Westfälische
Wilhelms-Universität
Münster

Fachbereich 10
Mathematik und
Informatik

Schwinger-Dyson equations and Ward identities in NC QFT on the example of scalar models

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Alexander Hock

Noncommutative space (Moyal) with a noncommutative product
$(f \star g)(x) \neq (g \star f)(x)$ for $f, g \in \mathcal{S}(\mathbb{R}^2)$

The vector space of Schwarz functions has a matrix basis
$\rightarrow$ "dynamical" matrix model

Schwinger-Dyson equations + Ward-Takahashi identities
$\rightarrow$ closed integral equation (for ϕ^3_D and ϕ^4_D in the large $V, \mathcal{N}$ limit)

ϕ^3 was solved for $D = 2, 4, 6$ by Grosse, Sako, Wulkenhaar ('17,'18)
ϕ^4 was solved for $D = 2$ by Panzer, Wulkenhaar (two weeks ago)

ϕ^3 and ϕ^4 are in the same class of models

Example ϕ_2^3

The action is

$$S[\phi] = V\left(\sum_{n,m=0}^{\mathcal{N}} \frac{H_{nm}}{2}\phi_{nm}\phi_{mn} + \kappa\sum_{n=0}^{\mathcal{N}}\phi_{nn} + \frac{\lambda}{3}\sum_{n,m,l=0}^{\mathcal{N}}\phi_{nm}\phi_{ml}\phi_{ln})\right),$$

where ϕ Hermitian matrix, $H_{nm} = E_n + E_m$, $E_m = \frac{m}{V} + \frac{\mu^2}{2}$ energy distribution, V deformation parameter of the Moyal plane and κ renormalization parameter.

$\mathcal{Z}[J] := \int \mathcal{D}\phi \exp\left(-S[\phi] + V\sum_{n,m} J_{nm}\phi_{mn}\right)$, $\phi_{nm} \leftrightarrow \frac{1}{V}\frac{\partial}{\partial J_{mn}}$

$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} \approx 1 + V\sum_n G_{|n|}J_{nn}$
$+\sum_{n,m}\left(\frac{V}{2}G_{|nm|}J_{nm}J_{mn} + \left(\frac{V^2}{2}G_{|n|}G_{|m|} + \frac{1}{2}G_{|n|m|}\right)J_{nn}J_{mm}\right) + ..$

Schwinger-Dyson equation + Ward-Takahashi identity

$$G_p = \frac{1}{H_{pp}}\left(-\kappa - \frac{\lambda}{V}\sum_{m=0}^{\mathcal{N}} G_{pm} - \frac{\lambda}{V^2} G_{p|p} - \lambda G_p^2\right)$$

&

$$G_{p_1p_2} = \frac{1}{H_{p_1p_2}}\left(1 + \lambda \frac{G_{p_1} - G_{p_2}}{E_{p_1} - E_{p_2}}\right)$$

Ward identity: $\mathcal{Z}$ should be invariant under $\phi \to U\phi U^\dagger$, $U \in \mathrm{U}(\mathcal{N})$

$\sum_{k=0}^{\mathcal{N}} \frac{\partial^2}{\partial J_{nk}\partial J_{km}} \mathcal{Z}[J] = \frac{V}{(E_n - E_m)} \sum_{k=0}^{\mathcal{N}} \left(J_{kn}\frac{\partial}{\partial J_{km}} - J_{mk}\frac{\partial}{\partial J_{nk}}\right)\mathcal{Z}[J]$

Limit $V, \mathcal{N} \to \infty$ with $\frac{\mathcal{N}}{V} = \Lambda^2$

$\lim \frac{1}{V}\sum_{m=0}^{\mathcal{N}} f(\frac{m}{V}) = \int_0^{\Lambda^2} f(x)$ with $x = \frac{m}{V}$

$\to$ closed integral equations

3-colour scalar model in 2D

3 scalar fields ϕ^a for $a \in \{1,2,3\}$ with the same energy distribution, where ϕ^a is a Hermitian matrix.
The action is definite [1]

$$S[\phi] = V\left(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}} \frac{H_{nm}}{2}\phi^a_{nm}\phi^a_{mn} + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}} \sigma_{abc}\phi^a_{nm}\phi^b_{ml}\phi^c_{ln}\right),$$

where $H_{nm} = E_n + E_m$, $E_n = \frac{\mu^2}{2} + \frac{n}{V}$, $\sigma_{abc} = |\varepsilon_{abc}|$ and V as deformation parameter of the Moyal space.

