The generalized Principle of Perturbative Agreement with applications to the thermal mass

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- The AQFT approach is based on the identification of a $*$-algebra $\mathscr{A}$ of physical observables.
- For free theories this construction is well under control.[Brunetti, Duetsch & Fredenhagen '09, F. & Rejzner '12-'14]
- Interacting theories are treated perturbatively and $\mathscr{A}$ is identified up to renormalization freedom. Physical requirements give restrictions on the possible choices.[Brunetti & Fredenhagen '00, Holland & Wald '01-'02-'05, B., Duetsch & F. '09]
- The Principle of the Perturbative Agreement [Hollands & Wald '05, Zahn '13] provides an example of such a requirement:

$$-\Box_g \phi + M^2 \phi = -\Box_g \phi + M^2 \phi$$

- The generalized PPA provides a generalization to PPA in case of higher order polynomial interactions:

$$-\Box_g \phi + M^2 \phi + \lambda \phi^3 = -\Box_g \phi + M^2 \phi + \lambda \phi^3$$

Functional approach

We deal with the real scalar Klein Gordon field on a globally hyperbolic spacetime $(\mathscr{M}, g)$.

$$\mathcal{S}_1(\phi) = \frac{1}{2}\int_{\mathscr{M}} f\left(\phi P_1 \phi - 2j\phi\right), \quad P_1\phi \doteq (-\Box_g + M_1)\phi,\ f \in \mathscr{D}(\mathscr{M}).$$

Dynamics is ruled by a linear differential hyperbolic operator $\Rightarrow \exists!\ \Delta_1^{R/A}$ retarded/advanced propagators

$$\Delta_1^{R/A} : \mathcal{D}(\mathscr{M}) \to \mathcal{E}(\mathscr{M}),$$

$$\begin{aligned} P_1\Delta_1 &= \Delta_1 P_1 = I_{\mathcal{D}(\mathscr{M})}, \quad \mathrm{supp}(\Delta_1^{R/A} f) \subseteq J^{\pm}(\mathrm{supp}(f)), \\ \Delta_1 &\doteq \Delta_1^R - \Delta_1^A \quad \text{causal propagator} \end{aligned}$$

Functional approach: the $*$-algebra $\mathscr{A}_1 = \mathscr{A}_1(\mathscr{M}, g)$ of observables is generated by functionals on field configurations. $\phi \in \mathscr{E}(\mathscr{M})$

[Brunetti, Duetsch & Fredenhagen '09]

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[Brunetti, Duetsch & Fredenhagen '09]

Actually $\mathscr{A}_1 = \mathrm{Alg}(\mathscr{F}_{\mu c}, \star_1, *)$ where

$$\begin{aligned}
\mathscr{F}_{\mu c} &\doteq \{F : \mathscr{E}(\mathscr{M}) \to \mathbb{C} |\ F \text{ smooth, compactly supported,} \\
&\qquad \mathrm{WF}(F^{(n)}(\phi)) \cap (V_+^n \cup V_-^n) = \emptyset\}, \\
(F \star_1 G)(\phi) &\doteq \sum_{n \geq 0} \frac{\hbar^n}{n!} \left\langle (\Delta_1^+)^{\otimes n}, F^{(n)}(\phi) \otimes G^{(n)}(\phi) \right\rangle \\
F^*(\phi) &\doteq \overline{F(\phi)},
\end{aligned}$$

being $\Delta_1^+ \in \mathscr{D}'(\mathscr{M}^2)$ an Hadamard bidistribution.

