Combinatorics of the Star Product in AQFT

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Changing K by a smooth, symmetric function gives an equivalent star product.

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The product $\star_{\mathcal{T}}$ will be the most useful here.

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$$\star_{\mathcal{T}} = 1 \quad 2 - i\hbar \; 1 \dashleftarrow 2 + \frac{(-i\hbar)^2}{2} \; 1 \substack{\dashleftarrow \\ \dashleftarrow} 2 + \frac{(-i\hbar)^3}{6} \; 1 \substack{\dashleftarrow \\ \dashleftarrow \\ \dashleftarrow} 2 + \dots$$

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- $e = \#$ edges;
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Its inverse is just

$$R^{-1}_{\mathcal{T},\lambda V}(G) = e^{-i\lambda V/\hbar} \cdot \left(e^{i\lambda V/\hbar} \star_{\mathcal{T}} G\right) \ .$$

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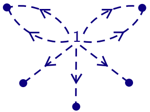

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- every unlabelled vertex has at least one ingoing edge and one outgoing edge;

Interacting Product

The Møller operator intertwines the interacting and non-interacting products.

$$F \star_{\mathcal{T},\mathrm{int}} G \doteq R^{-1}_{\mathcal{T},\lambda V}(R_{\mathcal{T},\lambda V} F \star_{\mathcal{T}} R_{\mathcal{T},\lambda V} G)$$

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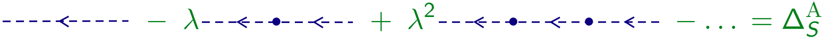

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This is a star product quantizing S.

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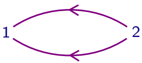
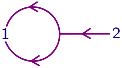
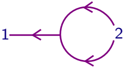
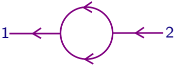
$B_2 = \frac{-1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $- \frac{1}{2}$  ,

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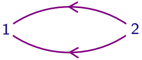
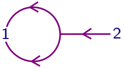
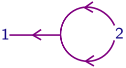
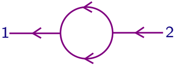
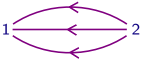
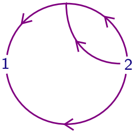
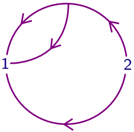
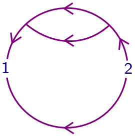
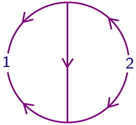
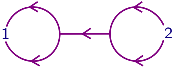
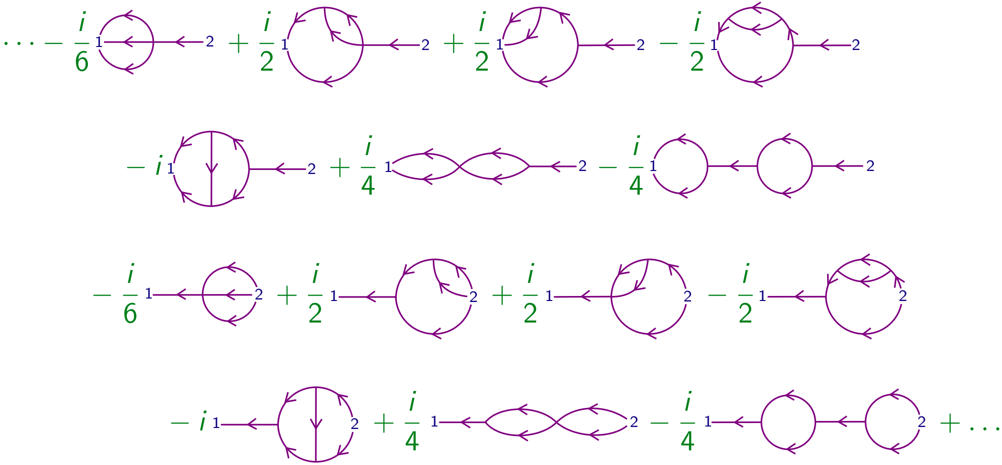
$B_2 = \frac{-1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $- \frac{1}{2}$  ,

$B_3 = \frac{i}{6}$  $- \frac{i}{2}$  $- \frac{i}{2}$  $+ \frac{i}{2}$ 

$+ i$  $- \frac{i}{4}$  $+ \frac{i}{4}$  $- \dots$



$\cdots - \frac{i}{6}$ 1 2 $+ \frac{i}{2}$ 1 2 $+ \frac{i}{2}$ 1 2 $- \frac{i}{2}$ 1 2
$- i$ 1 2 $+ \frac{i}{4}$ 1 2 $- \frac{i}{4}$ 1 2
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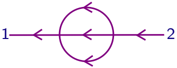
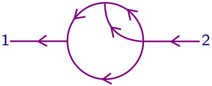
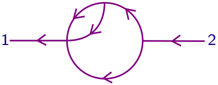
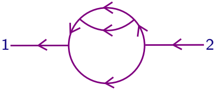
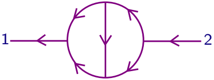
$\cdots + \frac{i}{6}$ 

$- \frac{i}{2}$ 

$- \frac{i}{2}$ 

$+ \frac{i}{2}$ 

$+ i$ 

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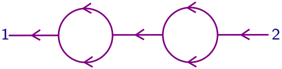
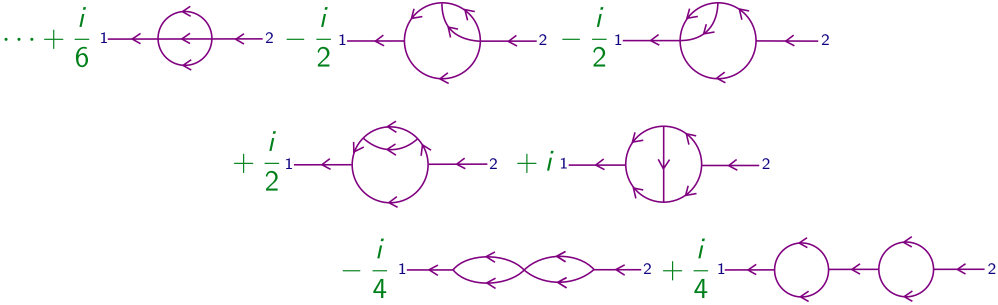
$+ \frac{i}{4}$ 



E. Hawkins, K. Rejzner:
The Star Product in Interacting Quantum Field Theory.
arXiv:1612.09157 [math-ph]