The universal C*-algebra of the electromagnetic field. Topological charges and non-linear fields

Detlev Buchholz, Fabio Ciolli, Guiseppe Ruzzi, Ezio Vasselli

Dedicated to Karl-Henning Rehren

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Motivation

Views of some philosophers on $\text{L}_{\text{agrangean}}$QFT versus $\text{A}_{\text{algebraic}}$QFT:

E. MacKinnon [Philosophy of Science 2005]:
In . . . AQFT the algebra of observables contains the physical content. This has not been extended to include local gauge theory.

D. Wallace [Studies in History and Philosophy of Science 2010]:
. . . to be lured away from the Standard Model by AQFT is sheer madness.

D.J. Baker [Philosophy of Science Archive 2015]:
Wallace is correct that, to some . . . questions (e.g., is there a photon field that interacts with an electron field?), LQFT provides the obvious right answer and present-day AQFT cannot.

Their conclusion: LQFT $\bigcap$ AQFT $= \emptyset$.

Motivation

Message of this talk:

AQFT description of the electromagnetic field exists.
Its relation to LQFT is well understood. Moreover, non-LQFT type representations of the field are also possible.

Outline

Electromagnetic field

Notation (Minkowski space $\mathbb{R}^4$):

- $\mathcal{D}_r(\mathbb{R}^4)$ real test functions with values in (skew) tensors of rank r
- $d : \mathcal{D}_r(\mathbb{R}^4) \to \mathcal{D}_{r+1}(\mathbb{R}^4)$ exterior derivative ("curl")
- $\delta : \mathcal{D}_r(\mathbb{R}^4) \to \mathcal{D}_{r-1}(\mathbb{R}^4)$ co-derivativ $\delta = - \star d \star$ ("divergence")

Electromagnetic field $F : \mathcal{D}_2(\mathbb{R}^4) \to \mathfrak{P}$ (local, covariant)

- $F(\delta h) = 0, \; h \in \mathcal{D}_3(\mathbb{R}^4)$ (homogeneous Maxwell equation)
- $j(f) \doteq F(df), \; f \in \mathcal{D}_1(\mathbb{R}^4)$ (inhomogeneous Maxwell equation)

Intrinsic vector potential $A : \mathcal{C}_1(\mathbb{R}^4) \doteq \delta \, \mathcal{D}_2(\mathbb{R}^4) \to \mathfrak{P}$

- $A(\delta g) \doteq F(g), \; g \in \mathcal{D}_2(\mathbb{R}^4)$ (homogeneous Maxwell equation ✓)
- $A(\delta df) = j(f), \; f \in \mathcal{D}_1(\mathbb{R}^4)$ (inhomogeneous Maxwell equation ✓)

Note: Localization region of $A(\delta g)$ determined by $\operatorname{supp} \delta g$. Is A local?

Intrinsic vector potential A is restrictedly local

Local Poincaré Lemma

$\mathcal{O}$ ***double cone:*** $0 \to \cdots \mathcal{D}_{r+1}(\mathcal{O}) \xrightarrow{\delta} \mathcal{D}_r(\mathcal{O}) \xrightarrow{\delta} \mathcal{D}_{r-1}(\mathcal{O}) \cdots \to 0$ *exact*

Concretely: If $\operatorname{supp} \delta g \subset \mathcal{O}$, there is a g' with $\delta g' = \delta g$ and $\operatorname{supp} g' \subset \mathcal{O}$.

Convenient to proceed to unitaries $V(\mathrm{a}, c) \mathrel{\widehat{=}} e^{i\mathrm{a}A(c)}$, $\mathrm{a} \in \mathbb{R}$, $c \in \mathcal{C}_1(\mathbb{R}^4)$

Notation: $c \in \mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$ *co-closed*

Universal algebra:

$\mathfrak{G}_0$: unitary group generated by $\{ V(\mathrm{a}, c) \}$, relations

$$V(\mathrm{a}_1, c) V(\mathrm{a}_2, c) = V(\mathrm{a}_1 + \mathrm{a}_2, c), \quad V(\mathrm{a}, c)^* = V(-\mathrm{a}, c), \quad V(0, c) = 1$$

$$V(\mathrm{a}_1, c_1) V(\mathrm{a}_2, c_2) = V(1, \mathrm{a}_1 c_1 + \mathrm{a}_2 c_2) \quad \text{if } \operatorname{supp} c_1 \times \operatorname{supp} c_2$$

$$\Rightarrow \quad \lfloor V(\mathrm{a}, c), \lfloor V(\mathrm{a}_1, c_1), V(\mathrm{a}_2, c_2) \rfloor \rfloor = 1 \text{ for any } c \text{ if } \operatorname{supp} c_1 \perp \operatorname{supp} c_2$$

$\mathfrak{V}_0$: complex linear span of the elements of $\mathfrak{G}_0$ (*-algebra); stable under action α_P, $P \in \mathcal{P}_+^\uparrow$ of Poincaré group, $\alpha_P(V(\mathrm{a}, c)) = V(\mathrm{a}, c_P)$.

