Background independence in gauge theories

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Motivation

- In perturbative QFT, one frequently splits the field

$$\Phi = \bar{\phi} + \phi$$

into a classical (background) part $\bar{\phi}$, and a dynamical perturbation ϕ, which is quantized. Examples are:
  - Spontaneous symmetry breaking: $\phi_i = v_i + \chi_i$.
  - Background field method in YM: $\mathcal{A} = \bar{\mathcal{A}} + A$.
  - (Perturbative) Quantum Gravity: $g = \bar{g} + h$.
- In which sense is the resulting theory independent of the split?
- Naively, an observable $F(\bar{\phi}, \phi)$ depends only on $\Phi = \bar{\phi} + \phi$ iff

$$\mathcal{D}_{\bar{\varphi}} F := (\bar{\delta}_{\bar{\varphi}} - \delta_{\bar{\varphi}}) F := \langle (\tfrac{\delta}{\delta \bar{\phi}} - \tfrac{\delta}{\delta \phi}) F, \bar{\varphi} \rangle = 0.$$

- Various problems in the implementation in quantum (gauge) theory:
  - The non-perturbative background field $\bar{\phi}$ enters the propagators, so the algebras of observables $\mathfrak{A}_{\bar{\phi}}$ depend on $\bar{\phi}$. How to define $\bar{\delta}_{\bar{\varphi}}$?
  - In gauge theories, the split independence of the action is broken by gauge fixing. The violation is BRST exact, background independence restored in observable algebra (BRST cohomology), in classical theory.
  - In quantum theory, anomalies might spoil background independence.

Outline

Outline

The setup

- The problem of defining the derivative w.r.t. the background field can be discussed in the context of the scalar field [Hollands 11, Collini 16].
- Let M globally hyperbolic, $\mathcal{R} \subset M$ compact, containing Cauchy surface, and λ compactly supported and constant on $\mathcal{R}$. Consider the action

$$S[\Phi] = -\int \left(\tfrac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \tfrac{1}{2}m^2\Phi^2 + \tfrac{1}{4!}\lambda\Phi^4\right) \mathrm{vol}.$$

- With $\Phi = \bar{\phi} + \phi$ and $\bar{\phi}$ on-shell, we obtain

$$S[\bar{\phi}, \phi] = \underbrace{-\tfrac{1}{2}\int \left(\partial_\mu \phi \partial^\mu \phi + (m^2 + \tfrac{1}{2}\lambda\bar{\phi}^2)\phi^2\right) \mathrm{vol}}_{S_0} \underbrace{- \int \left(\tfrac{1}{3!}\lambda\bar{\phi}\phi^3 + \tfrac{1}{4!}\lambda\phi^4)\right) \mathrm{vol}}_{S_{\mathrm{int}}}.$$

- S_0 for different backgrounds differ only in a compact region.
- The action is split invariant in the sense that

$$\frac{\delta}{\delta\bar{\phi}(x)} S = \frac{\delta}{\delta\phi(x)} S_{\mathrm{int}}.$$

- An infinitesimal variation $\bar{\varphi}$ of the on-shell background $\bar{\phi}$ is a solution to

$$(\Box - m^2 - \tfrac{1}{2}\lambda\bar{\phi})\bar{\varphi} = 0. \qquad (*)$$

- Set $\mathcal{S}_{\Phi^4}$ of solutions $\bar{\phi}$ as manifold with tangent spaces $T_{\bar{\phi}}\mathcal{S}_{\Phi^4} = Sol(*)$.