- Δ_1^+ is not unique: different choices of $\Delta_1^+, \widehat{\Delta_1^+}$ give to $*$-isomorphic algebras

$$\begin{aligned} \alpha_w : \widehat{\mathscr{A}_1} \to \mathscr{A}_1, \quad w &\doteq \Delta_1^+ - \widehat{\Delta_1^+} \\ F \mapsto \alpha_w(F) &\doteq \sum_{n\geq 0} \frac{\hbar^n}{n!} \left\langle w^{\otimes n}, F^{(2n)} \right\rangle , \end{aligned}$$

- $\mathscr{F}_{\mu c}$ contains local functionals, i.e.

$$\mathscr{F}_{\text{loc}} \doteq \{F : \mathscr{E}(\mathscr{M}) \to \mathbb{C} |\ F \text{ smooth, compactly supported,} \\ \text{supp}(F^{(n)}) \subseteq \text{D}_n,\ \text{WF}(F^{(n)}) \perp T^*\text{D}_n\}.$$

- $\mathscr{A}_1^{\text{reg}} \doteq \text{Alg}(\mathscr{F}_{\text{reg}}, \star_1, *)$ is the $*$-algebra generated by regular functionals ($\text{WF}(F^{(n)}) = \emptyset$).

Bogoliubov formula

Interacting models are typically described by local non linear perturbations

$$\mathcal{S}_{1,V} \doteq \mathcal{S}_1 + V, \qquad V \in \mathscr{F}_{\text{loc}}.$$

Perturbative approach: the interacting $*$-algebra $\widetilde{\mathscr{A}_{1,V}}$ is represented onto $\mathscr{A}_1$ via Bogoliubov formula.

$$\begin{aligned}
F \cdot_{T_1} G &\doteq F \star_1 G \qquad \text{if } F \gtrsim G, \\
\mathscr{R}^{\hbar}_{1,V} F &\doteq S(V)^{-1} \star_1 (S(V) \cdot_{T_1} F), \quad S(V) \doteq \exp_{T_1}\left(\frac{i}{\hbar} V\right) \\
F \star_{1,V} G &\doteq (\mathscr{R}^{\hbar}_{1,V})^{-1} \left(\mathscr{R}^{\hbar}_{1,V} F \star_1 \mathscr{R}^{\hbar}_{1,V} G\right) \\
F^{*_{1,V}} &\doteq (\mathscr{R}^{\hbar}_{1,V})^{-1} \left(\mathscr{R}^{\hbar}_{1,V}(F)^*\right).
\end{aligned}$$

- The time-ordered product is well defined on $\mathscr{A}_1^{\mathrm{reg}}$ by an exponential formula given by the Feynmann propagator $\Delta_1^F \doteq \Delta_1^+ + i\Delta_1^A$.
- $\cdot_{T_1}$ can be extended as a map

$$T_1 : \mathscr{F}_{\mathrm{mloc}} \doteq \bigoplus_n \mathscr{F}_{\mathrm{loc}}^{\otimes n} \to \mathscr{A}_1, \qquad T_1 = T(\mathscr{M}, g, M_1, j)$$

$$F_1 \cdot_{T_1} \ldots \cdot_{T_1} F_n \doteq T_1\left(T_1^{-1}(F_1) \otimes \cdots \otimes T_1^{-1}(F_n)\right)$$

  satisfying suitable axioms. [Brunetti & Fredenhagen '00, Holland & Wald '01-'02-'05, Brunetti, Duetsch & Fredenhagen '09]
  Non uniqueness is controlled by renormalization freedom.
- On $\mathscr{F}_{\mathrm{loc}}$, T_1 corresponds to the assignment of local and covariant Wick polynomials as element in $\mathscr{A}_1$.

- $\cdot_{T_1}$ is an associative product defined on

$$\mathscr{F}_{T_1 \mathrm{loc}} \doteq \left\{ \sum_{n \geq 0} F_{1,n} \cdot_{T_1} \ldots \cdot_{T_1} F_{n,n}, F_{k,l} \in \mathscr{F}_{\mathrm{reg}} \cup \mathscr{F}_{\mathrm{loc}} \right\}$$

$$\mathscr{R}^{\hbar}_{1,V} : \mathscr{F}_{T_1 \mathrm{loc}} \to \mathscr{A}_1$$

- $\widetilde{\mathscr{A}_{1,V}}$ is ill-defined even for the simplest cases of quadratic V.

$$F \star_{1,V} G \doteq (\mathscr{R}^{\hbar}_{1,V})^{-1} \left(\mathscr{R}^{\hbar}_{1,V} F \star_1 \mathscr{R}^{\hbar}_{1,V} G \right) .$$

- $\mathscr{A}_{1,V} \doteq \mathrm{Alg}(\mathscr{R}^{\hbar}_{1,V} \mathscr{F}_{T_1 \mathrm{loc}}, \star_1, *) \subseteq \mathscr{A}_1$ is a well-defined $*$-subalgebra.

