Construction of quantum integrable models with bound states

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Joint work with Yoh Tanimoto

Dear Henning,

As my Professor, Advisor, Mentor, Colleague and Collaborator you have been a very important influence during my career and I very much regret that today, I cannot be in York giving a talk in the honor of your 60th birthday and celebrating with you. I had tried everything to be in York, but due to complicated immigration law, I could not leave the country and therefore not participate in this event. I hereby send you my best birthday wishes.

Alles gute zum 60. Geburtstag!

Marcel Bischoff

Construction of models of QFT

A major problem of QFT is the difficulty of constructing interesting examples beyond formal perturbation theory.

- free theories
- $P(\phi)_2$ and ϕ_3^4 models
- conformal field theories in 2 dimensions
- integrable theories in 2 dimensions

Integrable models

- Bosons (no spin, with mass $\mu > 0$) in 1+1 dimensional Minkowski spacetime
- Two-momentum and rapidity:

$$p = p(\theta) = \mu(\cosh\theta, \sinh\theta)$$

- Two-particle scattering allows exchange of phase factor
  - two-particle scattering matrix $S(\theta_1 - \theta_2)$.
- multi-particle scattering matrix – product of two-particle scattering matrices (“factorizing S matrix”).
- The two-particle scattering function S is
  - a meromorphic function in the strip $0 < \operatorname{Im}\theta < \pi$
  - with certain symmetry properties,
  - $S = 1$: free field; $S = -1$: Ising model, other examples: sinh-Gordon model, Bullough-Dodd model.

Task: Given a function S, construct a corresponding quantum field theory.

Direct attempt

Previous attempt ("form factor programme"): Construct the *n*-point function of a local pointlike field $A(x)$.

$$\langle \Omega, A(x)A(0)\Omega \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\theta_1 \dots d\theta_n \, e^{-ix\cdot\sum_{k=1}^{n} p(\theta_k)} |\langle \Omega | A(0) | \theta_1, \dots, \theta_n \rangle_{\text{in}}|^2$$

Problem: Convergence of the series is extremly difficult to control.

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Problem: Convergence of the series is extremly difficult to control.

Here: Try constructing field operators with weaker localization properties first.

Deformed Hilbert space and deformed fields

The theory is constructed as a deformation of a free field:

- Zamolodchikov-Faddeev algebra (elements $z(\theta)$, $z^\dagger(\theta)$):

$$z(\theta_1)z(\theta_2) = S(\theta_1 - \theta_2)\, z(\theta_2)z(\theta_1)\,,$$
$$z^\dagger(\theta_1)z^\dagger(\theta_2) = S(\theta_1 - \theta_2)\, z^\dagger(\theta_2)z^\dagger(\theta_1)\,,$$
$$z(\theta_1)z^\dagger(\theta_2) = S(\theta_2 - \theta_1)\, z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2)\cdot \mathbf{1}.$$

  These act on an "S-symmetric" Fock space.

- Representation of the Poincaré group, including the space-time reflections J.
- Define

$$\phi(x) := \int d\theta \left(e^{ip(\theta)\cdot x} z^\dagger(\theta) + e^{-ip(\theta)\cdot x} z(\theta) \right).$$

- This field is not local:

$$[\phi(x), \phi(y)] \neq 0 \text{ even if } x \text{ spacelike separated from } y.$$

Local observables

- But, with $\phi'(x) := U(j)\phi(-x)U(j)$:

  $$[\phi(x), \phi'(y)] = 0 \text{ if } x \text{ spacelike separated to the left of } y.$$

  This assumes that S is analytic in the "physical strip" $0 < \operatorname{Im} \zeta < \pi$.
- Interpretation: $\phi(x)$ is localized in the wedge region $W_L + x$, and $\phi'(y)$ is localized in the wedge region $W_R - y$.
- Further wedge-local observables by relative locality / associated von Neumann algebras:

  $$\mathcal{A}(W_L + x) = \{\exp i\phi(f) \mid \operatorname{supp} f \subset W_L + x\}''$$

- Observables localized in bounded regions are obtained as intersections of von Neumann algebras

  $$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_L + x) \cap \mathcal{A}(W_R - y) \quad \text{where} \quad \mathcal{O} = W_L + x \cap W_R - y$$

- Result (Lechner 2006): Such observables exist for a large class of S.

Bound states

Now suppose that $S(\theta)$ has poles in the physical strip $0 < \mathrm{Im}\,\theta < \pi$.

