

Combinatorics of the Star Product in AQFT

Eli Hawkins Kasia Rejzner

The University of York

July 2, 2019

Classical Field Theory

Classical Field Theory

- M spacetime

Classical Field Theory

- M spacetime
- E vector bundle

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:

$$\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:

$$\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y) .$$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:

$$\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y) .$$

Denote $m(F, G)(\varphi) \doteq F(\varphi)G(\varphi)$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:
 $\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y).$$

Denote $m(F, G)(\varphi) \doteq F(\varphi)G(\varphi)$

A distribution K on $M \times M$ defines

$$D_K(F \otimes G)(\varphi_1, \varphi_2) \doteq \langle K, F^{(1)}(\varphi_1), G^{(1)}(\varphi_2) \rangle$$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:
 $\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y) .$$

Denote $m(F, G)(\varphi) \doteq F(\varphi)G(\varphi)$

A distribution K on $M \times M$ defines

$$D_K(F \otimes G)(\varphi_1, \varphi_2) \doteq \langle K, F^{(1)}(\varphi_1), G^{(1)}(\varphi_2) \rangle$$

and an *exponential star product*

$$F \star_K G \doteq m \circ e^{\hbar D_K}(F \otimes G) .$$

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:
 $\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y) .$$

Denote $m(F, G)(\varphi) \doteq F(\varphi)G(\varphi)$

A distribution K on $M \times M$ defines

$$D_K(F \otimes G)(\varphi_1, \varphi_2) \doteq \langle K, F^{(1)}(\varphi_1), G^{(1)}(\varphi_2) \rangle$$

and an *exponential star product*

$$F \star_K G \doteq m \circ e^{\hbar D_K}(F \otimes G) .$$

If $K(x, y) - K(y, x) = i\Delta_{S_0}(x, y)$, then this is a quantization for S_0 .

Classical Field Theory

- M spacetime
- E vector bundle
- $\mathcal{E} \doteq \Gamma(M, E)$ smooth sections
- S action

$S''(\phi)$ is linearized equation of motion operator about $\varphi \in \mathcal{E}$ (off shell).

- Δ_S^R retarded Green's function
- $\Delta_S^A(\varphi; x, y) \doteq \Delta_S^R(\varphi; y, x)$ advanced
- $\Delta_S \doteq \Delta_S^R - \Delta_S^A$
- Peierls bracket of $F, G : \mathcal{E} \rightarrow \mathbb{C}$:
 $\{F, G\}_S(\varphi) \doteq \langle \Delta_S[\varphi], F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \rangle$

Quantization of Free Theory

S_0 quadratic

$$\implies \Delta_{S_0}^R(\varphi; x, y) = \Delta_{S_0}^R(x, y) .$$

Denote $m(F, G)(\varphi) \doteq F(\varphi)G(\varphi)$

A distribution K on $M \times M$ defines

$$D_K(F \otimes G)(\varphi_1, \varphi_2) \doteq \langle K, F^{(1)}(\varphi_1), G^{(1)}(\varphi_2) \rangle$$

and an *exponential star product*

$$F \star_K G \doteq m \circ e^{\hbar D_K}(F \otimes G) .$$

If $K(x, y) - K(y, x) = i\Delta_{S_0}(x, y)$, then this is a quantization for S_0 .

Changing K by a smooth, symmetric function gives an equivalent star product.

Quantization Maps

Quantization Maps

A *quantization map*

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

Quantization Maps

A *quantization map*

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Quantization Maps

A *quantization map*

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Equivalent star products come from different choices of Q .

Quantization Maps

A quantization map

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Equivalent star products come from different choices of Q .

Ordering	K	Product
Symmetric	$\frac{i}{2}\Delta_{S_0} = \frac{i}{2}(\Delta_{S_0}^R - \Delta_{S_0}^A)$	Moyal-Weyl

Quantization Maps

A quantization map

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Equivalent star products come from different choices of Q .

Ordering	K	Product
Symmetric	$\frac{i}{2}\Delta_{S_0} = \frac{i}{2}(\Delta_{S_0}^R - \Delta_{S_0}^A)$	Moyal-Weyl
Normal	$\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + H$	Wick

Quantization Maps

A quantization map

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Equivalent star products come from different choices of Q .

