Quantum Field Theory on a Causal Set

Fay Dowker

Imperial College, London

York April 2017

QFT in curved spacetime

- A stepping stone to quantum gravity
- Furnishes us with a quantum gravity result: the entropy of a black hole!
- Consensus that in quantum gravity, spacetime is not a differentiable manifold at the Planck scale
- In same spirit as QFT in curved spacetime, hope to learn something by doing QFT on a background spacetime with no structure at smaller than Planck scales (though also expect pathologies because not the full theory)
- A Causal Set is such an entity. It provides a covariant cutoff.
- Explore issues such as "trans-Planckian" modes in Hawking radiation calculation and the nature of "entanglement entropy"

Plan

- The Causal Set approach to quantum gravity: atomic spacetime in which the fundamental degrees of freedom are discrete order relations. ('tHooft, Myrheim, Bombelli, Lee, Myer and Sorkin.)
- Scalar QFT on a finite causal set – Free, Gaussian.
- An attitude to QFT more generally (e.g. in the continuum) "From Green Function to Quantum Field"
- Retarded Green functions on causal sets: 2-d, 4-d Minkowski and 4-d deSitter/Anti-deSitter
- Results in 2d
- Discussion points: path integral, interacting fields, continuum theory, Hadamard condition & smoothing, entanglement entropy........

Quantum Gravity: what to keep? what to ditch? what's new?

The causal set approach claims that certain aspects of General Relativity and quantum theory will have direct counterparts in quantum gravity:

- the spacetime causal order from General Relativity,
- the path integral from quantum theory.

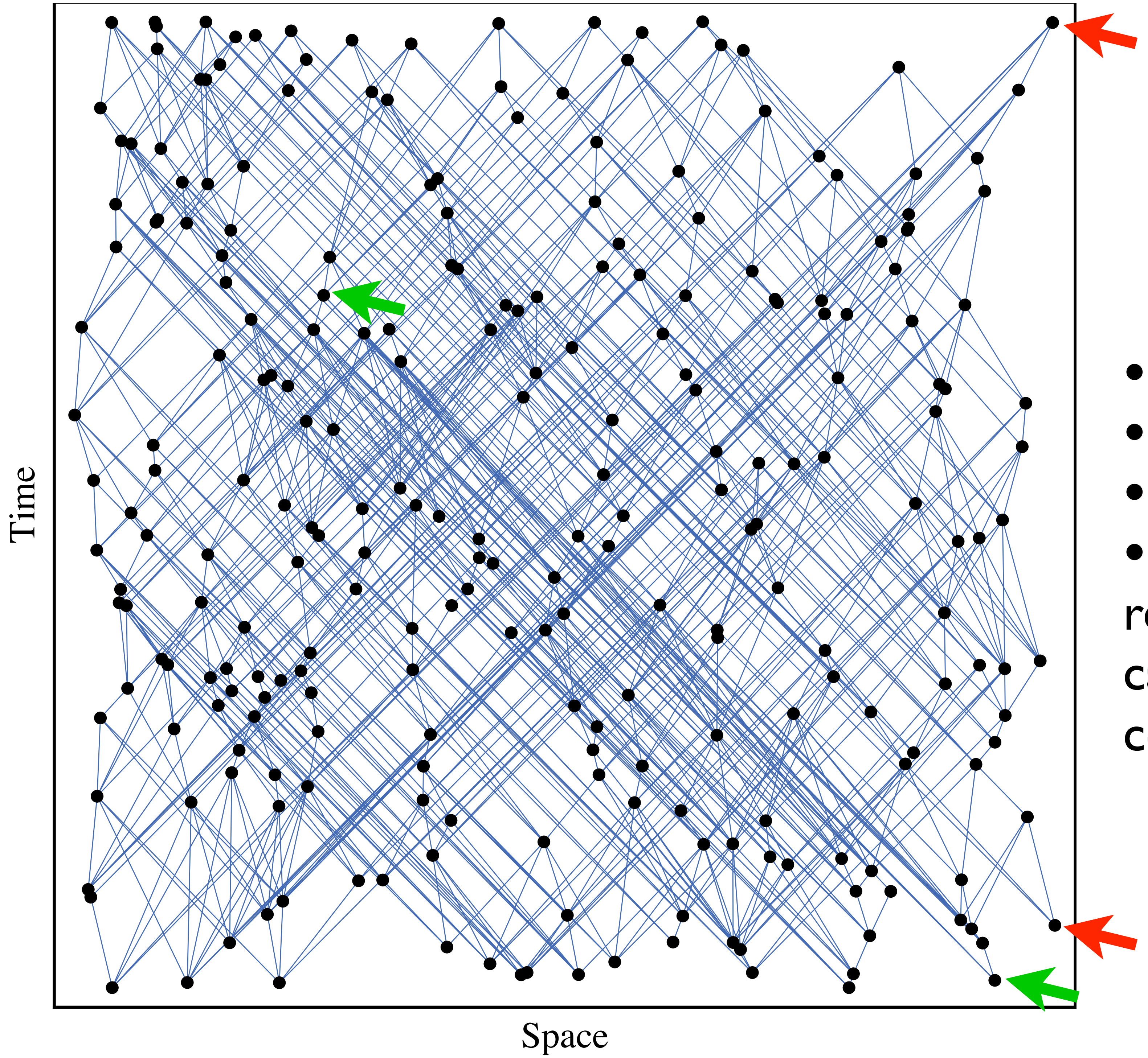
It makes one main new hypothesis about the nature of the physical world:

