Ground state for a massive scalar field in BTZ spacetime with Robin boundary conditions

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joint work with
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Bañados,Teitelboim, Zanelli black hole

It is a general, stationary, axisymmetric $(2+1)$ dimensional solution of the vacuum Einstein field equations with a negative cosmological constant $\Lambda = -1/\ell^2$.

$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 \big(d\phi + N^\phi(r) dt\big)^2$$

$$N(r)^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \qquad N^\phi(r) = -\frac{J}{2r^2} \qquad \mathrm{R} = -\frac{6}{\ell^2}$$

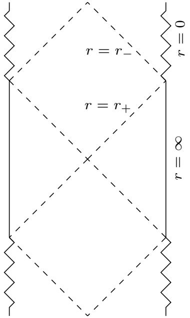
Bañados,Teitelboim, Zanelli black hole

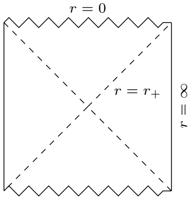
$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 \left(d\phi + N^\phi(r) dt\right)^2$$

- As a manifold it is diffeomorphic to $\mathbb{R} \times I \times \mathbb{S}^1$, $I \subset \mathbb{R}$ open interval

- For $M > 0$, $|J| \leq M\ell$ it has an outer and inner horizon $r = r_+, r_-$

$$r_\pm^2 = \frac{\ell^2}{2} \left(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}} \right)$$





- Two Killing vectors: ∂_t and ∂_ϕ

Bañados,Teitelboim, Zanelli black hole

$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 \left(d\phi + N^\phi(r) dt\right)^2$$

- $r = r_+$ is a Killing horizon for the Killing vector

$$\chi \doteq \partial_t - N^\phi(r_+)\partial_\phi = \partial_t + \Omega_{\mathcal{H}}\partial_\phi$$

- $\Omega_{\mathcal{H}}$ is the angular velocity of the horizon

- The Killing vector χ
  - timelike in the exterior region (r_+, ∞)
  - direction to foliate the spacetime in spacelike hypersurfaces

BTZ from AdS

$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 \big(d\phi + N^\phi(r) dt\big)^2$$

$$N(r)^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \qquad N^\phi(r) = -\frac{J}{2r^2}$$

- For $M = -1$, $J = 0$, one recovers the anti-de Sitter spacetime

$$ds^2 = -[1 + (r/\ell)^2] dt^2 + [1 + (r/\ell)^2]^{-1} dr^2 + r^2 d\phi^2 \ .$$

- BTZ can be obtained by an identification of boundaries of AdS$_3$, hence locally it is a region of constant curvature
- The BTZ black hole is locally isometric to AdS$_3$

Scalar field

- Real massive scalar field $P\Phi = (\Box_g - m^2 - \xi R)\Phi = 0$
- Dimensionless parameter $\mu^2 \doteq m^2\ell^2 + \xi R\ell^2$
- m and ξ st the Breitenlohner-Freedman bound holds: $\mu^2 \geqslant -1$

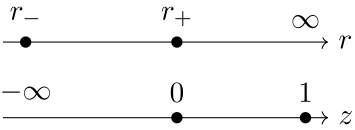
! **non** globally hyperbolic spacetime
(inital data + Boundary conditions)

- Using coordinates (t, r, ϕ)
- ∂_t and ∂_ϕ are Killing fields of the metric
- Fourier expansion of Φ

$$\Phi(t, r, \phi) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int d\omega \; e^{-i\omega t + ik\phi} \, \Psi_{\omega k}(r)$$

Change of radial variable

We focus on the exterior region $r_+ < r < \infty$. Boundary conditions will apply. We change variable as $z = \frac{r^2 - r_+^2}{r^2 - r_-^2}$



The radial mode of the field obeys

$$\left[z(1-z)\partial_z^2 + (1-z)\partial_z + \left(\frac{\ell^2(\omega r_+ - k r_-)^2}{4(r_+^2 - r_-^2)^2 z} - \frac{\ell^2(\omega r_- - k r_+)^2}{4(r_+^2 - r_-^2)^2} - \frac{\mu^2}{4(1-z)} \right) \right] \Psi_{\omega k}(z) = 0$$