Ordering	K	Product
Symmetric	$\frac{i}{2}\Delta_{S_0} = \frac{i}{2}(\Delta_{S_0}^R - \Delta_{S_0}^A)$	Moyal-Weyl
Normal	$\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + H$	Wick
Time-ordered	$-i\Delta_{S_0}^A$	$\star_{\mathcal{T}}$

Quantization Maps

A quantization map

$$Q : \{\text{Classical observables}\} \rightarrow \{\text{Quantum observables}\}$$

is a “choice of operator ordering”.

It induces a star product by

$$Q(F \star G) = Q(F)Q(G) .$$

Equivalent star products come from different choices of Q .

Ordering	K	Product
Symmetric	$\frac{i}{2}\Delta_{S_0} = \frac{i}{2}(\Delta_{S_0}^R - \Delta_{S_0}^A)$	Moyal-Weyl
Normal	$\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + H$	Wick
Time-ordered	$-i\Delta_{S_0}^A$	$\star_{\mathcal{T}}$

The product $\star_{\mathcal{T}}$ will be the most useful here.

Star Product by Graphs

Star Product by Graphs

A graph describes a multidifferential operator.

Star Product by Graphs

A graph describes a multidifferential operator.

- The vertex j represents (a derivative of) the j 'th argument.

Star Product by Graphs

A graph describes a multidifferential operator.

- The vertex j represents (a derivative of) the j 'th argument.
- $\text{---}\leftarrow\text{---} = \Delta_{S_0}^A$.

Star Product by Graphs

A graph describes a multidifferential operator.

- The vertex j represents (a derivative of) the j 'th argument.
- $\text{---}\leftarrow\text{---} = \Delta_{S_0}^A$.

E.g., $\text{---}\leftarrow\text{---} = m$, i.e., $m(F, G) = F \cdot G$.

Star Product by Graphs

A graph describes a multidifferential operator.

- The vertex j represents (a derivative of) the j 'th argument.
- $\text{---}\leftarrow\text{---} = \Delta_{S_0}^A$.

E.g., $\text{---}1\text{---}2 = m$, i.e., $m(F, G) = F \cdot G$.

$$\star_{\mathcal{T}} = \text{---}1\text{---}2 - i\hbar \text{---}1\text{---}\leftarrow\text{---}2 + \frac{(-i\hbar)^2}{2} \text{---}1 \left(\begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right) \text{---}2 + \frac{(-i\hbar)^3}{6} \text{---}1 \left(\begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right) \text{---}2 + \dots$$

$$= \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut } \gamma|} \gamma$$

- $e = \#$ edges;
- the sum is over γ with vertices 1 and 2 and edges from 2 to 1.

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_T G = F \cdot G .$$

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_{\mathcal{T}} G = F \cdot G .$$

So, the trivial pointwise product is the time-ordered product for $\star_{\mathcal{T}}$.

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_{\mathcal{T}} G = F \cdot G .$$

So, the trivial pointwise product is the time-ordered product for $\star_{\mathcal{T}}$.

This simplifies calculations.

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_{\mathcal{T}} G = F \cdot G .$$

So, the trivial pointwise product is the time-ordered product for $\star_{\mathcal{T}}$.

This simplifies calculations.

Møller Operator

Now consider the action $S = S_0 + \lambda V$.

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_{\mathcal{T}} G = F \cdot G .$$

So, the trivial pointwise product is the time-ordered product for $\star_{\mathcal{T}}$.

This simplifies calculations.

Møller Operator

Now consider the action $S = S_0 + \lambda V$.

The quantum Møller operator simplifies to

$$R_{\mathcal{T}, \lambda V}(F) = \left(e^{i\lambda V/\hbar} \right)^{\star_{\mathcal{T}}^{-1}} \star_{\mathcal{T}} \left(e^{i\lambda V/\hbar} \cdot F \right) .$$

Time-ordered Product

For $x, y \in M$,

$$x \not\prec y \implies \Delta_{S_0}^A(x, y) = 0 .$$

This implies that

$$\text{supp } F \not\prec \text{supp } G \implies F \star_{\mathcal{T}} G = F \cdot G .$$

So, the trivial pointwise product is the time-ordered product for $\star_{\mathcal{T}}$.

This simplifies calculations.

Møller Operator

Now consider the action $S = S_0 + \lambda V$.