The interacting algebras

- Quantization in the frameworks of perturbative AQFT [Brunetti & Fredenhagen 00] and of locally covariant QFT [Hollands & Wald 01; Brunetti, Fredenhagen & Verch 03], with $\bar{\phi}$ a geometric datum on the same footing as g.
- Star product $\star_{\bar{\phi}}$ on algebra $\mathbf{W}_{\bar{\phi}}$ of microcausal functionals.
- Renormalized time-ordered products $T_{\bar{\phi}}$ give rise to retarded products $R_{\bar{\phi}}$:

$$R_{\bar{\phi}}(e_{\otimes}^{iF}; e_{\otimes}^{iG}) := T_{\bar{\phi}}(e_{\otimes}^{iG})^{-1} \star_{\bar{\phi}} T_{\bar{\phi}}(e_{\otimes}^{iF} \otimes e_{\otimes}^{iG}).$$

  Here F, G are local functionals and $i := \frac{\mathrm{i}}{\hbar}$.
- Interacting time ordered products generated by

$$T_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iF}) := R_{\bar{\phi}}(e_{\otimes}^{iF}; e_{\otimes}^{iS_{\mathrm{int}}}).$$

- Interacting algebra $\mathbf{W}_{\bar{\phi}}^{\mathrm{int}}$ generated by $T_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iF})$ with $\operatorname{supp} F \subset \mathcal{R}$.
- Local algebras $\mathbf{W}_{\bar{\phi}}^{\mathrm{int}}(\mathcal{L})$ generated by $T_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iF})$ with $\operatorname{supp} F \subset \mathcal{L} \subset \mathcal{R}$.
- Interacting retarded products defined by

$$R_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iF}; e_{\otimes}^{iG}) := T_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iG})^{-1} \star_{\bar{\phi}} T_{\bar{\phi}}^{\mathrm{int}}(e_{\otimes}^{iF} \otimes e_{\otimes}^{iG}).$$

Retarded variation and perturbative agreement

- Actions S_0 coincide in past of $\operatorname{supp}\lambda$, so consider retarded Møller operator

$$\tau^{\mathrm{r}}_{\bar{\phi},\bar{\phi}'} : \mathbf{W}_{\bar{\phi}'} \to \mathbf{W}_{\bar{\phi}},$$

  identifying observables in past of $\operatorname{supp}\lambda$ [Hollands & Wald 01; Brennecke & Dütsch 08].
- Retarded variation as the infinitesimal version:

$$\delta^{\mathrm{r}}_{\bar{\varphi}} F := \partial_s (\tau^{\mathrm{r}}_{\bar{\phi},\bar{\phi}_s} F_s)|_{s=0}.$$

  Here $\bar{\varphi} = \partial_s \bar{\phi}_s|_{s=0}$ and $F_s \in \mathbf{W}_{\bar{\phi}_s}$. This a derivation.
- $\delta^{\mathrm{r}}_{\bar{\varphi}}$ is the appropriate replacement for $\bar{\delta}_{\bar{\varphi}}$.
- Perturbative agreement [Hollands & Wald 05] requires that it should not matter whether one includes quadratic terms in the free or interacting part of the action. For variations of $\bar{\phi}$, it implies

$$\delta^{\mathrm{r}}_{\bar{\varphi}} T(e_{\otimes}^{\frac{i}{\hbar}F}) = \tfrac{i}{\hbar} T(\bar{\delta}_{\bar{\varphi}} F \otimes e_{\otimes}^{\frac{i}{\hbar}F}) + \tfrac{i}{\hbar} R(e_{\otimes}^{\frac{i}{\hbar}F}; \bar{\delta}_{\bar{\varphi}} S_0). \qquad \text{(PA)}$$

- Renormalization condition, can be fulfilled [Collini 16; Drago, Hack & Pinamonti 17].
- (PA) implies

$$\delta^{\mathrm{r}}_{\bar{\varphi}} T^{\mathrm{int}}(e_{\otimes}^{\frac{i}{\hbar}F}) = \tfrac{i}{\hbar} T^{\mathrm{int}}(\bar{\delta}_{\bar{\varphi}} F \otimes e_{\otimes}^{\frac{i}{\hbar}F}) + \tfrac{i}{\hbar} R^{\mathrm{int}}(e_{\otimes}^{\frac{i}{\hbar}F}; \bar{\delta}_{\bar{\varphi}} S).$$