- Physically these poles correspond to "bound states", that is the "fusion" of two bosons.
    - Simplification: only one type of particle; two bosons of equal type fuse to form another boson of the same type.
- The momenta of the particles are related by $p(\theta_1) + p(\theta_2) = p(\theta_b)$, where θ_1, θ_2 and θ_b are the (complex) rapidities of the two fusing bosons and of the bound particle, respectively.
- The difference of the rapidities of the fusing bosons is the position of the pole on the rapidity complex plane: $\theta_1 - \theta_2 = i\lambda$ $(0 < \lambda < \pi)$.
    - If the particles have all equal masses, this is fulfilled if and only if $\theta_1 = \theta + \frac{i\pi}{3}$, $\theta_2 = \theta - \frac{i\pi}{3}$ and $\theta_b = \theta$ (that is, $\lambda = \frac{2\pi}{3}$.)

Bound states

- In Lechner's work, the commutator $[\phi'(f), \phi(g)]$ is seen to be zero by shifting an integral contour from $\mathbb{R}$ to $\mathbb{R} + i\pi$.
  - But due to the residue of S at the pole $\frac{2\pi i}{3}$, this is no longer true.
- We need to modify ϕ to get a wedge-local expression.

Wedge-local model with bound states

The properties of the two-particle scattering function are

- Unitarity: $S(-\theta) = \overline{S(\theta)} = S(\theta)^{-1}$.
- Crossing symmetry: $S(i\pi - \theta) = S(\theta)$.
- Bootstrap equation: $S(\theta) = S(\theta + \frac{i\pi}{3})S(\theta - \frac{i\pi}{3})$.

Example for such a function S: Bullough-Dodd model.

$$S(\zeta, B) = f_{\frac{2}{3}}(\zeta) f_{\frac{B}{3}-\frac{2}{3}}(\zeta) f_{-\frac{B}{3}}(\zeta),$$

where

$$f_a(\zeta) := \frac{\tanh \frac{1}{2}(\zeta + i\pi a)}{\tanh \frac{1}{2}(\zeta - i\pi a)}, \quad 0 < B < 1.$$

Wedge-local model with bound states

We introduce, on the S-symmetric Fock space, the "bound state operator".

On the single-particle Hilbert space $\mathcal{H}_1$:

$$\mathrm{Dom}(\chi_1(f)) :=$$

$$\{\xi \in \mathcal{H}_1 : \xi(\theta) \text{ has an } L^2\text{-bounded analytic continuation to } \theta - \frac{i\pi}{3}\},$$

$$(\chi_1(f)\xi)(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \frac{i\pi}{3}\right)\xi\left(\theta - \frac{i\pi}{3}\right),$$

where $R := \mathrm{res}_{\zeta = \frac{2\pi i}{3}} S(\zeta)$. Note: $\chi_1(f)$ realizes the idea that the state of one elementary particle ξ is fused with f^+ into the same species of particle.

On the S-symmetric Fock space: (P projector onto this space)

$$\chi_n(f) := nP_n(\chi_1(f) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})P_n,$$

$$\chi(f) = \bigoplus_{n=0}^{\infty} \chi_n(f).$$

Wedge-local model with bound states

Note: As a consequence of crossing symmetry, the two-particle scattering function has another pole at $\theta' = i\pi - \frac{2i\pi}{3} = \frac{i\pi}{3}$ with residue

$$R' := \text{res}_{\zeta=\frac{i\pi}{3}} S(\zeta).$$

As a consequence of the properties of S, one finds that $R' = -R$ and that R is purely imaginary.

Wedge-local model with bound states

We define a new field

$$\tilde{\phi}(f) = \phi(f) + \chi(f),$$

where

$$\phi(f) = z^{\dagger}(f_{+}) + z(f_{-})$$

$(f_{\pm}(\theta) = \int d\theta\, e^{\pm ip(\theta)\cdot x} f(x))$.
We can introduce the reflected field as $\tilde{\phi}'(g) := J\tilde{\phi}(jg)J$.

Weak wedge commutativity

Consider the following linear space of vectors:
$\Psi \in \mathrm{Dom}(\tilde{\phi}(f)) \cap \mathrm{Dom}(\tilde{\phi}'(g))$ such that
$\prod_j S\left(\theta - \theta_j + \frac{\pi i}{3}\right) \Psi_n(\theta, \theta_1, \cdots, \theta_{n-1})$ and
$\prod_j S\left(\theta - \theta_j + \frac{2\pi i}{3}\right) \Psi_n(\theta, \theta_1, \cdots, \theta_{n-1})$ have $L^2(\mathbb{R}^{n-1})$-valued bounded analytic continuations in θ to $\theta \pm \epsilon i$ for some $\epsilon > 0$.

Theorem

Let f and g be real test functions supported in W_L and W_R, respectively. Then, for each Φ, Ψ in the linear space above, it holds that

$$\langle \tilde{\phi}(f)\Phi, \tilde{\phi}'(g)\Psi \rangle = \langle \tilde{\phi}'(g)\Phi, \tilde{\phi}(f)\Psi \rangle.$$

- Note: It is the commutator of χ with its reflected operator χ' that cancels the contribution of the residues coming from the commutator between ϕ and ϕ', mentioned before.