Hypergeometric equation

With the ansatz $\Psi_{\omega\kappa}(z) = z^{\alpha}(1-z)^{\beta}F_{\omega\kappa}(z)$

$$\alpha = -i\frac{\ell(\omega - \Omega_H \kappa)r_+}{2(r_+^2 - r_-^2)} \,, \quad \beta = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right)$$

we obtain an hypergeometric equation

$$z(1-z)\partial_z^2 F_{\omega k} + [c - (a+b+1)z]\partial_z F_{\omega k} - abF_{\omega k} = 0$$

with

$$\begin{cases} a = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} + i\ell\,\frac{\omega - k}{r_+ - r_-}\right) \\ b = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} + i\ell\,\frac{\omega + k}{r_+ + r_-}\right) \\ c = 1 - i\frac{\ell(\omega - \Omega_H \kappa)r_+}{r_+^2 - r_-^2} \end{cases}$$

Hypergeometric equation - comment 1

With the ansatz $\Psi_{\omega\kappa}(z) = z^{\alpha}(1-z)^{\beta}F_{\omega\kappa}(z)$

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Hypergeometric equation - comment 2

With the ansatz $\Psi_{\omega\kappa}(z) = z^{\alpha}(1-z)^{\beta}F_{\omega\kappa}(z)$

$$\alpha = -i\frac{\ell(\omega - \Omega_H\kappa)r_+}{2(r_+^2 - r_-^2)}\ , \qquad \beta = \frac{1}{2}\left(1+\sqrt{1+\mu^2}\right)$$

we obtain an hypergeometric equation

$$z(1-z)\partial_z^2 F_{\omega k} + [c-(a+b+1)z]\partial_z F_{\omega k} - abF_{\omega k} = 0$$

with

$$\begin{cases} a = \frac{1}{2}\left(1+\sqrt{1+\mu^2} + i\ell\,\frac{(\omega-\Omega_H\kappa)-(1-\Omega_{\mathcal{H}})k}{r_+ - r_-}\right) \\ b = \frac{1}{2}\left(1+\sqrt{1+\mu^2} + i\ell\,\frac{(\omega-\Omega_H\kappa)+(1+\Omega_{\mathcal{H}})k}{r_+ + r_-}\right) \\ c = 1 - i\frac{\ell(\omega-\Omega_H\kappa)r_+}{r_+^2 - r_-^2} \end{cases}$$

Hypergeometric equation - comment 2

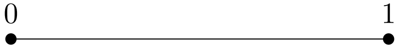
With the ansatz $\Psi_{\omega\kappa}(z) = z^{\alpha}(1-z)^{\beta} F_{\omega\kappa}(z)$

$$\alpha = -i\frac{\ell(\omega - \Omega_H \kappa) r_+}{2(r_+^2 - r_-^2)}\ , \quad \beta = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right)$$

- What is ω? $\mathcal{L}_{\partial_t}\Phi = -i\omega\Phi \qquad \Phi \propto e^{i\omega t} e^{i\kappa\phi}\Psi_{\omega\kappa}(z)$
- $\tilde{\omega} := \omega - \Omega_H \kappa$
- What is $\tilde{\omega}$? $\mathcal{L}_{\chi}\Phi = -i\tilde{\omega}\Phi$
- $\chi = \partial_t + \Omega_{\mathcal{H}}\partial_\phi$ is the Killing vector defining the event horizon

Hypergeometric solutions

Solutions: Gaussian hypergeometric functions $F(p, q, s; z)$



- For $\mu^2 \geqslant -1$ and $\mu^2 \neq (n-1)^2 - 1,\ n = 1, 2, 3, \dots$

$$\Psi_1(z) = z^\alpha (1-z)^\beta F(a, b, a+b-c+1; 1-z)$$
$$\Psi_2(z) = z^\alpha (1-z)^{1-\beta} F(c-a, c-b, c-a-b+1; 1-z)$$

- For $\mu^2 = (n-1)^2 - 1,\ n = 2, 3, \dots$

Hypergeometric solutions

Solutions in the general case $\mu^2 \geqslant -1$ and $\mu^2 \neq (n-1)^2 - 1$

$$\Psi_1(z) = z^\alpha (1-z)^\beta F(a, b, a+b-c+1; 1-z)$$
$$\Psi_2(z) = z^\alpha (1-z)^{1-\beta} F(c-a, c-b, c-a-b+1; 1-z)$$