$$\mathcal{Z}[J] := \int \left(\prod_{a=1}^{3} \mathcal{D}\phi^a\right) \exp\left(-S[\phi] + V\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}} J^a_{nm}\phi^a_{mn}\right)$$

$$\phi^a_{nm} \leftrightarrow \frac{1}{V}\frac{\partial}{\partial J^a_{mn}}$$

[1] arXiv:1804.06075 with R. Wulkenhaar

Correlation functions are expansion coefficients of $\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$.
The first terms are

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} =: 1 + \sum_{a=1}^{3} \sum_{n,m=0}^{\mathcal{N}} \left(\frac{V}{2} G^{aa}_{|nm|} J^a_{nm} J^a_{mn} + \frac{1}{2} G^{a|a}_{|n|m|} J^a_{nn} J^a_{mm} \right)$$

$$+ \sum_{a,b,c=1}^{3} \sum_{n,m,l=0}^{\mathcal{N}} \sigma_{abc} \left(\frac{V}{3} G^{abc}_{|nml|} J^a_{nm} J^b_{ml} J^c_{ln} + \frac{1}{2} G^{a|bc}_{|n|ml|} J^a_{nn} J^b_{ml} J^c_{lm} \right.$$

$$\left. + \frac{1}{6V} G^{a|b|c}_{|n|m|l|} J^a_{nn} J^b_{mm} J^c_{ll} \right) + \dots$$

Example of a Schwinger-Dyson equation

$$G^{aa}_{p_1p_2} = \frac{1}{1+p_1+p_2} + \frac{\lambda}{H_{p_1p_2}} \sum_{b,c=1}^{3} \sigma_{abc}$$

$$\times \left(\frac{1}{V} \sum_{n=0}^{\mathcal{N}} G^{abc}_{p_1p_2n} + \frac{1}{V^2} G^{a|bc}_{p_1|p_1p_2} + \frac{1}{V^2} G^{a|bc}_{p_2|p_1p_2} + \frac{\delta_{p_1p_2}}{V^3} G^{a|b|c}_{p_1|p_1|p_2} \right).$$

One gets a tower of D-S equations.
Try to decouple the tower in the large $\mathcal{N}, V$ limit with Ward-Takahashi identities

Ward-Takahashi identities

$$S[\phi] = V\left(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}}\frac{H_{nm}}{2}\phi_{nm}^{a}\phi_{mn}^{a} + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}}\sigma_{abc}\phi_{nm}^{a}\phi_{ml}^{b}\phi_{ln}^{c}\right)$$

The identity is obtained by the invariance of $\mathcal{Z}[J]$ under $\mathrm{U}(\mathcal{N})$ transformation

Ward-Takahashi identities

$$S[\phi] = V\left(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}} \frac{H_{nm}}{2}\phi_{nm}^a\phi_{mn}^a + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}} \sigma_{abc}\phi_{nm}^a\phi_{ml}^b\phi_{ln}^c\right)$$

The identity is obtained by the invariance of $\mathcal{Z}[J]$ under $\mathrm{U}(\mathcal{N})$ transformation ($\phi \to U\phi U^\dagger$).

For the $\phi_D^{3/4}$ model

$$\sum_{k=0}^{\mathcal{N}}\left(\frac{\partial^2}{\partial J_{nk}\partial J_{km}} - \frac{V}{(E_n - E_m)}\left(J_{kn}\frac{\partial}{\partial J_{km}} - J_{mk}\frac{\partial}{\partial J_{nk}}\right)\right)\mathcal{Z}[J] = 0.$$

Ward-Takahashi identities

$$S[\phi] = V\left(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}} \frac{H_{nm}}{2}\phi^a_{nm}\phi^a_{mn} + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}} \sigma_{abc}\phi^a_{nm}\phi^b_{ml}\phi^c_{ln}\right)$$

The identity is obtained by the invariance of $\mathcal{Z}[J]$ under $\mathrm{U}(\mathcal{N})$ transformation ($\phi^a \to U\phi^a U^\dagger$, $\forall a$).