Outlook

- The fields $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ do not preserve their domains, especially one cannot iterate them on the vacuum more than once.
- The Reeh-Schlieder property is difficult to verify since the domain of the field is not invariant. If we assume the existence of nice self-adjoint extensions, Reeh-Schlieder can be shown.
- The field $\tilde{\phi}(f)$ is a polarization-free generator, but non-temperate.
- To show: $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ have self-adjoint extensions and they strongly commute. (some progress by Y. Tanimoto)
- Apply Haag-Ruelle scattering theory.
- Construction of Haag-Kastler nets: prove the modular nuclearity condition for the associated wedge-local nets and for separations of wedges larger than a minimal distance.

Other models

Other interesting models with poles in the physical strip contain more than one type of particle.

- $Z(N)$ model: has particles labelled $1, \ldots, N-1$, where $N-j$ is the anti-particle of j ($\bar{j} = N - j$) with fusion rules $(jk) = j + k \bmod N$.
- Thirring model: has particles $s, \bar{s}$ ("soliton", "anti-soliton") and a finite number of bound states of s and $\bar{s}$ called b_k ("breathers").

$$
\begin{aligned}
(s, \bar{s}) &= b_k, & (s, b_k) &= s, & (\bar{s}, b_k) &= \bar{s}, \\
(b_k, b_l) &= b_{k+l}, & (b_{k+l}, b_k) &= b_l.
\end{aligned}
$$

Other models

What changes in these models:

- There is a larger single particle space, i.e., there are several types of creators and annihilators: $z_\alpha, z_\alpha^\dagger$.
- The two-particle scattering function is a matrix $S_{\gamma\delta}^{\alpha\beta}(\zeta)$.
- Fusion angles can be more complicated $\theta_{\alpha\beta}^{\gamma} = \theta_{(\alpha\beta)}^{\gamma} + \theta_{(\beta\alpha)}^{\gamma}$.
- Wedge local fields and associated local nets in the case without poles in the physical strip have been worked out by Lechner-Schützenhofer and Alazzawi

$$\phi(f) = z_\alpha(Jf_{-\alpha}) + z_\alpha^\dagger(f_{+\alpha}).$$

- In the case with poles in the physical strip, the proof of wedge-locality requires a new form of the bootstrap equation:

$$S_{\gamma\nu}^{\mu\hat{\gamma}}(\zeta)\eta_{\alpha\beta}^{\gamma} = \eta_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} S_{\alpha k}^{\mu\hat{\alpha}}(\zeta + i\theta_{(\alpha\beta)}^{\gamma}) S_{\beta\nu}^{k\hat{\beta}}(\zeta - i\theta_{(\beta\alpha)}^{\gamma}),$$

where the matrix η is related to the residue of S.

Other models

... and the Yang-Baxter equation:

$$S^{\alpha\beta}_{\beta'\alpha'}(\theta)S^{\alpha'\gamma}_{\gamma'\alpha''}(\theta+\theta')S^{\beta'\gamma'}_{\gamma''\beta''}(\theta') = S^{\beta\gamma}_{\gamma'\beta'}(\theta')S^{\alpha\gamma'}_{\gamma''\alpha'}(\theta+\theta')S^{\alpha'\beta'}_{\beta''\alpha''}(\theta).$$

together with a new χ, acting on $\mathcal{H}_1$ as

$$(\chi_1(f)\xi)_\gamma(\theta) := \sum_{\alpha\beta} \eta^{\gamma}_{\alpha\beta} f^{+}_{\alpha}(\theta + i\theta_{(\alpha\beta)})\xi_\beta(\theta - i\theta_{(\beta\alpha)}).$$

- What we have so far:
  - In the $Z(N)$-Ising model and in the Affine-Toda field theories certain components of the fields $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ weakly commute on a dense domain.
  - We can prove an analogous result in a “deformed” version of sine-Gordon model with CDD factors, if we restrict ourselvies to only two breathers.
  - We are currently investigating the Thirring model.

Summary and outlook

- We have investigated integrable models where the two-particle scattering function has poles in the physical strip.
- We have modified Lechner's definition of wedge-local field by adding an extra term "χ".
- In some models, e.g. Bullough-Dodd, this again yields a wedge-local quantity.
- In other models, e.g. $Z(N)$ and sine-Gordon, we have obtained partial results.
- Operator theoretic properties of $\tilde{\phi}$ are a difficult issue and are under investigation.
- Questions concerning the construction of Haag-Kastler nets (modular nuclearity condition) and the application of Haag-Ruelle scattering theory for scalar S-matrices are work in progress.