- fundamental discreteness of spacetime.

'tHooft; Myrheim; Bombelli, Lee, Meyer and Sorkin.

A causal set that is well approximated by 2d Minkowski space

A
Transitive
Acyclic
Directed
Graph



- Hasse
- combinatorial
- nonlocal (&causal)
- **statistical** relation between causet and continuum: **N~V**

On Planckian scales, this is what Minkowski space is (usual quantum caveats)
For any causal Lorentzian space-time M $\exists$ causal sets approximated by M
Unification: (Manifold, metric) is replaced by Causal Set

Free Scalar Field I

In continuum (globally hyperbolic), a typical treatment is

1. $\Box\phi = 0$ (for convenience, take $m = 0$)
2. $\Box G = \delta$, the retarded Green function
3. $\Delta^{xy} := G^{xy} - \widetilde{G}^{xy}$ (where $\widetilde{\cdot}$ denotes transpose).
4. $[\phi^x, \phi^y] = i\Delta^{xy}$ (covariant form of CCR)
5. positive frequency $\rightarrow a, a^\dagger$
6. Vacuum state and Fock space from $a|0\rangle = 0$
7. $W^{xy} := \langle 0|\phi^x\phi^y|0\rangle$
8. All n-point functions from Wick's formula (assume $\langle 0|\phi|0\rangle = 0$.)

Free Scalar Field II

On causal set, may or may not have $\Box$. Certainly don't have positive frequency. But note, if $[\phi^x, \phi^y] = i\Delta^{xy}$ then $[\phi^x, \Box\phi^y] = 0$

So instead of the e.o.m. use

$$[\phi^x, \sum_y \omega^y \phi^y] = 0 \quad \forall x \quad \Rightarrow \sum_y \omega^y \phi^y = 0$$

$$\sum_y \Delta^{xy} \omega^y = 0 \quad \forall x \quad \Rightarrow \sum_y \omega^y \phi^y = 0$$

Indeed,

$$\mathrm{ker}\Delta \perp \mathrm{ker}\Box$$

$$\mathrm{im}\Delta = \mathrm{ker}\Box$$

Free Scalar Field on finite Causal Set (Johnston)

1. Retarded Green function G
2. $\Delta^{xy} := G^{xy} - \widetilde{G}^{xy}$
3. $[\phi^x, \phi^y] = i\Delta^{xy}$

$i\Delta$ is a skew-symmetric, Hermitian matrix. Its non-zero eigenvalues come in pairs, $\pm\lambda_k$, with eigenvectors $i\Delta u_k = \lambda u_k$, $i\Delta \bar{u}_k = -\lambda \bar{u}_k$.

The conditions on ϕ are satisfied if

$$\phi^x = \sum_k u_k^x a_k + \bar{u}_k^x a_k^\dagger$$

and

$$[a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0, \quad [a_k, a_l^\dagger] = \lambda_k \delta_{kl} \,.$$

Then $a_k|0\rangle = 0 \ \forall k$ gives

$$\langle 0|\phi^x \phi^y|0\rangle = W^{xy} = \sum_k \lambda_k u_k^x \bar{u}_k^y$$

the *positive part* of $i\Delta$.