Principal solution at $z = 1$: Ψ_1
The unique solution st $\lim_{z\to 1} \Psi_1(z)/\Psi(z) = 0$ for every other Ψ

- For $-1 \leqslant \mu^2 < 0$, both solutions are $L^2((z_0, 1), d\nu)$
- For $\mu^2 \geqslant 0$ only Ψ_1 is $L^2((z_0, 1), d\nu)$

With $d\nu = \sqrt{|g|}\, g^{tt} dr d\phi$ over a spacelike hypersurface Σ_t of constant t

Boundary conditions

- **Aim** - Identify *all the possible boundary conditions* that can be applied at the boundary $z = 1$, aka $r = \infty$.

- Imposing the physical principle of zero energy flux at infinity is equivalent to impose **Robin boundary conditions** to the scalar field at infinity.

Boundary conditions

Regular ODE at the boundary

- Construct a convenient basis of fundamental solution $\{\varphi_1, \varphi_2\}$
- Identify the *principal solution* at the boundary, let it be φ_1
- Write a general solution in the form

$$\Psi(z) = \mathcal{N}[\cos(\zeta)\varphi_1(z) + \sin(\zeta)\varphi_2(z)] \quad \zeta \in [0, \pi)$$

- The most general homogeneous boundary condition is then a *Robin boundary condition* in the form

$$\cos(\zeta)\Psi(1) + \sin(\zeta)\Psi'(1) = 0$$

Boundary conditions

$$\Psi(z) = \mathcal{N}[\cos(\zeta)\varphi_1(z) + \sin(\zeta)\varphi_2(z)]$$
$$\cos(\zeta)\Psi(1) + \sin(\zeta)\Psi'(1) = 0$$

- The case which selects the principal solution φ_1, namely $\zeta = 0$, corresponds to the *Dirichlet boundary condition*, $\Psi(1) = 0$
- The case $\zeta = \frac{\pi}{2}$ corresponds to the *Neumann boundary condition*

Boundary conditions

Singular ODE at the boundary

- Basis of fundamental solution $\{\Psi_1, \Psi_2\}$
- Identify the *principal solution* Ψ_1
- Define $\mathcal{W}_z[u,v] \doteq u(z)v'(z) - v(z)u'(z)$
- For a solution $\Psi_{\omega k}$, a Robin boundary condition at $z = 1$ is

$$\lim_{z\to 1} \{\cos(\zeta)\mathcal{W}_z[\Psi_{\omega k}, \Psi_1](z) + \sin(\zeta)\mathcal{W}_z[\Psi_{\omega k}, \Psi_2](z)\} = 0\,, \zeta \in [0,\pi)$$

- The solution is given by

$$\Psi_{\omega k}(z) = \mathcal{N}_{\omega k}\left[\cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(z)\right]$$

Boundary conditions

Natural generalization of the standard Robin boundary conditon

$$\Psi_{\omega k}(z) = \mathcal{N}_{\omega k}\left[\cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(z)\right]$$

$$\lim_{z\to 1}\left\{\cos(\zeta)\mathcal{W}_z[\Psi_{\omega k}, \Psi_1](z) + \sin(\zeta)\mathcal{W}_z[\Psi_{\omega k}, \Psi_2](z)\right\} = 0$$

- ▷ $\zeta = 0$ corresponds to the standard *Dirichlet boundary conditions*
- ! A Robin boundary condition at $z = 1$ leads to a well-posed problem if Ψ_1 and Ψ_2 are L^2 near $z = 1$ with $d\nu = \sqrt{|g|}\, g^{tt} dr d\phi$

A brief recap

- Up to now everything is classical
  - Scalar field on spacetime with symmetries
  - Mode expansion
  - Radial mode solutions
- We aim to quantize the system
- We want to build the Two Point Function
- Possibly a Hadamard ground state
- Green operators and CCR

Two Point Function and causal propagator

Bidistribution $\Lambda_2 \in \mathcal{D}'(M \times M)$ such that

$$(P \otimes \mathbb{I})\Lambda_2 = (\mathbb{I} \otimes P)\Lambda_2 = 0 \quad \textbf{(equations of motion)}$$