PPA for mass/curvature variation

$(\mathscr{M}, g, M, j) \mapsto T(\mathscr{M}, g, M, j)$ is locally covariant

$$\begin{array}{ccc} (\mathscr{M}_1, g, M, j) & \overset{T}{\rightsquigarrow} & T_1 = T(\mathscr{M}_1, g, M, j) \\ \Big\downarrow \psi & & \Big\downarrow \mathscr{A}(\psi) \\ (\mathscr{M}_2, g', M', j') & \overset{T}{\rightsquigarrow} & T_2 = T(\mathscr{M}_2, g', M', j') \end{array}$$

PPA for mass/curvature variation

What if we change M?

$$
\begin{array}{ccc}
(\mathscr{M}, g, M_1, j) & \overset{T}{\rightsquigarrow} & T_1 = T(\mathscr{M}, g, M_1, j) \\
\downarrow & & \downarrow \text{?!?} \\
(\mathscr{M}, g, M_2, j) & \overset{T}{\rightsquigarrow} & T_2 = T(\mathscr{M}, g, M_2, j)
\end{array}
$$

PPA for mass/curvature variation

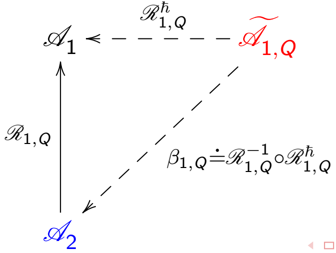
Assume $Q \in \mathscr{F}_{\text{loc}}$ is such that $Q(\phi) \doteq \frac{1}{2}\int_{\mathscr{M}} M^2\phi^2$.

$$\mathcal{S}_{1,Q} = \mathcal{S}_1 + Q = \mathcal{S}_2 = \frac{1}{2}\int_{\mathscr{M}} f\left(\phi P_2\phi - 2j\phi\right).$$

$\widetilde{\mathscr{A}}_{1,Q}$ and $\mathscr{A}_2$ carry the same physical information.

$\Downarrow$

"$\widetilde{\mathscr{A}}_{1,Q} \simeq \mathscr{A}_2$" for a suitable choice of renormalization freedom.



PPA for mass/curvature variation

$$
\begin{array}{ccc}
(\mathscr{M}, g, M_1, j) & \overset{T}{\rightsquigarrow} & T_1 = T(\mathscr{M}, g, M_1, j) \\
\downarrow & & \downarrow \beta_{1,Q} \\
(\mathscr{M}, g, M_2, j) & \overset{T}{\rightsquigarrow} & T_2 = (\mathscr{M}, g, M_2, j)
\end{array}
$$

Principle of Perturbative Agreement (Hollands & Wald '05)

As a map $(\mathscr{M}, g, M, j) \to T(\mathscr{M}, g, M, j)$, T is said to satisfies the Principle of Perturbative Agreement if

$$T_2 = \beta_{1,Q} \circ T_1 \quad \text{on } \mathscr{F}_{\text{mloc}} \quad \beta_{1,Q} \doteq \mathscr{R}_{1,Q}^{-1} \circ \mathscr{R}_{1,Q}^{\hbar}.$$

Classical Møller operator

The classical Møller operator intertwines the dynamics of $\mathscr{A}_2$ and $\mathscr{A}_1$ by identifying them in the past.