The other way around (Sorkin: From Green Function to Quantum Field)

1 Retarded Green function G

2 $\Delta^{xy} := G^{xy} - \widetilde{G}^{xy}$

3 Diagonalise $i\Delta$ and W is the positive part

3 $W = \frac{i\Delta + \sqrt{-\Delta^2}}{2}$

3 (a) $W - \overline{W} = i\Delta$; (b) $W\overline{W} = 0$; and (c) $W \geq 0$

Furnishes W, the "Sorkin-Johnston" state directly from $i\Delta$.

From W, can then construct a Fock space representation – indeed this can be done for any Gaussian state W, not just the SJ state.

To calculate SJ state, need G

Given a *massless* Green function on a causal set, one can build the massive one (and vice versa):

$$\begin{aligned} \Box G &= \delta \\ G_m &:= G + m^2 G^2 + m^4 G^3 + m^6 G^4 + \dots \\ (\Box - m^2) G &= \delta \end{aligned}$$

For a causal set, for any G_m^{xy}, this series terminates as there are only finitely many elements in the causal interval between x and y.

Known in special cases

For 2d and 4d Minkowski space, we have natural candidates for the retarded Green functions. Recall in the continuum:

$$G^{xy} = -\frac{1}{2} \quad \text{if} \quad y \prec x \quad (0 \quad \text{otherwise}) \quad d = 2$$

$$G^{xy} = \frac{1}{2}\delta(\tau_{xy}^2) \quad \text{if} \quad y \prec x \quad (0 \quad \text{otherwise}) \quad d = 4$$

Causal set:

$$K^{xy} := -\frac{1}{2} C_{xy} \quad \text{the causal matrix} \quad d = 2$$

$$K^{xy} := \frac{\sqrt{\rho}}{2\pi\sqrt{6}} L^{xy} \quad \text{the link matrix} \quad d = 4$$

deSitter space (Ahmed, FD, Surya)

More causal set Green functions? Yes, in deSitter space.
Note first that R is constant in deSitter space and so any coupling to the Ricci tensor in the e.o.m, $(\Box - \xi R)\phi = 0$, is like a mass term. So if we know G for the massless conformally coupled field, we know it for any mass and any coupling ξ.
In $d = 4$ for conformally flat spacetimes $g = \Omega^2 \eta$ the conformally coupled massless Green function is related to that in $\mathbb{M}^4$ by

$$G^{xy}_{0,\xi_c} = \frac{1}{\Omega^x} \; G^{xy}_{\mathbb{M}} \; \frac{1}{\Omega^y}$$

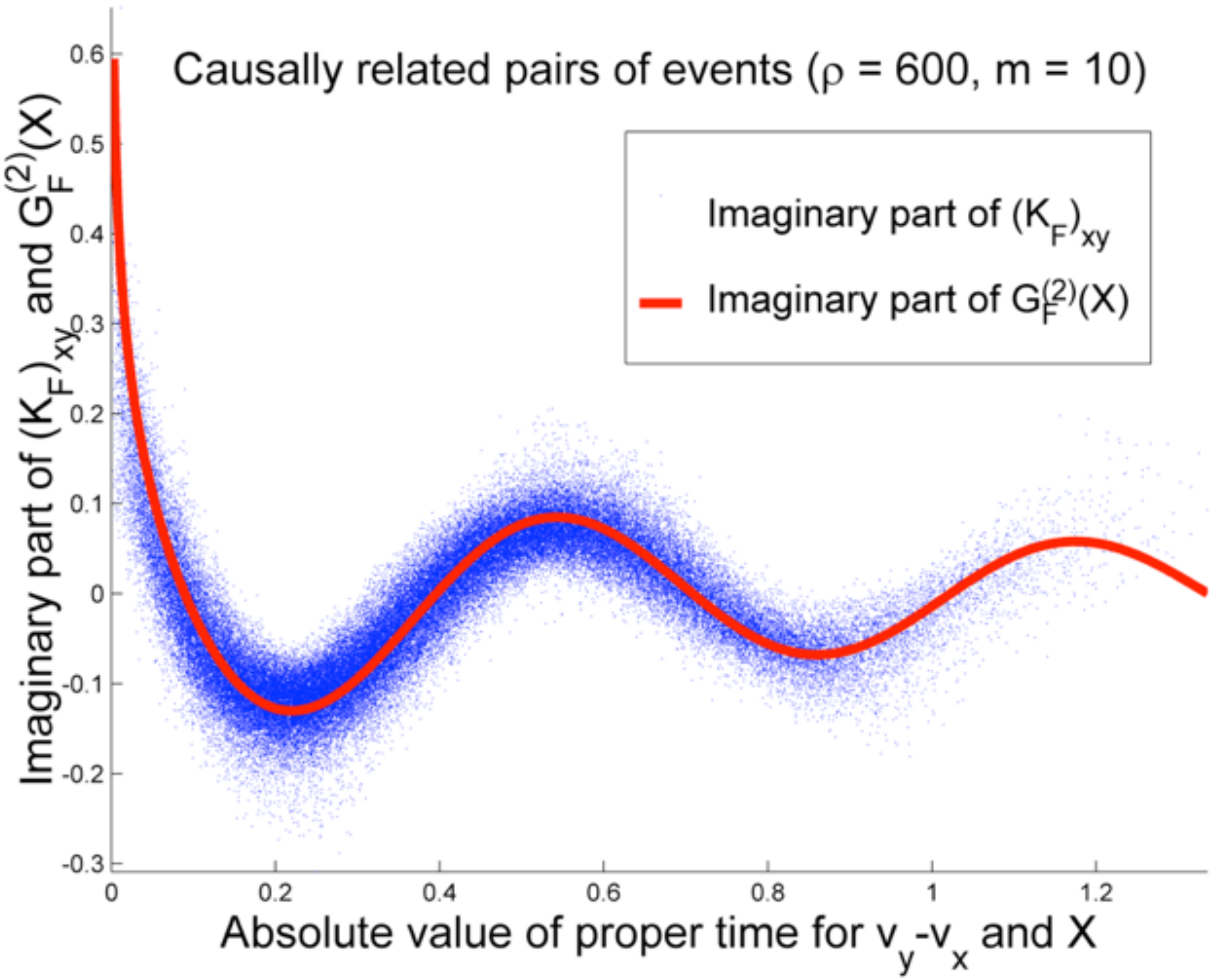
Using this one can show that the same causal set expression as before