We assume that Λ_2 admits a mode expansion

$$\Lambda_2(x, x') = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_0^\infty d\tilde{\omega} \sum_{k \in \mathbb{Z}} e^{-i(\tilde{\omega}(t-t') - k(\phi - \phi') - i\epsilon)} \widehat{\Lambda}_{\tilde{\omega}k}(z, z'),$$

The CCR impose that the antisymmetric part of Λ_2 is proportional to the causal propagator: $iE(x, x') = \Lambda_2(x, x') - \Lambda_2(x', x)$

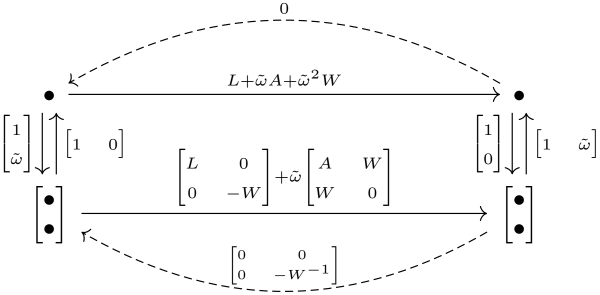
$$\frac{1}{2\pi} \int_0^\infty d\tilde{\omega}\ \tilde{\omega}\ \widehat{\Lambda}_{\tilde{\omega}k}(z, z') = \frac{\delta(z - z')}{J(z)}$$

Quadratic operator pencil

No standard eigenvalue problem: How to reconstruct the delta?

$$(L + \tilde{\omega} A + \tilde{\omega}^2 W)\Psi = 0$$

$$\begin{bmatrix} L & 0 \\ 0 & -W \end{bmatrix} Z = -\tilde{\omega} \begin{bmatrix} A & W \\ W & 0 \end{bmatrix} Z \,, \quad Z = \begin{pmatrix} \Psi \\ \tilde{\omega}\Psi \end{pmatrix}$$



1. Appendices of the present work
2. I. Khavkine https://arxiv.org/pdf/1711.00585.pdf

Resolution of identity

$$\delta(z - z') = \oint_{\gamma_\infty} d\tilde{\omega}\, \tilde{\omega}\, \mathcal{G}_{\tilde{\omega}}(z, z') J(z)$$

- Where

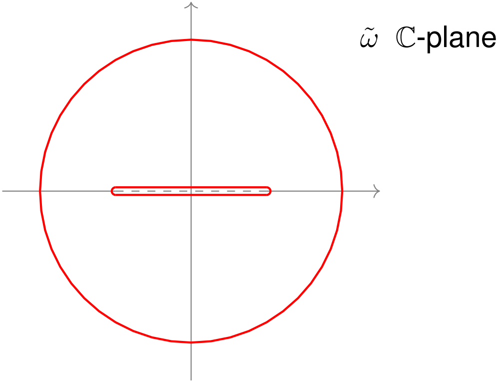
$$(L \otimes \mathbb{1})\mathcal{G}_{\tilde{\omega}} = (\mathbb{1} \otimes L)\mathcal{G}_{\tilde{\omega}} = \delta(z - z')$$

$$\mathcal{G}_{\tilde{\omega}}(z, z') = C_{\tilde{\omega}}[\theta(z - z')u_{\tilde{\omega}}(z')v_{\tilde{\omega}}(z) + \theta(z' - z)u_{\tilde{\omega}}(z)v_{\tilde{\omega}}(z')]$$

- We need $u_{\tilde{\omega}}(z) \in L^2$ near the horizon $z = 0$
- $u_{\tilde{\omega}}(z) \propto z^\alpha = z^{-i\frac{\ell\tilde{\omega}r_+}{2(r_+^2 - r_-^2)}}$
- Branch choice $u_{\tilde{\omega}}(z) = \begin{cases} u^+_{\tilde{\omega}}(z), & \mathrm{Im}\,\widetilde{\omega} > 0 \\ u^-_{\tilde{\omega}}(z), & \mathrm{Im}\,\widetilde{\omega} < 0 \end{cases}$
- Arbitrariness in the definition of $\tilde{\omega} = +\sqrt{\tilde{\omega}^2}$ ...or ...$\tilde{\omega} = -\sqrt{\tilde{\omega}^2}$