Lemma

Let ω be the functional on $\mathfrak{V}_0$ fixed by linear extension from

$$\omega(V) = \begin{cases} 0 & \text{for } V \in \mathfrak{G}_0 \backslash \{1\} \\ 1 & \text{for } V = 1\,. \end{cases}$$

It defines a faithful state on $\mathfrak{V}_0$ with GNS representation $(\pi, \mathcal{H})$.

Definition: C*-norm on $\mathfrak{V}_0$

$$\|A\| \doteq \sup_{\pi,\mathcal{H}} \|\pi(A)\|_{\mathcal{H}}\,, \quad A \in \mathfrak{V}_0\,.$$

Completion: $\mathfrak{V}$ (universal C*-algebra of electromagnetic field)

It brings about a local, Poincaré covariant net

$$\mathcal{O} \mapsto \mathfrak{V}(\mathcal{O}) \doteq \text{C*-algebra}\,\{V(\mathrm{a}, c) : \mathrm{a} \in \mathbb{R},\ c \in \mathcal{C}_1(\mathcal{O})\} \subset \mathfrak{V}$$

But: $\mathfrak{V}$ **not** a primitive algebra (not a theory).

Vacuum states

Strategy: Take quotient of $\mathfrak{V}$ with regard to *suitable ideals*.

Definition: A pure state $\omega \in \mathfrak{V}^*$ describes a *vacuum state* if it is invariant under Poincaré transformations and a ground state.

Fact

Let ω be a vacuum state on $\mathfrak{V}$ with GNS representation $(\pi, \mathcal{H}, \Omega)$. There exists a continuous unitary representation U_π of $\mathcal{P}_+^\uparrow$ such that

$$U_\pi(P)\pi(V)U_\pi(P)^{-1} = \pi \circ \alpha_P(V)\,, \quad P \in \mathcal{P}_+^\uparrow,\ V \in \mathfrak{V}\,.$$

Note: $\ker \pi$ is a Poincaré invariant ideal of $\mathfrak{V}$. Thus $\mathfrak{V}/\ker \pi$ defines a net satisfying all Haag-Kastler axioms (a theory).

Remark: Vacuum states on $\mathfrak{V}$ are fixed by the generating functionals

$$\mathcal{C}_1(\mathbb{R}^4) \ni c \mapsto \omega(V(1, c))\,.$$

Vacuum states

Definition: A state $\omega \in \mathfrak{V}^*$ is said to be regular if the functions

$$\mathrm{a}_1, \dots, \mathrm{a}_n \mapsto \omega(V(\mathrm{a}_1, c_1) \cdots V(\mathrm{a}_n, c_n))$$

are smooth for any $c_1, \dots, c_n \in \mathcal{C}_1(\mathbb{R}^4)$ and $n \in \mathbb{N}$.

Fact: Let $(\pi, \mathcal{H}, \Omega)$ be the GNS representation induced by a regular ω.

- There exist selfadjoint operators $A_\pi(c)$ with common core $\mathcal{D} \subset \mathcal{H}$ such that
$$\pi(V(\mathrm{a}, c)) = e^{i\mathrm{a}\, A_\pi(c)}, \quad c \in \mathcal{C}_1(\mathbb{R}^4).$$
- The operators A_π are *spacelike linear*, *i.e.* one has on $\mathcal{D}$

$$\mathrm{a}_1 A_\pi(c_1) + \mathrm{a}_2 A_\pi(c_2) = A_\pi(\mathrm{a}_1 c_1 + \mathrm{a}_2 c_2) \quad \text{if } \operatorname{supp} c_1 \times \operatorname{supp} c_2 .$$

Definition: A state $\omega \in \mathfrak{V}^*$ is of type Linear if it is regular and the resulting potential A_π is fully linear on $\mathcal{C}_1(\mathbb{R}^4)$.

