# A semiclassical singularity theorem

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# Outline

1 Introduction

2 Weakened conditions

3 A condition obeyed by quantum fields

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# Introduction

**Definition**

A spacetime is singular if it possesses at least one incomplete geodesic.

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A spacetime is singular if it possesses at least one incomplete geodesic.

## Singularity theorem structure

1. **Causality condition**
   There is a Cauchy hypersurface
2. **The initial or boundary condition**
   There exists a trapped surface (null geodesics) or a spatial slice with negative expansion (timelike goedesics)

# Introduction

3. **The energy condition**

Penrose (Null geodesics)

Null Convergence Condition | Null Energy Condition

$$R_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \qquad T_{\mu\nu}\ell^\mu\ell^\mu \geq 0$$

Hawking (Timelike geodesics)

Timelike Convergence Condition | Strong Energy Condition

$$R_{\mu\nu}U^\mu U^\nu \geq 0 \qquad T_{\mu\nu}(U^\mu U^\nu - g^{\mu\nu}/(n-2)) \geq 0$$

$\Rightarrow$ Then the spacetime is geodesically incomplete.

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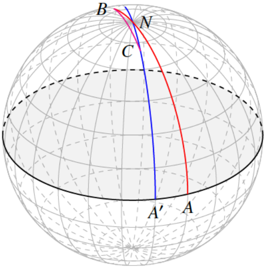

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$$I[V] = \int_0^\tau \left( R_{\mu\nu\alpha\beta} \overbrace{U^\mu}^{\text{tangent vector}} \underbrace{V^\nu}_{\text{variation vector}} V^\alpha U^\beta - \frac{DV^\mu}{dt}\frac{DV_\mu}{dt} \right) dt - \overbrace{K_{\mu\nu}}^{\text{Extrinsic curvature}} V^\mu V^\nu |_{\gamma(0)}$$

Whether $\gamma$ is, or is not, a local maximum of the length functional, among amounts to the absence, or presence, of a focal point.

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**Focal point test**

$I[V] \geq 0$ for some $V^\mu \implies \exists$ focal point in $(0, \tau]$

# The formation of focal points

With $V^\mu = fv^\mu$ where $f$ smooth function that obeys $f(0) = 1$, $f(\tau) = 0$

$$\int_0^\tau \left((n-1)\dot{f}^2 - f^2 R_{\mu\nu} U^\mu U^\nu\right) dt \leq -K|_{\gamma(0)},$$

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## Hawking's singularity theorem

- Energy condition: $R_{\mu\nu} U^\mu U^\nu \geq 0$
- Initial condition: For $f(t) = 1 - t/\tau$, $K|_{\gamma(0)} < 0$ and $\tau \geq (n-1)/|K|_{\gamma(0)}|$
- Causality condition: The existence of a compact Cauchy surface which implies that there are no focal points

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All pointwise energy conditions are violated by quantum fields

# Singularity theorems with weakened energy conditions

- Energy condition

$$\int_0^\tau f(t)^2 R_{\mu\nu} U^\mu U^\nu|_{\gamma(t)}\, dt \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2,$$

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and $R_{\mu\nu} U^\mu U^\nu \geq 0$ for $[0, \tau_0]$.

- Initial contraction

$$-K|_{\gamma(0)} > \inf_{\varphi} J_1[\varphi] + \inf_{f} J_2[f]$$

$$J_1[\varphi] = \int_0^{\tau_0} \left( (1-\varphi^2) R_{\mu\nu} \overset{\text{only for } [0,\tau_0]}{\overset{\downarrow}{U^\mu}} U^\nu + Q_0 \varphi^2 + Q_m (\varphi^{(m)})^2 \right) dt$$

$$J_2[f] = \int_{\tau_0}^{\tau} \left( (n-1)\dot{f}^2 + Q_0 f^2 + Q_m (f^{(m)})^2 \right) dt.$$

- Causality condition: Existence of a compact Cauchy surface

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# Initial contraction

If $Q_0\tau_0^2 \ll 1$ and $Q_m/\tau_0^{2(m-1)} \ll 1$ we can show that for initial extrinsic curvature obeying

$$-K|_{\gamma(0)} > \sqrt{4(n-1)A_m B_m Q_0}$$

a focal point is formed within a timescale

$$\tau \sim \sqrt{\frac{(n-1)B_m}{A_m Q_0}}\,.$$

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| $m$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $A_m$ | $1/3$ | $13/35$ | $181/462$ | $521/1287$ |
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## Advantages