Resolution of identity

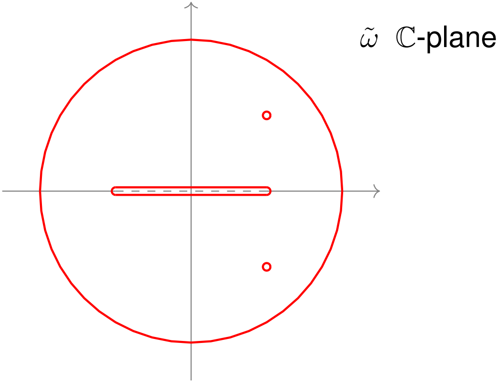
$$\delta(z - z') = \oint d\tilde{\omega}\ \tilde{\omega}\ C_{\tilde{\omega}} u_{\tilde{\omega}}(z_<) v_{\tilde{\omega}}(z_>) J(z)$$



$$u_{\tilde{\omega}}(z) = \begin{cases} u^+_{\tilde{\omega}}(z), & \mathrm{Im}\,\widetilde{\omega} > 0 \\ u^-_{\tilde{\omega}}(z), & \mathrm{Im}\,\widetilde{\omega} < 0 \end{cases}$$

Resolution of identity

$$\delta(z-z') = \oint d\tilde{\omega}\ \tilde{\omega}\ C_{\tilde{\omega}} u_{\tilde{\omega}}(z_<) v_{\tilde{\omega}}(z_>) J(z) + \sum \text{Res}$$



$$u_{\tilde{\omega}}(z) = \begin{cases} u^{+}_{\tilde{\omega}}(z), & \operatorname{Im}\widetilde{\omega} > 0 \\ u^{-}_{\tilde{\omega}}(z), & \operatorname{Im}\widetilde{\omega} < 0 \end{cases}$$

Ground state $\mu^2 \in (-1, 0)$

- ▷ Boundary condition at $z = 1$
- ▷ A different TPF for each Robin boundary condition

1) $\zeta \in [0, \zeta_*)$, $\zeta_* > \frac{\pi}{2}$: No additional poles, just the branch cut on the $\tilde{\omega}$ real axis

$$\Lambda_2^{\zeta}(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} \int_0^\infty \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(\tilde{t} - \tilde{t}' - i\epsilon)} \frac{(A\overline{B} - \overline{A}B)\, C}{|\cos(\zeta)B - \sin(\zeta)A|^2} \Psi_\zeta(z)\Psi_\zeta(z')$$

- ◦ ground state built only out of positive $\widetilde{\omega}$-frequencies
- ◦ using the results of Sahlmann - Verch (2000) it is Hadamard

Bound states $\mu^2 \in (-1, 0)$

- Boundary condition at $z = 1$
- A different TPF for each Robin boundary condition

2) $\zeta \in [\zeta_*, \pi)$, $\zeta_* > \frac{\pi}{2}$: One additional pole

$$\Lambda_2^\zeta(x, x') = \lim_{\epsilon \to 0+} \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} \int_0^\infty \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(\tilde{t} - \tilde{t}' - i\epsilon)} \frac{(A\overline{B} - \overline{A}B)\, C}{|\cos(\zeta)B - \sin(\zeta)A|^2} \Psi_\zeta(z)\Psi_\zeta(z')$$

$$+ i \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} \left(e^{-i\tilde{\omega}_\zeta(\tilde{t} - \tilde{t}')} + e^{-i\overline{\tilde{\omega}_\zeta}(\tilde{t} - \tilde{t}')} \right) \Re\big[CD(\tilde{\omega})\Psi_\zeta(z)\Psi_\zeta(z')\big]\big|_{\tilde{\omega} = \tilde{\omega}_\zeta}$$

- No ground state
- Hadamard?

To summarize

- Scalar field on rotating BTZ
- Symmetries and mode decomposition
- Solutions for different mass values (with or without boundary conditions)
- All possible Robin boundary conditions
- Construction of TPF in all mass ranges and for all boundary conditions
- Two regimes of boundary conditions:
  - Ground state and Hadamard
  - Bound states

Ongoing

Characterization of Hawking radiation for the scalar field in BTZ

- 3D generalization of the Moretti-Pinamonti's approach
- local computation
- scaling limit towards the Killing horizon
- thermal nature of the quantum correlation functions