Familiar examples

Determination of vacuum states $\omega \in \mathfrak{V}^*$ of type L

(1) *Vanishing current:* (recall $A_\pi(\delta df) = j_\pi(f) \overset{!}{=} 0, \; f \in \mathcal{D}_1(\mathbb{R}^4)$)

Lemma

Let ω be a vacuum state of type L with vanishing current. Then

$$c \mapsto \omega(V(1, c)) = e^{-const\,\langle c,c\rangle}\,, \quad c \in \mathcal{C}_1(\mathbb{R}^4),$$

(free electromagnetic field in Fock space representation, const ≥ 0).

Recall: $\langle c, c\rangle = -(2\pi)^{-3} \int dp\, \theta(p_0)\delta(p^2)\, \hat{c}(-p) \cdot \hat{c}(p)$

Note: Apart from the numerical value of Planck's constant, the state is algebraically fixed without any Lagrangean.

(2) *Central (external) currents:* $\left(A_\pi(\delta df) = j_\pi(f) \overset{!}{\in} \mathbb{C}1,\ f \in \mathcal{D}_1(\mathbb{R}^4)\right)$

Lemma

Let j_π be extendable to $\Box^{-1}\mathcal{D}_1(\mathbb{R}^4) \supset \mathcal{D}_1(\mathbb{R}^4)$. Then

$$\gamma(V(1,c)) \doteq e^{\,i\,j_\pi(\Box^{-1}c)}\, V(1,c)\,, \quad c \in \mathcal{C}_1(\mathbb{R}^4)$$

defines an automorphism of $\mathfrak{V}$.

Note: γ changes currents by adding to them the external current j_π.

Corollary

Let ω_0 be a vacuum state on $\mathfrak{V}$. Then

$$\omega(V(1,c)) \doteq e^{ij_\pi(\Box^{-1}c)}\, \omega_0(V(1,c))\,, \quad c \in \mathcal{C}_1(\mathbb{R}^4)$$

describes the state with (additional) external current j_π.

Familiar examples

(3) *Quantum currents (QED, electroweak sector, quarks, SUSY):*

[O. Steinmann] Renormalized perturbation theory yields formal power series for generating functions

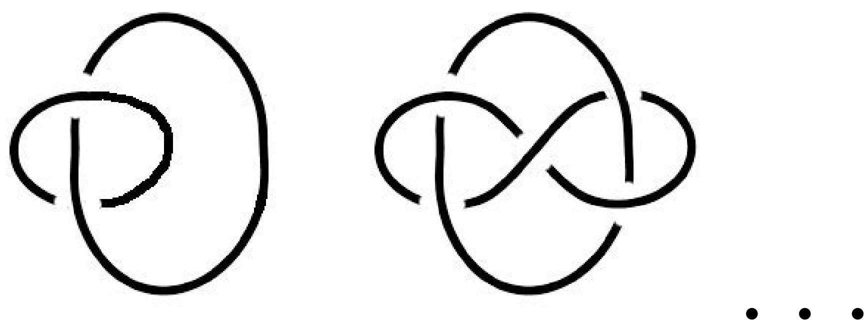
$$f \mapsto \omega(V(1, c)), \quad c \in \mathcal{C}_1(\mathbb{R}^4).$$

Existence of corresponding states $\omega \in \mathfrak{V}^*$ of type L: open problem.

But: Regular vacuum states $\omega \in \mathfrak{V}^*$ (not of type L) exist for given current.

Topological charges

Let $c_1, c_2 \in \mathcal{C}_1(\mathbb{R}^4)$ have their supports in linked spacelike loops, *e.g.*



Then:

$$\lfloor V(\mathrm{a}_1, f_1), V(\mathrm{a}_2, f_2) \rfloor \quad \text{element of the center of } \mathfrak{V}\,.$$

Question: Do there exist **regular** states $\omega \in \mathfrak{V}$ for which

$$\pi(\lfloor V(\mathrm{a}_1, c_1), V(\mathrm{a}_2, c_2) \rfloor) \neq 1 \quad \textit{i.e.} \quad [A_\pi(c_1), A_\pi(c_2)] \neq 0$$

(states carrying a ”topological charge”)?

Lemma

Let $\omega \in \mathfrak{V}$ be a regular state with GNS representation $(\pi, \mathcal{H}, \Omega)$. If the underlying potential A_π is ***fully linear*** *on $\mathcal{C}_1(\mathbb{R}^4)$, then*

$$[A_\pi(c_1), A_\pi(c_2)] = 0$$

for c_1, c_2 having support in spacelike separated linked loops $\mathcal{L}_1, \mathcal{L}_2$.