$$K^{xy} := \frac{\sqrt{\rho}}{2\pi\sqrt{6}} L^{xy}$$

works: its mean equals G^{xy}_{0,ξ_c} in the limit $\rho \to \infty$.
Also works for globally hyperbolic patch of Anti-deSitter. Any more?

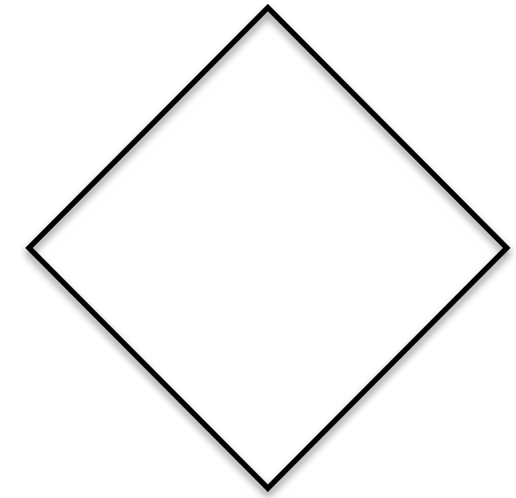
2d massive field (Johnston)

For a causet that is a sprinkling in a diamond in 1+1 Minkowski spacetime, Feynman propagator of free massive scalar field in 2 dims (Johnston)



1+1 diamond,   m=0

(Afshordi, Buck, FD, Rideout, Sorkin, Yazdi)