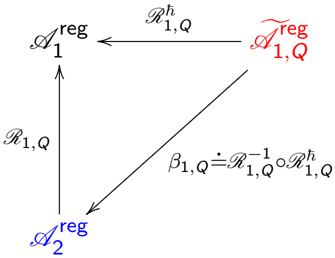
$$\mathscr{R}_{1,Q} : \mathscr{A}_2 \to \mathscr{A}_1, \quad \mathscr{R}_{1,Q} F(\phi) \doteq F(R_{1,Q}\phi), \quad P_2 \circ R_{1,Q} = P_1.$$

Theorem (D.,Hack, Pinamonti)

- $\Delta_2^R = R_{1,Q} \circ \Delta_1^R, \quad \Delta_2^A = \Delta_1^A \circ R_{1,Q}^\dagger;$
- $\Delta_2^{(+)} = R_{1,Q} \circ \Delta_1^{(+)} \circ R_{1,Q}^\dagger;$
- $\mathscr{R}_{1,Q} : \mathscr{A}_2 \to \mathscr{R}_{1,Q}(\mathscr{A}_2) \subset \mathscr{A}_1$ *is a* $*$*-isomorphism.*

From now on the $\star_2$-product on $\mathscr{A}_2$ will be the one induced by $\star_1$ via $\mathscr{R}_{1,Q}$.

Perturbative Agreement for $\mathscr{F}_{\text{reg}}$



Proposition

- $\widetilde{\mathscr{A}}^{\text{reg}}_{1,Q} = \text{Alg}(\mathscr{F}_{\text{reg}}, \star_{1,Q}, *_{1,Q})$ *is well-defined.*
- $\beta_{1,Q} : \widetilde{\mathscr{A}}^{\text{reg}}_{1,Q} \to \mathscr{A}_2^{\text{reg}}$ *is a $*$-isomorphism.*

Characterization of $\beta_{1,Q}$ on $\mathscr{F}_{\mathrm{reg}}$

Let $F_f(\phi) \doteq \int f\phi \in \mathscr{F}_{\mathrm{reg}}$. Consider $\beta_{1,Q} : \widetilde{\mathscr{A}}^{\mathrm{reg}}_{1,Q} \to \mathscr{A}^{\mathrm{reg}}_2$.

Theorem

- $\mathscr{R}^{\hbar}_{1,Q} F_f = \mathscr{R}_{1,Q} F_f$ ($\Longrightarrow \beta_{1,Q} F_f = F_f$);
- $[F_f, F_g]_{\star_{1,Q}} = \beta^{-1}_{1,Q}[\beta_{1,Q} F_f, \beta_{1,Q} F_g]_{\star_2} = i\hbar\Delta_2(f,g)$.
- *The* $\star_{1,Q}$*-product on* $\widetilde{\mathscr{A}}^{\mathrm{reg}}_{1,Q}$ *is given by an exponential formula with*

$$\Delta^+_{1,Q} \doteq \Delta^+_2 + \Delta^F_1 - \Delta^F_2 .$$

- $\beta_{1,Q} = \alpha_{d_{1,Q}}$, *where* $d_{1,Q} = \Delta^F_2 - \Delta^F_1$.
- *The Principle of Perturbative Agreement holds on* $\mathscr{F}_{\mathrm{mreg}}$.

Extension on $\mathscr{F}_{\mathrm{mloc}}$

$$\beta_{1,Q}(F) = \alpha_{d_{1,Q}}(F) \quad \text{for } F \in \mathscr{F}_{\mathrm{loc}}?$$

$d_{1,Q} = \Delta_2^F - \Delta_1^F$ is at least logarithmically divergent on D_2.
Perturbative expansion:

$$\begin{aligned}
\Delta_2^F &= \Delta_2^+ + i\Delta_2^A \\
&= R_{1,Q} \circ \Delta_1^+ R_{1,Q}^\dagger + i\Delta_1^A \circ R_{1,Q}^\dagger, \\
R_{1,Q} &= (I + \Delta_1^R Q^{(1)})^{-1} = \sum (-\Delta_1^R Q^{(1)})^n \\
\Delta_2^F - \Delta_1^F &\sim_n i^n \Delta_1^F (Q^{(1)} \Delta_1^F)^n + A.
\end{aligned}$$

It is enough to renormalize Δ_1^F.