The quantum Møller operator simplifies to

$$R_{\mathcal{T}, \lambda V}(F) = \left(e^{i\lambda V/\hbar} \right)^{\star_{\mathcal{T}}^{-1}} \star_{\mathcal{T}} \left(e^{i\lambda V/\hbar} \cdot F \right) .$$

Its inverse is just

$$R_{\mathcal{T}, \lambda V}^{-1}(G) = e^{-i\lambda V/\hbar} \cdot \left(e^{i\lambda V/\hbar} \star_{\mathcal{T}} G \right) .$$

Møller by Graphs

Møller by Graphs

An unlabelled vertex • represents V .

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{(i\lambda)^3}{6\hbar^3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \dots$$

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{(i\lambda)^3}{6\hbar^3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \dots$$
$$= \sum_{\gamma} \frac{(-i\hbar)^{-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- $v = \#$ unlabelled vertices;
- the sum is over γ with only unlabelled vertices and no edges.

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} + \frac{(i\lambda)^3}{6\hbar^3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \dots$$
$$= \sum_{\gamma} \frac{(-i\hbar)^{-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- $v = \#$ unlabelled vertices;
- the sum is over γ with only unlabelled vertices and no edges.

$$\text{Recall, } F \star_{\mathcal{T}} G = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut } \gamma|} \gamma(F, G)$$

sum over γ with edges from 2 to 1.

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} \cdot R_{T,\lambda V}^{-1}(G) = e^{i\lambda V/\hbar} \star_T G$$

$$\begin{aligned} e^{i\lambda V/\hbar} &= 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \bullet\bullet + \frac{(i\lambda)^3}{6\hbar^3} \bullet\bullet\bullet + \dots \\ &= \sum_{\gamma} \frac{(-i\hbar)^{-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma \end{aligned}$$

- $v = \#$ unlabelled vertices;
- the sum is over γ with only unlabelled vertices and no edges.

$$\text{Recall, } F \star_T G = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut } \gamma|} \gamma(F, G)$$

sum over γ with edges from 2 to 1.

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \bullet\bullet + \frac{(i\lambda)^3}{6\hbar^3} \bullet\bullet\bullet + \dots$$
$$= \sum_{\gamma} \frac{(-i\hbar)^{-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- $v = \#$ unlabelled vertices;
- the sum is over γ with only unlabelled vertices and no edges.

Recall, $F \star_{\mathcal{T}} G = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut} \gamma|} \gamma(F, G)$

sum over γ with edges from 2 to 1.

$$e^{i\lambda V/\hbar} \cdot R_{\mathcal{T}, \lambda V}^{-1}(G) = e^{i\lambda V/\hbar} \star_{\mathcal{T}} G$$
$$= \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma) - v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma(G)$$

sum over γ with edges from 1 to unlabelled vertices.

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \bullet\bullet + \frac{(i\lambda)^3}{6\hbar^3} \bullet\bullet\bullet + \dots$$

$$= \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- $v = \#$ unlabelled vertices;
- the sum is over γ with only unlabelled vertices and no edges.

Recall, $F \star_{\mathcal{T}} G = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut} \gamma|} \gamma(F, G)$

sum over γ with edges from 2 to 1.

$$e^{i\lambda V/\hbar} \cdot R_{\mathcal{T}, \lambda V}^{-1}(G) = e^{i\lambda V/\hbar} \star_{\mathcal{T}} G$$

$$= \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma) - v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma(G)$$

sum over γ with edges from 1 to unlabelled vertices.

$$R_{\mathcal{T}, \lambda V}^{-1}(G) = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma) - v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma(G)$$

sum over *connected* γ with edges from 1 to unlabelled vertices.

Møller by Graphs

An unlabelled vertex \bullet represents V .

$$e^{i\lambda V/\hbar} = 1 + i\lambda\hbar^{-1} \bullet + \frac{(i\lambda)^2}{2\hbar^2} \bullet\bullet + \frac{(i\lambda)^3}{6\hbar^3} \bullet\bullet\bullet + \dots$$

$$= \sum_{\gamma} \frac{(-i\hbar)^{-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- $v = \#\text{unlabelled vertices}$;
- the sum is over γ with only unlabelled vertices and no edges.

Recall, $F \star_{\mathcal{T}} G = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)}}{|\text{Aut} \gamma|} \gamma(F, G)$

sum over γ with edges from 2 to 1.