Simultaneous transformation of all 3 fields

$$\sum_{a=1}^{3}\sum_{k=0}^{\mathcal{N}}\left(\frac{\partial^2}{\partial J^a_{nk}\partial J^a_{km}} - \frac{V}{(E_n - E_m)}\left(J^a_{kn}\frac{\partial}{\partial J^a_{km}} - J^a_{mk}\frac{\partial}{\partial J^a_{nk}}\right)\right)\mathcal{Z}[J] = 0.$$

Ward-Takahashi identities

$$S[\phi] = V\Big(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}}\frac{H_{nm}}{2}\phi^a_{nm}\phi^a_{mn} + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}}\sigma_{abc}\phi^a_{nm}\phi^b_{ml}\phi^c_{ln}\Big)$$

The identity is obtained by the invariance of $\mathcal{Z}[J]$ under $\mathrm{U}(\mathcal{N})$ transformation ($\phi^a \to U\phi^a U^\dagger$, fix a).

Transformation of one field

$$\sum_{k=0}^{\mathcal{N}}\frac{\partial^2}{\partial J^a_{nk}\partial J^a_{km}} + \frac{V}{E_n - E_m}\sum_{k=0}^{\mathcal{N}}\left(J^a_{mk}\frac{\partial}{\partial J^a_{nk}} - J^a_{kn}\frac{\partial}{\partial J^a_{km}}\right)$$
$$= \frac{\lambda}{V(E_n - E_m)}\sum_{k,l=0}^{\mathcal{N}}\sum_{c,d=1}^{3}\left(\sigma_{acd}\frac{\partial^3}{\partial J^a_{nk}\partial J^c_{kl}\partial J^d_{lm}} - \sigma_{acd}\frac{\partial^3}{\partial J^c_{nk}\partial J^d_{kl}\partial J^a_{lm}}\right)$$

Ward-Takahashi identities

$$S[\phi] = V\left(\sum_{a=1}^{3}\sum_{n,m=0}^{\mathcal{N}} \frac{H_{nm}}{2}\phi^a_{nm}\phi^a_{mn} + \frac{\lambda}{3}\sum_{a,b,c=1}^{3}\sum_{n,m,l=0}^{\mathcal{N}} \sigma_{abc}\phi^a_{nm}\phi^b_{ml}\phi^c_{ln}\right)$$

The identity is obtained by the invariance of $\mathcal{Z}[J]$ under $\mathrm{U}(\mathcal{N})$ transformation ($\phi^a \to \phi'^a(\phi^a, \phi^b)$, $\phi^b \to \phi'^b(\phi^b, \phi^a)$).

Mixed Transformation

$$\sum_{k=0}^{\mathcal{N}} \frac{\partial^2}{\partial J^a_{nk}\partial J^b_{km}} + \frac{V}{E_n - E_m}\sum_{k=0}^{\mathcal{N}}\left(J^b_{mk}\frac{\partial}{\partial J^a_{nk}} - J^a_{kn}\frac{\partial}{\partial J^b_{km}}\right)$$

$$= \frac{\lambda}{V(E_n - E_m)}\sum_{k,l=0}^{\mathcal{N}}\sum_{c,d=1}^{3}\left(\sigma_{bcd}\frac{\partial^3}{\partial J^a_{nk}\partial J^c_{kl}\partial J^d_{lm}} - \sigma_{acd}\frac{\partial^3}{\partial J^c_{nk}\partial J^d_{kl}\partial J^b_{lm}}\right)$$

- Using the Ward identities twice
- Performing the large $V, \mathcal{N}$ limit
- Closed integral equations are derived for this special model

Nonlinear closed integral equation of the 2-point function for the 3-colour model

$$
\begin{aligned}
G^{aa}_{p_1p_2} =& \frac{1}{1+p_1+p_2} + \frac{\lambda^2}{(1+p_1+p_2)\,(p_1-p_2)} \\
&\times \left(3\,G^{aa}_{p_1p_2} \int_0^{\Lambda^2} dq\, \left(G^{aa}_{qp_2} - G^{aa}_{p_1q} \right) \right. \\
&\left. - \int_0^{\Lambda^2} dq\, \frac{G^{aa}_{p_1q} - G^{aa}_{p_1p_2}}{q-p_2} + \int_0^{\Lambda^2} dq\, \frac{G^{aa}_{p_2q} - G^{aa}_{p_1p_2}}{q-p_1} \right)
\end{aligned}
$$

