f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}} = f_2^{\vec{k}}, \qquad g_2^{\vec{k}} = -f_1^{\vec{k}},$$

- Its relation with any other invariant $\widetilde{J}$ given by

$$\widetilde{a}_{\vec{k}}^{(x,y)}(\eta) = \kappa_{\vec{k}}^{f}(\eta)\, a_{\vec{k}}^{(x,y)}(\eta) + \lambda_{\vec{k}}^{f}(\eta)\, \bar{b}_{\vec{k}}^{(x,y)}(\eta)$$

$$\widetilde{\bar{b}}_{\vec{k}}^{(x,y)}(\eta) = \kappa_{\vec{k}}^{g}(\eta)\, \bar{b}_{\vec{k}}^{(x,y)}(\eta) + \lambda_{\vec{k}}^{g}(\eta)\, a_{\vec{k}}^{(x,y)}(\eta)$$

where

$$\lambda_{\vec{k}}^{h} = \frac{\widetilde{h}_1^{\vec{k}} h_2^{\vec{k}} - \widetilde{h}_2^{\vec{k}} h_1^{\vec{k}}}{h_2^{\vec{k}} k_1^{\vec{k}} - h_1^{\vec{k}} k_2^{\vec{k}}}, \qquad \{h, k\} := \{f, g\} \text{ as a set.}$$

Unitary equivalence

- Reference J_R:

$$f_1^{\vec{k}} = \frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}} = \sqrt{1-(f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}} = f_2^{\vec{k}}, \qquad g_2^{\vec{k}} = -f_1^{\vec{k}},$$

- The Bogoliubov transformation defined by the previous sequence of transformations is unitarily implementable in the Fock space if and only if

$$\sum_{\vec{k}} |\lambda_{\vec{k}}^{f}(\eta)|^2 < \infty, \qquad \sum_{\vec{k}} |\lambda_{\vec{k}}^{g}(\eta)|^2 < \infty, \qquad \forall\, \eta.$$

- Notice that

$$|\lambda_{\vec{k}}^{h}| = |\tilde{h}_1^{\vec{k}} h_2^{\vec{k}} - \tilde{h}_2^{\vec{k}} h_1^{\vec{k}}| \Longrightarrow |\lambda_{\vec{k}}^{f}| = |\lambda_{\vec{k}}^{g}|$$

Uniqueness

- Reference J_R:

$$f_1^{\vec{k}} = \frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}} = \sqrt{1-(f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}} = f_2^{\vec{k}}, \qquad g_2^{\vec{k}} = -f_1^{\vec{k}},$$

- Take $\tilde{J}$ to admit a unitarily implementable dynamics

$$\tilde{f}_l^{\vec{k}} = (-1)^{l+1}\frac{me^{\alpha}}{2\omega_k} e^{i\tilde{F}_{\tilde{l}}^{\vec{k}}} + \vartheta_{\tilde{f},l}^{\vec{k}}, \quad \vec{k} \in \mathbb{Z}_l^3, \quad \{l,\tilde{l}\} = \{1,2\}, \quad \sum_{\vec{k}\in\mathbb{Z}_l^3} |\vartheta_{\tilde{f},l}^{\vec{k}}|^2 < \infty,$$

and $\tilde{f}_l^{\vec{k}}$ s.q.s. in the possible complementary sublattices.

Uniqueness

- Reference J_R:

$$f_1^{\vec{k}} = \frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}} = \sqrt{1-(f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}} = f_2^{\vec{k}}, \qquad g_2^{\vec{k}} = -f_1^{\vec{k}},$$

- Take $\widetilde{J}$ to admit a unitarily implementable dynamics

$$\widetilde{f}_l^{\vec{k}} = (-1)^{l+1} \frac{me^{\alpha}}{2\omega_k} e^{i\widetilde{F}_{\tilde{l}}^{\vec{k}}} + \vartheta_{\widetilde{f},l}^{\vec{k}}, \quad \vec{k} \in \mathbb{Z}_l^3, \quad \{l,\tilde{l}\} = \{1,2\}, \quad \sum_{\vec{k}\in\mathbb{Z}_l^3} |\vartheta_{\widetilde{f},l}^{\vec{k}}|^2 < \infty,$$

and $\widetilde{f}_l^{\vec{k}}$ s.q.s. in the possible complementary sublattices.