The Fedosov connection

- The Møller operator provides local trivializations for the algebra bundle

$$\mathbf{W}^{\mathrm{int}}_{\Phi^4} := \sqcup_{\bar\phi} \mathbf{W}^{\mathrm{int}}_{\bar\phi} \to \mathcal{S}_{\Phi^4}, \qquad \mathbf{W}^{\mathrm{int}}_{\Phi^4}(\mathcal{L}) := \sqcup_{\bar\phi} \mathbf{W}^{\mathrm{int}}_{\bar\phi}(\mathcal{L}) \to \mathcal{S}_{\Phi^4}.$$

- We require connection $\mathfrak{D}_{\bar\varphi}$ on $\mathbf{W}^{\mathrm{int}}_{\Phi^4}$ to be derivation respecting localization,

$$\mathfrak{D}_{\bar\varphi}\Gamma(\mathbf{W}^{\mathrm{int}}_{\Phi^4}(\mathcal{L})) \subset \Gamma(\mathbf{W}^{\mathrm{int}}_{\Phi^4}(\mathcal{L})). \qquad \text{(LocCond)}$$

- Retarded variation $\delta^{\mathrm{r}}_{\bar\varphi}$ violates this, but

$$\mathfrak{D}_{\bar\varphi} := \delta^{\mathrm{r}}_{\bar\varphi} - \delta_{\bar\varphi}$$

is a flat (Fedosov) connection [Hollands 11; Collini 16]:

$$\mathfrak{D}_{\bar\varphi} T^{\mathrm{int}}(e_{\otimes}^{\imath F}) = \imath T^{\mathrm{int}}(\underbrace{\{\bar\delta_{\bar\varphi} - \delta_{\bar\varphi}\}F}_{\mathcal{D}_{\bar\varphi}F} \otimes e_{\otimes}^{\imath F}) + \imath R^{\mathrm{int}}(e_{\otimes}^{\imath F}; \underbrace{\{\bar\delta_{\bar\varphi} S - \delta_{\bar\varphi} S_{\mathrm{int}}\}}_{=0}),$$

$$[\mathfrak{D}_{\bar\varphi}, \mathfrak{D}_{\bar\varphi'}] = \mathfrak{D}_{\lfloor\bar\varphi,\bar\varphi'\rfloor}.$$

- One-to-one correspondence of classically b.i. functionals $\mathcal{D}_{\bar\varphi}F = 0$ and quantum b.i. sections $\mathfrak{D}_{\bar\varphi} T^{\mathrm{int}}(e_{\otimes}^{\imath F}) = 0$.
- Flat sections of $\mathbf{W}^{\mathrm{int}}_{\Phi^4}$ provide consistent assignment of observables to different backgrounds.
- Existence of a flat connection as criterion for background independence.

Outline

The setup

- Dynamical quantity: Connection $\mathcal{A}$ on principal G bundle $P \to M$.
- Split
$$\mathcal{A} = \bar{\mathcal{A}} + A$$
into a background connection $\bar{\mathcal{A}}$ and dynamical vector potential A.
- $\bar{\mathcal{A}}$ on-shell w.r.t. Yang-Mills action (in neighborhood of $\mathcal{R}$),
$$\bar{\nabla}^\mu \bar{F}_{\mu\nu} = 0.$$
- Action for the perturbation A
$$S_{\mathrm{YM}} = -\tfrac{1}{4} \int \left\{ (\bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + \lambda [A_\mu, A_\nu])^2 + 2 \bar{F}^I_{\mu\nu} [A^\mu, A^\nu]^I \right\} \mathrm{vol},$$
with λ compactly supported and equal to 1 on $\mathcal{R}$.
- Split independent in $\mathcal{R}$:
$$\tfrac{\delta}{\delta \bar{\mathcal{A}}(x)} S_{\mathrm{YM}} = \tfrac{\delta}{\delta A(x)} S_{\mathrm{YM,int}}, \qquad x \in \mathcal{R}.$$
- Consider $\bar{\mathcal{A}}$ as geometric datum and require local covariance w.r.t. it [Z. 12].
- Under background gauge transformation, A transforms in the adjoint:
$$\bar{\mathcal{A}} \mapsto \bar{\mathcal{A}}^g = \mathrm{ad}_{g^{-1}} \circ \bar{\mathcal{A}} + g^* \theta, \qquad A \mapsto \mathrm{ad}_{g^{-1}} A.$$