- Allows us to estimate the timescale of formation of the focal point
- Simpler generalization of the theorem for weakened energy conditions

# Non-minimally coupled scalar field

Stress energy tensor for non-minimally coupled scalar fields

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) + \frac{1}{2} g_{\mu\nu}(\mu^2 \phi^2 - (\nabla\phi)^2) + \xi(g_{\mu\nu}\Box_g - \nabla_\mu \nabla_\nu - G_{\mu\nu})\phi^2$$

The main observable of interest will be the effective energy density (EED)

$$\rho_U = T_{\mu\nu} U^\mu U^\nu - \frac{1}{n-2} T \,.$$

As a quantum field, $\rho_U$ may be defined by

$$\rho_U(f) = T_{\mu\nu}\left(\left(U^\mu U^\nu - \frac{g^{\mu\nu}}{n-2}\right) f\right),$$

# Quantization

- Introduction of a unital *-algebra $\mathscr{A}(M)$ on our manifold $M$
- Generated by the objects $\Phi(f)$, $f \in \mathscr{D}(M)$ where $\mathscr{D}(M)$ is the space of complex-valued, compactly-supported, smooth functions on M

# Quantization

- Introduction of a unital *-algebra $\mathscr{A}(M)$ on our manifold $M$
- Generated by the objects $\Phi(f)$, $f \in \mathscr{D}(M)$ where $\mathscr{D}(M)$ is the space of complex-valued, compactly-supported, smooth functions on M
- We only consider Hadamard states on our algebra $W(x, y) = \langle\Phi(x)\Phi(y)\rangle_{\omega} : \mathscr{D}(M) \times \mathscr{D}(M) \to \mathbb{C}$
- The smeared local Wick polynomials of the form

  $$\langle : \nabla^{(r)}\Phi\nabla^{(s)}\Phi : (f)\rangle_{\omega}$$

  are part of an extended algebra
- We need a prescription for finding algebra elements that qualify as local and covariant Wick powers. This might be done in various ways, expressing finite renormalisation freedoms. Hollands and Wald (2014) set out a list of axioms that we follow.

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Absolute QEIs

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Absolute QEIs

$$\langle :\rho:(f)\rangle_\omega \geq -\langle \mathfrak{Q}(f)\rangle_\omega$$

Difference QEIs

$$\langle :\rho:_{\omega_0}(f)\rangle_\omega = \langle \rho(f)\rangle_{\underset{\substack{\uparrow \\ \text{state of interest}}}{\omega}} - \langle \rho(f)\rangle_{\underset{\substack{\uparrow \\ \text{reference state}}}{\omega_0}} \geq -\langle \overbrace{\mathfrak{Q}_{\omega_0}(f)}^{\text{unbounded operator}}\rangle_\omega \,.$$

# Quantum strong energy inequality

Our aim is to establish QEI lower bounds on the averaged EED along timelike geodesic $\gamma$,

$$\langle :\rho_U: \circ\, \gamma \rangle_\omega (f^2) = \int d\tau f^2(\tau) \langle :\rho_U: \rangle_\omega (\gamma(\tau)) \,,$$

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$$\langle :\rho_U: \rangle_\omega = [\hat{\rho}_1 :W:] + [\hat{\rho}_2 :W:] + \left( \xi \mathcal{R}_\xi - \frac{1-2\xi}{n-2} \mu^2 \right) [:W:]$$

along $\gamma$, where $:W: = W - W_0$ and the operators $\hat{\rho}_i$ are given by

$$\begin{aligned} \boxed{\hat{\rho}_1} &= \left( 1 - 2\xi \frac{n-1}{n-2} \right) (\nabla_U \otimes \nabla_U) + \frac{2\xi}{n-2} \sum_{a=1}^{n-1} (\nabla_{e_a} \otimes \nabla_{e_a}) \, , \\ \boxed{\hat{\rho}_2} &= -2\xi (\mathbb{1} \otimes_{\mathfrak{s}} U^\mu U^\nu \nabla_\mu \nabla_\nu) \, , \\ \mathcal{R}_\xi &= \frac{2\xi}{n-2} R - R_{\mu\nu} U^\mu U^\nu \, . \end{aligned}$$