- For $\vec{k}$ in the $l=1$ sublattices, we respectively have:

$$|\lambda_{\vec{k}}^{f}| = |\vartheta_{\widetilde{f},1}^{\vec{k}}| + O(\omega_n^{-2}), \qquad |\lambda_{\vec{k}}^{f}| = |\widetilde{f}_1^{\vec{k}}| + o(|\widetilde{f}_1^{\vec{k}}|)$$

Uniqueness

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and $\tilde{f}_l^{\vec{k}}$ s.q.s. in the possible complementary sublattices.

- For $\vec{k}$ in the $l=2$ sublattices, $|\lambda_{\vec{k}}^f| = O(1) \longrightarrow$ not s.q.s!

However...

Uniqueness

- Reference J_R:

$$f_1^{\vec{k}}=\frac{me^{\alpha}}{2\omega_k}, \qquad f_2^{\vec{k}}=\sqrt{1-(f_1^{\vec{k}})^2}, \qquad g_1^{\vec{k}}=f_2^{\vec{k}}, \qquad g_2^{\vec{k}}=-f_1^{\vec{k}},$$

- Take $\tilde{J}$ to admit a unitarily implementable dynamics

$$\tilde{f}_l^{\vec{k}}=(-1)^{l+1}\frac{me^{\alpha}}{2\omega_k}e^{i\tilde{F}_{\tilde{l}}^{\vec{k}}}+\vartheta^{\vec{k}}_{\tilde{f},l}, \quad \vec{k}\in\mathbb{Z}_l^3, \quad \{l,\tilde{l}\}=\{1,2\}, \quad \sum_{\vec{k}\in\mathbb{Z}_l^3}|\vartheta^{\vec{k}}_{\tilde{f},l}|^2<\infty,$$

and $\tilde{f}_l^{\vec{k}}$ s.q.s. in the possible complementary sublattices.

- For $\vec{k}$ in the $l=2$ sublattices, $|\lambda^f_{\vec{k}}|=O(1) \longrightarrow$ not s.q.s!

Can be understood as due to a reversal in the convention of particles and antiparticles for an infinite collection of modes.

Uniqueness

- Reference $\widetilde{J}_R$ same as J_R for $\vec{k}$ in the $l=1$ sublattices, but $f_1^{\vec{k}} \leftrightarrow g_1^{\vec{k}},\ f_2^{\vec{k}} \leftrightarrow g_2^{\vec{k}}$, i.e., particles $\leftrightarrow$ antiparticles, $\vec{k}$ in $l=2$.

- Take $\widetilde{J}$ to admit a unitarily implementable dynamics

$$\widetilde{f}_l^{\vec{k}} = (-1)^{l+1} \frac{m e^{\alpha}}{2\omega_k} e^{i \widetilde{F}_{\tilde{l}}^{\vec{k}}} + \vartheta_{\widetilde{f},l}^{\vec{k}}, \quad \vec{k} \in \mathbb{Z}_l^3, \quad \{l, \tilde{l}\} = \{1,2\}, \quad \sum_{\vec{k} \in \mathbb{Z}_l^3} |\vartheta_{\widetilde{f},l}^{\vec{k}}|^2 < \infty,$$

and $\widetilde{f}_l^{\vec{k}}$ s.q.s. in the possible complementary sublattices.

- For $\vec{k}$ in the $l=1$ sublattices, we respectively have

$$|\lambda_{\vec{k}}^f| = |\vartheta_{\widetilde{f},1}^{\vec{k}}| + O(\omega_n^{-2}), \qquad |\lambda_{\vec{k}}^f| = |\widetilde{f}_l^{\vec{k}}| + o(|\widetilde{f}_l^{\vec{k}}|)$$

- For $\vec{k}$ in the $l=2$ sublattices, we now have

$$|\lambda_{\vec{k}}^f| = |\vartheta_{\widetilde{f},2}^{\vec{k}}| + O(\omega_n^{-2}), \qquad |\lambda_{\vec{k}}^f| = |\widetilde{f}_2^{\vec{k}}| + o(|\widetilde{f}_2^{\vec{k}}|)$$

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Unitary equiv.!

Conclusions

- Combined criteria of invariance of the vacuum under symmetries of flat homogeneous and isotropic cosmologies + unitary implementation of the dynamics $\longrightarrow$ unique Fock quantization of the Dirac field.

- Uniqueness attained given a convention of particles and antiparticles.

- The part of the dynamics that can be unitarily implementable is uniquely determined $\longrightarrow$ extraction of explicitly time-dependent functions from the dominant parts of the field.

- Similar results and characterizations apply as well for the spherical case and for conformally ultrastatic spacetimes in $2+1$ dimensions.