Conclusion: Standard “Wightman type” treatment of gauge fields excludes from the outset topological charges based on linked loops.

Elements of proof:

- Given c, exhibit special loop functions $l \simeq c$ in the same co-cohomology class relative to localization region $\mathcal{L}$ of c
- **Linearity** of A_π and a "Causal Poincaré Lemma" imply

$$[A_\pi(c_1), A_\pi(c_2)] = [A_\pi(l_1), A_\pi(l_2)]$$

- By deformation arguments one gets

$$[A_\pi(l_1), A_\pi(l_2)] = [A_\pi(\widehat{l_1}), A_\pi(\widehat{l_2})]$$

  where $\widehat{l_1}, \widehat{l_2}$ localized in loops $\widehat{\mathcal{L}_1}, \widehat{\mathcal{L}_2} \subset \mathbb{R}^3$
- Homology of simple loops in $\mathbb{R}^3 \backslash \widehat{\mathcal{L}}$ and deformation argument yields

$$[A_\pi(\widehat{l_1}), A_\pi(\widehat{l_2})] = [A_\pi(\widehat{l_2}), A_\pi(\widehat{l_1})] = -[A_\pi(\widehat{l_1}), A_\pi(\widehat{l_2})] = 0\,. \qquad \blacksquare$$

(4) *Regular vacuum state with non-trivial topological charge*

Starting point: Vacuum state ω_0 of type L with *vanishing current* and GNS representation $(\pi_0, \mathcal{H}_0, \Omega_0)$.

Note: Potential A_0 linear on $\mathcal{C}_1(\mathbb{R}^4)$, hence topological charge vanishes.

Lemma

There exists a regular vacuum state ω_T on $\mathfrak{V}$ with GNS representation $(\pi_T, \mathcal{H}_0, \Omega_0)$ which carries a non-trivial topological charge.

Remark: The corresponding (spacelike linear) potential A_T transforms covariantly under the original unitary representation U_0 of $\mathcal{P}_+^\uparrow$.

Elements of proof:

- Let $c \in \mathcal{C}_1(\mathbb{R}^4)$ and let $g_c \in \mathcal{D}_2(\mathbb{R}^4)$ be any co-primitive, *i.e.* $\delta g_c = c$. Then $A_0(\delta g_c)$, $A_0(\delta \star g_c)$ and $\overline{g_c} \doteq \int dx\, g_c(x)$ are independent of the choice of g_c.
- Split $c \in \mathcal{C}_1(\mathbb{R}^4)$ into a sum of functions with disjoint connected supports, $c = \sum_n c_n$, and define

$$A_T(c) \doteq A_0\big(\sum_n (\theta_+(\overline{g_{c_n}}^2)\, \delta g_{c_n} + \theta_-(\overline{g_{c_n}}^2)\, \delta \star g_{c_n})\big)\,.$$

- Exhibit functions c_1, c_2 with support on spacelike separated linked loops such that

$$[A_T(c_1), A_T(c_2)] = [A_0(\delta g_{c_1}), A_0(\delta \overset{\Downarrow}{\star} g_{c_2})] \neq 0\,.$$

- Functional $\omega_T(V(1,c)) \doteq \langle \Omega_0, e^{iA_T(c)}\Omega_0 \rangle$, $c \in \mathcal{C}_1(\mathbb{R}^4)$, defines the state. ■

(5) *Regular vacuum states for given ”electric currents”*

Starting point: Any current $J : \mathcal{D}_1(\mathbb{R}^4) \to \mathfrak{P}$ satisfying all Wightman axioms on $(\mathcal{H}_J, \Omega_J)$ and having suitable domain properties.

Lemma

There exists a regular vacuum state ω_J on $\mathfrak{V}$ with GNS representation $(\pi_J, \mathcal{H}_J, \Omega_J)$ such that the corresponding (spacelike linear) potential A_J satisfies $A_J(\delta df) = J(f)$ for $f \in \mathcal{D}_1(\mathbb{R}^4)$.

Remark: The topological charges of these states vanish, but forming ”s-products” $A_{JT} = A_J \otimes 1_T + 1_J \otimes A_T$ with vacuum vector $\Omega_J \otimes \Omega_T \in \mathcal{H}_J \otimes \mathcal{H}_T$ one obtains states ω_{JT} on $\mathfrak{V}$ for the given current, carrying a non-trivial topological charge.