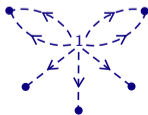
$$e^{i\lambda V/\hbar} \cdot R_{\mathcal{T}, \lambda V}^{-1}(G) = e^{i\lambda V/\hbar} \star_{\mathcal{T}} G$$

$$= \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma(G)$$

sum over γ with edges from 1 to unlabelled vertices.

$$R_{\mathcal{T}, \lambda V}^{-1}(G) = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} \lambda^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma(G)$$

sum over *connected* γ with edges from 1 to unlabelled vertices.



Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T}, \text{int}} G \doteq R_{\mathcal{T}, \lambda V}^{-1} (R_{\mathcal{T}, \lambda V} F \star_{\mathcal{T}} R_{\mathcal{T}, \lambda V} G)$$

Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T}, \text{int}} G \doteq R_{\mathcal{T}, \lambda V}^{-1} (R_{\mathcal{T}, \lambda V} F \star_{\mathcal{T}} R_{\mathcal{T}, \lambda V} G)$$

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-\lambda)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T}, \text{int}} G \doteq R_{\mathcal{T}, \lambda V}^{-1} (R_{\mathcal{T}, \lambda V} F \star_{\mathcal{T}} R_{\mathcal{T}, \lambda V} G)$$

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-\lambda)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;

Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T}, \text{int}} G \doteq R_{\mathcal{T}, \lambda V}^{-1} (R_{\mathcal{T}, \lambda V} F \star_{\mathcal{T}} R_{\mathcal{T}, \lambda V} G)$$

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-\lambda)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;

Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T}, \text{int}} G \doteq R_{\mathcal{T}, \lambda V}^{-1} (R_{\mathcal{T}, \lambda V} F \star_{\mathcal{T}} R_{\mathcal{T}, \lambda V} G)$$

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-\lambda)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;
- the vertices can be ordered with 1 lowest, 2 highest, and edges pointing down.

Resummation

These include arbitrary chains of bivalent vertices.

Resummation

These include arbitrary chains of bivalent vertices.



Resummation

These include arbitrary chains of bivalent vertices.



Resummation

These include arbitrary chains of bivalent vertices.

$$- \text{---} \leftarrow \text{---} \text{---} - \lambda \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} + \lambda^2 \text{---} \leftarrow \bullet \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} - \dots$$

Resummation

These include arbitrary chains of bivalent vertices.

$$-\text{---}\leftarrow\text{---} - \lambda \text{---}\leftarrow\bullet\text{---}\leftarrow\text{---} + \lambda^2 \text{---}\leftarrow\bullet\text{---}\leftarrow\bullet\text{---}\leftarrow\text{---} - \dots = \Delta_S^A$$

Resummation

These include arbitrary chains of bivalent vertices.

$$\text{---} \leftarrow \text{---} \leftarrow \text{---} - \lambda \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} + \lambda^2 \text{---} \leftarrow \bullet \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} - \dots = \Delta_S^A$$

- This accounts for all bivalent vertices in $\star_{\mathcal{T}, \text{int}}$.

Resummation

These include arbitrary chains of bivalent vertices.

$$\text{---} \leftarrow \text{---} \leftarrow \text{---} - \lambda \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} + \lambda^2 \text{---} \leftarrow \bullet \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} - \dots = \Delta_S^A$$

- This accounts for all bivalent vertices in $\star_{\mathcal{T},\text{int}}$.
- A valency $r > 2$ vertex represents $\lambda V^{(r)} = S^{(r)}$.

Resummation

These include arbitrary chains of bivalent vertices.

$$\text{---} \leftarrow \text{---} \text{---} - \lambda \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} + \lambda^2 \text{---} \leftarrow \bullet \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} - \dots = \Delta_S^A$$

- This accounts for all bivalent vertices in $\star_{\mathcal{T},\text{int}}$.
- A valency $r > 2$ vertex represents $\lambda V^{(r)} = S^{(r)}$.

Now let:

- Unlabelled vertices represent derivatives of S ;

Resummation

These include arbitrary chains of bivalent vertices.

$$\begin{array}{c}
 \text{---} \leftarrow \text{---} \quad - \quad \lambda \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} \quad + \quad \lambda^2 \text{---} \leftarrow \bullet \text{---} \leftarrow \bullet \text{---} \leftarrow \text{---} \quad - \quad \dots = \Delta_S^A
 \end{array}$$

- This accounts for all bivalent vertices in $\star_{\mathcal{T}, \text{int}}$.
- A valency $r > 2$ vertex represents $\lambda V^{(r)} = S^{(r)}$.