Can be used perturbatively

$$
G^{aa}_{p_1p_2} = \sum_{n=0}^{\infty} \lambda^n G_{n,\,p_1p_2}^{\;\;aa}
$$

3-Colour model, 2-point function perturbative result ($\Lambda^2=\infty$)

$$
\begin{aligned}
G^{aa}_{p_1p_2} =& \frac{1}{1+p_1+p_2} + \frac{2\lambda^2 \log(\frac{1+p_1}{1+p_2})}{(1+p_1+p_2)^2(p_1-p_2)} \\
&+ \frac{2\lambda^4}{(1+p_1+p_2)^2(p_1-p_2)}\Bigg(\frac{3\log(\frac{1+p_1}{1+p_2})^2}{(1+p_1+p_2)(p_1-p_2)} \\
&+ \frac{2\log(1+p_1)}{p_1(1+2p_1)(1+p_1)} - \frac{2\log(1+p_2)}{p_2(1+2p_2)(1+p_2)} \\
&- \frac{(1+2p_2)\left(\frac{\pi^2}{6}+2\mathrm{Li}_2(\frac{p_1}{1+p_1})\right)}{(1+2p_1)^2(1+p_1+p_2)} + \frac{(1+2p_1)\left(\frac{\pi^2}{6}+2\mathrm{Li}_2(\frac{p_2}{1+p_2})\right)}{(1+2p_2)^2(1+p_1+p_2)}\Bigg)
\end{aligned}
$$

$$
\Big[\Big\{ \log(1+p_1)f_1(p_1,p_2) + \pi^2\log(1+p_1)f_2(p_1,p_2) + \log(1+p_1)^2 f_3(p_1,p_2)
$$

$$
+ \Big(\mathrm{Li}_2\big(\tfrac{p_1}{1+p_1}\big) + \tfrac{\pi^2}{6}\Big) f_4(p_1,p_2) + \mathrm{Li}_2\big(\tfrac{p_1}{1+p_1}\big)\log\big(\tfrac{1+p_1}{1+p_2}\big) f_5(p_1,p_2)
$$

$$
+ \Big(\mathrm{Li}_3(-p_1) + \mathrm{Li}_3\big(\tfrac{p_1}{1+p_1}\big) + \mathrm{Li}_2\big(\tfrac{p_1}{1+p_1}\big)\log(1+p_1) + \tfrac{\log(1+p_1)^3}{6}
$$

$$
- \tfrac{\pi^2\log(1+p_1)}{6}\Big) f_6(p_1,p_2)
$$

$$
+ \Big(\mathrm{Li}_3\big(\tfrac{p_1}{1+p_1}\big) + \tfrac{\pi^2\log(1+p_1)}{3}\Big) f_7(p_1,p_2)\Big\} + \{p_1 \leftrightarrow p_2\}
$$

$$
+ \log\big(\tfrac{1+p_1}{1+p_2}\big)^3 f_8(p_1,p_2) + \log(1+p_1)\log(1+p_2) f_9(p_1,p_2) + \pi^2 f_{10}(p_1,p_2)
$$

$$
+ \pi^2\log(2) f_{11}(p_1,p_2) + \zeta(3) f_{12}(p_1,p_2)\Big]\lambda^6 + \mathcal{O}(\lambda^8)
$$

- The closed equation gives the sum over all diagrams at a certain order
- The result is known up to the third order and fits perfectly with the loop expansion
- Polylogs and the ζ function appear
- At higher order Harmonic Polylogs are supposed to appear
- Closed integral equations are given for any N-point function

Summary

- Moyal product leads to matrix models for scalar fields
- The Ward identity is the same in the $\phi^{3/4}$ model
- Schwinger-Dyson equation $+$ Ward identity
  $\rightarrow$ closed equation in the large $V, \mathcal{N}$ limit
- Interaction of more fields leads to complicated Ward identities
- However, closed equations exist

Outlook

- Perturbation to higher order $\rightarrow$ exact solution?
- Relation to the higher boundary sector
- Extension to 4,6 dimension

Thank you for your attention!