So currents do **not** exclude from the outset the existence of topological charges.

Topological charges: Examples

Elements of proof:

- A_J defined on subspace $\delta d\mathcal{D}_1(\mathbb{R}^4) \subset \mathcal{C}_1(\mathbb{R}^4)$ by inhomogeneous Maxwell equation. Task: exhibit *spacelike linear* extension to full space.
- Given $c \in \mathcal{C}_1(\mathbb{R}^4)$, decompose it into a sum of functions c_n with disjoint connected supports, $c = \sum_n c_n$. If $c_n \in \delta d\mathcal{D}_1(\mathbb{R}^4)$ write $c_n = c_{n\smile}$, otherwise write $c_n = c_{n\frown}$. Resulting unique decomposition: $c = c_\smile + c_\frown$.
- Given $c_\smile = \delta d f_\smile$, its pre-image $f_\smile \in \mathcal{D}_1(\mathbb{R}^4)$ is unique up to some element of $d\mathcal{D}_0(\mathbb{R}^4)$. One can therefore consistently define

$$A_J(c) = \underbrace{A_J(c_\smile)}_{\doteq J(f_\smile)} + \underbrace{A_J(c_\frown)}_{\doteq 0} \doteq J(f_\smile)\,, \quad c \in \mathcal{C}_1(\mathbb{R}^4)\,.$$

- A_J turns out to be "local", Poincaré covariant under action of underlying unitary representation U_J and $[A_J(c_1), A_J(c_2)] = 0$ if c_1, c_2 have support in spacelike separated linked loops.
- Functional $\omega_J(V(1,c)) \doteq \langle \Omega, e^{iJ(f_\smile)}\Omega \rangle$, $c \in \mathcal{C}(\mathbb{R}^4)$, defines desired state. ■

(6) *Multiplets of electromagnetic fields*

(Short distance limit of asymptotically free non-abelian gauge theories *etc.*)

Example: Universal algebra $\mathfrak{V}_2$ based on $\mathcal{C}_1(\mathbb{R}^4) \oplus \mathcal{C}_1(\mathbb{R}^4)$; its generating unitary elements are $V_2(\mathrm{a}, c)$ where $\mathrm{a} \in \mathbb{R}$, $c = c_u \oplus c_d$.

Let $c_{u/d} = \delta g_{u/d}$ with $g_{u/d} \in \mathcal{D}_2(\mathbb{R}^4)$. Given $-1 \leq \zeta \leq 1$, put

$$\langle c_1 , c_2 \rangle_\zeta \doteq \langle g_{1u}, g_{2u} \rangle + \langle g_{1d}, g_{2d} \rangle + \zeta \langle g_{1u}, \overset{\Downarrow}{\star} g_{2d} \rangle - \zeta \langle g_{1d}, \overset{\Downarrow}{\star} g_{2u} \rangle$$

where $\langle \cdot \, , \cdot \rangle$ is one-particle scalar product in the free Maxwell theory.

Lemma

Let $-1 \leq \zeta \leq 1$. The functional ω_ζ on $\mathfrak{V}_2$ given by

$$\omega_\zeta(V_2(1, c)) \doteq e^{-\langle c,c \rangle_\zeta /2} , \quad c \in \mathcal{C}_1(\mathbb{R}^4) \oplus \mathcal{C}_1(\mathbb{R}^4)$$

*is a vacuum state of **type L** which carries a non-trivial charge if $\zeta \neq 0$.*

Similar results hold for multiplets of more fields.

Summary

- Universal C*-algebra $\mathfrak{V}$ of the electromagnetic field in $\mathbb{R}^4$ **exists**
  Extension to multiplets of fields straight forward
- Any relativistic QFT involving electromagnetic field leads to a specific vacuum representation π of $\mathfrak{V}$
- $\mathfrak{V}/\ker \pi$ defines net satisfying all Haag-Kastler axioms (primitivity)
- Possible topological features of intrinsic vector potential *A* encoded in center of $\mathfrak{V}$
- Non-trivial topological charges are accompanied by restricted (spacelike) linear potential *A*; many examples
  (Aharonov-Bohm type effects for photons?)
- Algebra $\mathfrak{V}$ has sufficiently rich structure in order to compute examples of vacua (no ”quantization” needed)
- Meaningful starting point for study of existence problems and structural analysis (IR problems *etc*)