Gauge fixing

- Action invariant under dynamical gauge transformations, infinitesimally
$$\delta A_\mu = \bar{\nabla}_\mu \chi + \lambda[A_\mu, \chi].$$
- Need to gauge fix. Use BV-BRST formalism: Introduce fields $(C, \bar{C}, B)$ and anti-fields $(A^\ddagger_\mu, C^\ddagger, \bar{C}^\ddagger, B^\ddagger)$, and define
$$S_{\mathrm{sc}} := -\int \underbrace{(\bar{\nabla}_\mu C + \lambda[A_\mu, C])}_{sA_\mu} A^{\mu\ddagger} + \underbrace{\tfrac{1}{2}\lambda[C, C]}_{sC} C^\ddagger + \underbrace{B}_{s\bar{C}} \bar{C}^\ddagger.$$
- It generates the BRST transformation via the anti-bracket
$$(F, G) := \int \frac{\delta^R F}{\delta \Phi^i(x)} \frac{\delta^L G}{\delta \Phi^\ddagger_i(x)} - \frac{\delta^R F}{\delta \Phi^\ddagger_i(x)} \frac{\delta^L G}{\delta \Phi^i(x)}$$
- Choose gauge-fixing fermion
$$\Psi := \int \bar{C}\left(\bar{\nabla}^\mu A_\mu + \tfrac{1}{2} B\right) \mathrm{vol}$$
  and perform "canonical transformation"
$$S := e^{(-,\Psi)}(S_{\mathrm{YM}} + S_{\mathrm{sc}}).$$
- Define BV differential
$$sF := (S, F).$$

Background independence

- Due to the explicit background dependence of the gauge fixing fermion, the gauge fixed action is not split independent

$$\tfrac{\delta}{\delta \bar{A}(x)} S - \tfrac{\delta}{\delta A(x)} S_{\mathrm{int}} = s \tfrac{\delta}{\delta \bar{A}(x)} \Psi, \qquad x \in \mathcal{R}.$$

  Violation is s exact, not relevant for observables (cohomology of s).

- Set of solutions $\mathcal{S}_{\mathrm{YM}}$ of YM equation is a manifold, away from special (symmetric) solutions [Arms 81]. Away from such singularities, $T_{\bar{A}} \mathcal{S}_{\mathrm{YM}}$ is space of solutions $\bar{a}$ to YM equation linearized around $\bar{A}$.
- Conjugating "split differential" $\mathcal{D}_{\bar{a}} := \bar{\delta}_{\bar{a}} - \delta_{\bar{a}}$ with gauge fixing trafo yields

$$\hat{\mathcal{D}}_{\bar{a}} := e^{(-,\Psi)} \circ \mathcal{D}_{\bar{a}} \circ e^{-(-,\Psi)} = \mathcal{D}_{\bar{a}} - (-, \mathcal{D}_{\bar{a}} \Psi).$$

- It fulfills

$$\hat{\mathcal{D}}_{\bar{a}} \circ s - s \circ \hat{\mathcal{D}}_{\bar{a}} = 0,$$
$$[\hat{\mathcal{D}}_{\bar{a}}, \hat{\mathcal{D}}_{\bar{a}'}] - \hat{\mathcal{D}}_{\lfloor \bar{a}, \bar{a}' \rfloor} = 0.$$

- $\hat{\mathcal{D}}_{\bar{a}}$ flat and well-defined on s cohomology. Can be used to characterize background independent classical observables.