Now let:

- Unlabelled vertices represent derivatives of S ;
- $\text{---} \leftarrow \bullet = \Delta_S^A$.

Nonperturbative Expression

$$\star\tau_{,\text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

Nonperturbative Expression

$$\star\tau_{\text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

Nonperturbative Expression

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;

Nonperturbative Expression

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;

Nonperturbative Expression

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;
- unlabelled vertices have valency *at least 3*;

Nonperturbative Expression

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;
- unlabelled vertices have valency *at least 3*;
- the vertices can be ordered with 1 lowest, 2 highest, and edges pointing down.

Nonperturbative Expression

$$\star_{\mathcal{T}, \text{int}} = \sum_{\gamma} \frac{(-i\hbar)^{e(\gamma)-v(\gamma)} (-1)^{v(\gamma)}}{|\text{Aut}(\gamma)|} \gamma$$

- γ has labelled vertices 1 and 2;
- every unlabelled vertex has at least one ingoing edge and one outgoing edge;
- unlabelled vertices have valency *at least 3*;
- the vertices can be ordered with 1 lowest, 2 highest, and edges pointing down.

This is a star product quantizing S .

$$\star_{\mathcal{T},\text{int}} = m + \hbar B_1 + \hbar^2 B_2 + \hbar^3 B_3 + \dots$$

$$\star_{\mathcal{T},\text{int}} = m + \hbar B_1 + \hbar^2 B_2 + \hbar^3 B_3 + \dots$$

$$B_1 = -i \text{ 1} \longleftarrow \text{2} \text{ ,}$$

$$\star \mathcal{T}_{,\text{int}} = m + \hbar B_1 + \hbar^2 B_2 + \hbar^3 B_3 + \dots$$

$$B_1 = -i \text{ 1} \longleftarrow \text{2} \text{ ,}$$

$$B_2 = \frac{-1}{2} \text{ 1} \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \text{ 2} + \frac{1}{2} \text{ 1} \begin{array}{c} \circlearrowleft \\ \longleftarrow \end{array} \text{ 2} + \frac{1}{2} \text{ 1} \longleftarrow \begin{array}{c} \circlearrowright \\ \text{ 2} \end{array} - \frac{1}{2} \text{ 1} \longleftarrow \begin{array}{c} \circlearrowleft \\ \longleftarrow \end{array} \text{ 2} \text{ ,}$$

$$\star \mathcal{T}_{,\text{int}} = m + \hbar B_1 + \hbar^2 B_2 + \hbar^3 B_3 + \dots$$

$$B_1 = -i \text{ 1 } \longleftarrow \text{ 2 } ,$$

$$B_2 = \frac{-1}{2} \text{ 1 } \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \text{ 2 } + \frac{1}{2} \text{ 1 } \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} \longleftarrow \text{ 2 } + \frac{1}{2} \text{ 1 } \longleftarrow \begin{array}{c} \circlearrowright \\ \circlearrowright \end{array} \text{ 2 } - \frac{1}{2} \text{ 1 } \longleftarrow \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} \longleftarrow \text{ 2 } ,$$

$$B_3 = \frac{i}{6} \text{ 1 } \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \text{ 2 } - \frac{i}{2} \text{ 1 } \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowleft \end{array} \text{ 2 } - \frac{i}{2} \text{ 1 } \begin{array}{c} \circlearrowright \\ \circlearrowright \\ \circlearrowright \end{array} \text{ 2 } + \frac{i}{2} \text{ 1 } \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowright \end{array} \text{ 2 }$$

$$+ i \text{ 1 } \begin{array}{c} \circlearrowleft \\ \circlearrowleft \\ \circlearrowright \\ \circlearrowright \end{array} \text{ 2 } - \frac{i}{4} \text{ 1 } \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \text{ 2 } + \frac{i}{4} \text{ 1 } \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} \longleftarrow \begin{array}{c} \circlearrowright \\ \circlearrowright \end{array} \text{ 2 } - \dots$$