Theorem (D.,Hack, Pinamonti)

- $\beta_{1,Q} : \mathscr{F}_{T_1 \mathrm{loc}} \to \mathscr{F}_{\mu c}$ *is a deformation* $\alpha_{d_{1,Q}}$.
- $T_2 \doteq \beta_{1,Q} \circ T_1$ *defines a time ordered map for* $\mathscr{A}_2$.
- *It holds the cocycle condition*

$$\beta_{1,Q_3} = \beta_{2,Q_3} \circ \beta_{1,Q_2}.$$

- *Fixing* $T_1 = T_1(\mathscr{M}, g, M = 0, j)$ *the map*

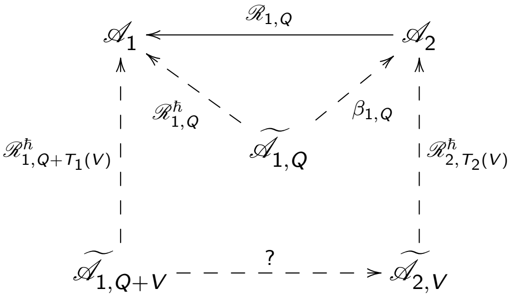
$$T(\mathscr{M}, g, M_2, j) \doteq \beta_{1,Q} \circ T_1(\mathscr{M}, g, 0, j)$$

satisfies the Perturbative Agreement for mass/curvature variation.

The perturbative construction is “exact”.

generalized PPA

$$-\Box_g\phi + M^2\phi + \lambda\phi^3 = -\Box_g\phi + M^2\phi + \lambda\phi^3$$



generalized Principle of Perturbative Agreement

On $\mathscr{F}_{T_1\mathrm{loc}}$, $\mathscr{R}^\hbar_{1,Q+T_1(V)} = \mathscr{R}_{1,Q} \circ \mathscr{R}^\hbar_{2,T_2(V)} \circ \beta_{1,Q}$.

Applications: thermal mass

Theorem (Fredenhagen & Lindner '14)

Let $\mathscr{A}_1 = \mathscr{A}_1(\mathbb{M}^4, \eta)$, $V \in \mathscr{F}_{\text{loc}}$, ω^β a β-KMS for $\mathscr{A}_1$ w.r.t. $\{\alpha_t\}_t$. If the truncated functions of ω^β satisfy suitable space-like decay behaviour, then there exists a β-KMS state ω^β_V on $\widetilde{\mathscr{A}_{1,V}}$ w.r.t. $\{\alpha^V_t\}_t$ constructed out of ω^β.

This construction applies for the case of massive Klein Gordon field.

What about the massless case?

Applications: thermal mass

Theorem (Fredenhagen & Lindner '14)

Let $\mathscr{A}_1 = \mathscr{A}_1(\mathbb{M}^4, \eta)$, $V \in \mathscr{F}_{\text{loc}}$, ω^β a β-KMS for $\mathscr{A}_1$ w.r.t. $\{\alpha_t\}_t$. If the truncated functions of ω^β satisfy suitable space-like decay behaviour, then there exists a β-KMS state ω^β_V on $\widetilde{\mathscr{A}_{1,V}}$ w.r.t. $\{\alpha^V_t\}_t$ constructed out of ω^β.

This construction applies for the case of massive Klein Gordon field.

$$\text{gPPA: } \widetilde{\mathscr{A}_{1,V}} \simeq \widetilde{\mathscr{A}_{1+Q,V-Q}}, \quad Q = \text{“virtual” mass.}$$

Theorem (D.,Hack,Pinamonti)

In the above hypothesis, the pull-back of the β-KMS state ω^β_{V-Q} on $\mathscr{A}_{2,V-Q}$ defines a β-KMS state on $\mathscr{A}_{1,V}$.