Outline

The quantum BV differential and anomalies

- Construction of interacting algebra $\mathbf{W}^{\mathrm{int}}$ analogously to scalar case.
- The free BV differential s_0 fulfills the anomalous Ward identity [Hollands 07]

$$s_0 T(e_\otimes^{\frac{i}{\hbar}F}) = \frac{i}{\hbar} T(\{s_0 F + \tfrac{1}{2}(F,F) + A(e_\otimes^F)\} \otimes e_\otimes^{\frac{i}{\hbar}F})$$

  with the anomaly A of order $\hbar$ and subject to consistency conditions.
- We assume absence of gauge anomalies, i.e.,

$$A(e_\otimes^{S_{\mathrm{int}}}) = 0.$$

- The interacting BV differential s^{int} is on-shell equal to $-\frac{i}{\hbar}[Q^{\mathrm{int}}, -]_\star$ [Fröb 18]. Its cohomology in $\mathbf{W}^{\mathrm{int}}$ are the observables in the interacting theory.
- s^{int} fulfils the interacting anomalous Ward identity [Taslimi Tehrani 17; Fröb 18]

$$s^{\mathrm{int}} T^{\mathrm{int}}(e_\otimes^{\frac{i}{\hbar}F}) = \frac{i}{\hbar} T^{\mathrm{int}}(\{sF + \tfrac{1}{2}(F,F) + A^{\mathrm{int}}(e_\otimes^F)\} \otimes e_\otimes^{\frac{i}{\hbar}F})$$

  with the interacting anomaly

$$A^{\mathrm{int}}(e_\otimes^F) := A(e_\otimes^F \otimes e_\otimes^{S_{\mathrm{int}}}).$$

- Generators $T^{\mathrm{int}}(e_\otimes^{\frac{i}{\hbar}F})$ of observables are characterized by

$$sF + \tfrac{1}{2}(F,F) + A^{\mathrm{int}}(e_\otimes^F) = 0. \qquad \text{(genObs)}$$

Background independence I

- Connection $\mathfrak{D}_{\bar{a}}$ should be well-defined and flat on s^{int} cohomology, i.e.,

$$\mathfrak{D}_{\bar{a}} \circ s^{\mathrm{int}} - s^{\mathrm{int}} \circ \mathfrak{D}_{\bar{a}} = 0,$$
$$\left([\mathfrak{D}_{\bar{a}}, \mathfrak{D}_{\bar{a}'}] - \mathfrak{D}_{\lfloor \bar{a}, \bar{a}' \rfloor}\right) \operatorname{Ker} s^{\mathrm{int}} \in \operatorname{Im} s^{\mathrm{int}}.$$

- First guess: Replace

$$\hat{\mathcal{D}}_{\bar{a}} = \bar{\delta}_{\bar{a}} - (-, \bar{\delta}_{\bar{a}}\Psi) - \delta_{\bar{a}} + (-, \delta_{\bar{a}}\Psi) \quad \to \quad \mathfrak{D}^0_{\bar{a}} := \delta^{\mathrm{r}}_{\bar{a}} - \delta_{\bar{a}} + (-, \delta_{\bar{a}}\Psi).$$

- Perturbative agreement w.r.t. changes in $\bar{\mathcal{A}}$ may be assumed [Z. 14.]. Then

$$\mathfrak{D}^0_{\bar{a}} T^{\mathrm{int}}(e_\otimes^{\frac{i}{\hbar}F}) = \tfrac{i}{\hbar} T^{\mathrm{int}}(\mathcal{D}^0_{\bar{a}} F \otimes e_\otimes^{\frac{i}{\hbar}F}) + \tfrac{i}{\hbar} R^{\mathrm{int}}(e_\otimes^{\frac{i}{\hbar}F}; \underbrace{\mathcal{D}^0_{\bar{a}} S_{\mathrm{int}} + \bar{\delta}_{\bar{a}} S_0}_{\underline{s}(\bar{\delta}_{\bar{a}}\Psi)}) \qquad (*)$$

with $\mathcal{D}^0_{\bar{a}} := \mathcal{D}_{\bar{a}} + (-, \delta_{\bar{a}}\Psi)$ and $\underline{s}$ coinciding with s on $\mathcal{R}$.