$$\begin{aligned}
& \dots - \frac{i}{6} \text{ (circle with two counter-propagating arrows) } \leftarrow 2 + \frac{i}{2} \text{ (circle with two clockwise arrows) } \leftarrow 2 + \frac{i}{2} \text{ (circle with two counter-clockwise arrows) } \leftarrow 2 - \frac{i}{2} \text{ (circle with two clockwise arrows) } \leftarrow 2 \\
& - i \text{ (circle with two vertical arrows) } \leftarrow 2 + \frac{i}{4} \text{ (figure-eight with two counter-propagating arrows) } \leftarrow 2 - \frac{i}{4} \text{ (two circles with two counter-propagating arrows) } \leftarrow 2 \\
& - \frac{i}{6} \text{ (circle with two counter-propagating arrows) } \leftarrow 2 + \frac{i}{2} \text{ (circle with two clockwise arrows) } \leftarrow 2 + \frac{i}{2} \text{ (circle with two counter-clockwise arrows) } \leftarrow 2 - \frac{i}{2} \text{ (circle with two clockwise arrows) } \leftarrow 2 \\
& - i \text{ (circle with two vertical arrows) } \leftarrow 2 + \frac{i}{4} \text{ (figure-eight with two counter-propagating arrows) } \leftarrow 2 - \frac{i}{4} \text{ (two circles with two counter-propagating arrows) } \leftarrow 2 + \dots
\end{aligned}$$

$$\dots + \frac{i}{6} \left[\text{Diagram 1} \right] - \frac{i}{2} \left[\text{Diagram 2} \right] - \frac{i}{2} \left[\text{Diagram 3} \right]$$

Diagram 1: A circle with a horizontal line through its center. The line has an arrow pointing left. The circle has two curved arrows: one on the top half pointing left, and one on the bottom half pointing right.

Diagram 2: A circle with a horizontal line through its center. The line has an arrow pointing left. The circle has two curved arrows: one on the top half pointing left, and one on the bottom half pointing right. A diagonal line segment is drawn from the top-right to the center, with an arrow pointing towards the center.

Diagram 3: A circle with a horizontal line through its center. The line has an arrow pointing left. The circle has two curved arrows: one on the top half pointing left, and one on the bottom half pointing right. A diagonal line segment is drawn from the top-left to the center, with an arrow pointing towards the center.

$$+ \frac{i}{2} \left[\text{Diagram 4} \right] + i \left[\text{Diagram 5} \right]$$

Diagram 4: A circle with a horizontal line through its center. The line has an arrow pointing left. The circle has two curved arrows: one on the top half pointing left, and one on the bottom half pointing right. A diagonal line segment is drawn from the top-left to the top-right, with an arrow pointing from left to right.

Diagram 5: A circle with a horizontal line through its center. The line has an arrow pointing left. The circle has two curved arrows: one on the top half pointing left, and one on the bottom half pointing right. A vertical line segment is drawn from the top to the bottom, with an arrow pointing from top to bottom.

$$- \frac{i}{4} \left[\text{Diagram 6} \right] + \frac{i}{4} \left[\text{Diagram 7} \right]$$

Diagram 6: A horizontal line with an arrow pointing left. Two pairs of curved arrows are drawn above and below the line, each pair forming a lens shape that points towards the center of the line.

Diagram 7: Two circles connected by a horizontal line. The line has an arrow pointing left. Each circle has a curved arrow on its top half pointing left and a curved arrow on its bottom half pointing right.

$$\begin{aligned}
& \dots + \frac{i}{6} \text{ (circle with counter-clockwise arrow) } \leftarrow 2 - \frac{i}{2} \text{ (circle with clockwise arrow and diagonal line) } \leftarrow 2 - \frac{i}{2} \text{ (circle with clockwise arrow and diagonal line) } \leftarrow 2 \\
& + \frac{i}{2} \text{ (circle with clockwise arrow and diagonal line) } \leftarrow 2 + i \text{ (circle with clockwise arrow and vertical line) } \leftarrow 2 \\
& - \frac{i}{4} \text{ (figure-eight diagram) } \leftarrow 2 + \frac{i}{4} \text{ (two circles in series) } \leftarrow 2
\end{aligned}$$

E. Hawkins, K. Rejzner:

The Star Product in Interacting Quantum Field Theory.

arXiv:1612.09157 [math-ph]