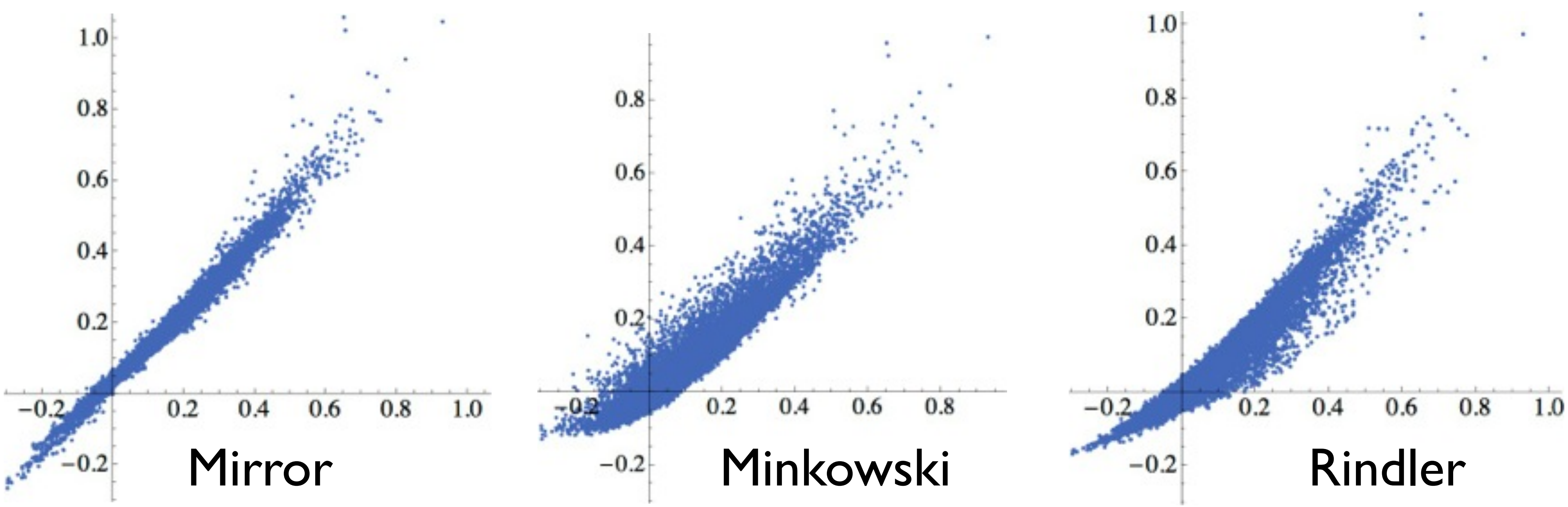


This is a sick theory! Has an IR divergence. Finite region imposes a cutoff.

Conjecture: SJ state is close to Minkowski in middle, Rindler in side corner

Can solve for SJ modes exactly (!) and find W in continuum as well as find W for a sprinkling and in both cases:

Result: SJ state looks like state between two static mirrors at corners



Correlation plots of real(W) for causet (vertical axis) vs known state

Discussion points 1

- Attitude: the distinguished state, the SJ state, is part of the theory. Different from algebraic approach.
- Sorkin reformulated causal set QFT as a path integral actually double path integral or "Schwinger-Keldysh" path integral:

$$
\begin{aligned}
D(\xi,\bar{\xi}) = \langle \delta(\phi^1-\xi^1)\delta(\phi^2-\xi^2)\dots \\
\delta(\phi^N-\xi^N)\delta(\phi^N-\bar{\xi}^N)\dots\delta(\phi^2-\bar{\xi}^2)\delta(\phi^1-\bar{\xi}^1)\rangle \\
= (\text{constant })\exp\{i\xi_-\Box\xi_+ + \dots\}
\end{aligned}
$$

- It is natural to introduce interactions by adding $i\lambda\xi^4 - i\lambda\bar{\xi}^4$ to the exponent above. Calculate n-point functions perturbatively...
- Entropy of spacetime region from W (Sorkin)

$$S = \sum_\lambda \lambda \log|\lambda|$$

where sum is over solutions of

$$Wv = \lambda\, i\Delta v$$

Discussion points 2

- Entanglement entropy of small diamond in larger diamond gives right (CFT) answer in continuum but without further modification, it gives much too big an answer for the causal set (Saravani, Sorkin, Yazdi).
- Yazdi has found a way to "regulate" the causal set entanglement entropy in 1+1. But is the unregulated result telling us that the vacuum is actually highly degenerate?
- The SJ state can be defined in the continuum in a globally hyperbolic spacetime of finite volume.
- When there is a global timelike Killing vector, the SJ state is the usual vacuum state.
- The continuum SJ state in a finite volume region is not Hadamard in general (Fewster & Verch)

Discussion points 3

- Smoothing of the boundary can render the SJ state Hadamard. Brum & Fredenhagen gave a technique based on embedding the region in a larger spacetime. Sorkin has a scheme (for spacelike boundaries) based on replacing

  $$dV(x) \to dV'(x) = r(x)\sqrt{-g(x)}d^4x$$

  where fix a smooth $F(t)$ such that $F(0) = 0$ and $F(\epsilon) = 1$ and let

  $$r(x) = F\left(\frac{V_+(x)V_-(x)}{V_+(x) + V_-(x)}\right)$$

  and $V_+(x)$ ($V_-(x)$) is the volume of the causal future (past) of x.

  Question: will this work for null boundaries too? e.g. causal intervals.
- Other kinds of fields? (Fewster & Rejzner)