- Second term in (*) spoils localization property (LocCond). But
  - vanishes for $\bar{a}$ supported in future of $\mathcal{R}$,
  - equals a commutator for $\bar{a}$ supported in past of $\mathcal{R}$,
  - is formally s^{int} exact for $\bar{a}$ supported in $\mathcal{R}$ and F fullfilling (genObs).
- Choose η with $\operatorname{supp} \eta \subset J^-(\mathcal{R})$, $\eta = 1$ on $J^-(\mathcal{R}) \setminus \mathcal{R}$, and define

$$\mathfrak{D}_{\bar{a}} := \mathfrak{D}^0_{\bar{a}} + \tfrac{i}{\hbar}[T^{\mathrm{int}}(\underline{s}(\bar{\delta}_{\eta\bar{a}}\Psi)), -]_\star.$$

Background independence II

- Provided that

$$A^{\mathrm{int}}(\bar{\delta}_a \Psi) = 0, \qquad \operatorname{supp} a \subset \mathcal{R}, \qquad \text{(bgAnomaly)}$$

  $\mathfrak{D}_{\bar{a}}$ is well-defined and flat on s^{int} cohomology. For F fulfilling (genObs)

$$\mathfrak{D}_{\bar{a}} T^{\mathrm{int}}(e_{\otimes}^{\frac{i}{\hbar}F}) = \tfrac{i}{\hbar} T^{\mathrm{int}}(\{\hat{\mathcal{D}}_{\bar{a}} F + A^{\mathrm{int}}(\bar{\delta}_{\bar{a}} \Psi \otimes e_{\otimes}^{F})\} \otimes e_{\otimes}^{\frac{i}{\hbar}F}) \quad \operatorname{mod} \ \operatorname{Im} s^{\mathrm{int}}$$

- Condition $\hat{\mathcal{D}}_{\bar{a}} F = 0$ for background independent classical functionals quantum corrected to

$$\hat{\mathcal{D}}_{\bar{a}} F + A^{\mathrm{int}}(\bar{\delta}_{\bar{a}} \Psi \otimes e_{\otimes}^{F}) = 0.$$

- The condition (bgAnomaly) can be fulfilled in YM theory in $D = 4$.
- Proof uses that cohomology of s is trivial for Lie algebra valued one-forms of mass dimension 3.

Outline

Summary & Outlook

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  - Existence of a flat connection on the observable bundle over background configurations as criterion for background independence.
  - Established for (pure) Yang-Mills in $D = 4$.
- Outlook:
  - Spontaneous symmetry breaking.
  - Non-renormalizable theories, such as gravity.
  - Fedosov quantization.

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THANK YOU VERY MUCH FOR YOUR ATTENTION!

Appendix: Comparison with other approaches

- In the literature, background independence is typically discussed within the Riemannian path integral or with an infinitesimal background field.
- The former is formal and connection to Lorentzian signature unclear.
- The latter misses non-perturbative aspects and depends on the choice of a reference connection.
- A further common shortcoming is that they do not provide a means to compare observables on different backgrounds.
- Unclear (to me) whether the criterion of triviality of the interacting relative Cauchy evolution [Brunetti, Fredenhagen & Rejzner 13] is better in that respect.
- Existence of a flat connection as criterion for background independence in other settings:
  - QM [Reuter 98].
  - String field theory [Witten 93; Sen & Zwiebach 93].