# Point-splitting technique

$\boxed{\hat{\rho}_2}$

$$\int d\tau f(\tau)^2 \left[(\mathbb{1} \otimes_{\mathfrak{s}} U^\mu U^\nu \nabla_\mu \nabla_\nu) F\right](\gamma(\tau))$$

$$= -\boxed{\int d\tau \left[(\partial \otimes \partial)\left((f \otimes f)\phi^* F\right)\right](\tau)} + \int d\tau f'(\tau)^2 \left[F\right](\gamma(\tau))$$

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$\boxed{\hat{\rho}_1}$

$$\int d\tau \left[\!\left[(\partial^k \otimes \partial^k)\left((f \otimes f)\phi^*(Q \otimes Q){:}W{:}\right)\right]\!\right](\tau)$$

$$= \int_0^\infty \frac{d\alpha}{\pi} \alpha^{2k} \left(\left(\phi^*((Q \otimes Q)W)\right)(\bar{f}_\alpha, f_\alpha) - \left(\phi^*((Q \otimes Q)W_0)(\bar{f}_\alpha, f_\alpha)\right)\right)$$

The two terms in the integrand are non-negative and decay rapidly as $\alpha \to +\infty$ for any Hadamard state $\omega$.

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The two terms in the integrand are non-negative and decay rapidly as $\alpha \to +\infty$ for any Hadamard state $\omega$.
$\Rightarrow$ The state dependent part can be discarded

**Theorem**

*For non-minimally coupled scalar field with coupling constant $\xi \in [0, \xi_c]$, $\gamma$ a timelike geodesic, for all Hadamard states $\omega$, the normal-ordered effective energy density obeys the QSEI*

$$\int d\tau f^2(\tau)\langle{:}\rho_U{:}\rangle_\omega(\gamma(\tau)) \geq -\left[\mathfrak{Q}_A(f)\mathbb{1} + \langle{:}\Phi^2{:} \circ \gamma\rangle_\omega(\mathfrak{Q}_B(f) + \mathfrak{Q}_C(f))\right] ,$$

*where*

$$\mathfrak{Q}_A(f) = \int_0^\infty \frac{d\alpha}{\pi} \left(\phi^*(\hat{\rho}_1 W_0)(\bar{f}_\alpha, f_\alpha) + 2\xi\alpha^2\phi^* W_0(\bar{f}_\alpha, f_\alpha)\right) ,$$

$$\mathfrak{Q}_B[f](\tau) = \frac{1-2\xi}{n-2}\mu^2 f^2(\tau) + 2\xi(f'(\tau))^2 ,$$

*and*

$$\mathfrak{Q}_C[f](\tau) = f^2(\tau)\xi\left(R_{\mu\nu}U^\mu U^\nu - \frac{2\xi}{n-2}R\right)(\tau) .$$

(CJ Fewster, E-A K, 2018)

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$$\langle :\rho_U:_{\omega_0}(f)\rangle_\omega(\gamma(\tau)) \geq -\left[\boxed{\mathfrak{Q}_A(f,\omega_0)}\mathbb{1} + \boxed{\langle :\Phi^2:_{\omega_0} \circ \gamma\rangle_\omega}\left(\boxed{\mathfrak{Q}_B(f)} + \boxed{\mathfrak{Q}_C(f)}\right)\right]$$

- $\mathfrak{Q}_A(f,\omega_0)$: *Dependence on $f$ and the reference state*
- $\mathfrak{Q}_B(f)$: *Dependence on $f$*
- $\langle :\Phi^2:_{\omega_0} \circ \gamma\rangle_\omega$: *Dependence on the state of interest and the reference state*
- $\mathfrak{Q}_C(f)$: *Curvature terms dependent on $f$*

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# The semiclassical Einstein equation

- The singularity theorems require a geometric assumption
- In the case of classical fields we can use the Einstein equation
- When we are treating quantum fields on a classical curved background we can instead use the semiclassical Einstein equation.

$$\langle :T_{\mu\nu}: \rangle_\omega = 8\pi G_{\mu\nu} \,.$$

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**Problem**

In order to use an QSEI and the SEE for general curved spacetimes we need a QSEI, where the EED is renormalized by subtracting the Hadamard parametrix.

## The semiclassical Einstein equation

There is evidence that in situations where the curvature is bounded we can find a uniform length which is small compared to local curvature length scales and then the Hadamard parametrix approximates that of flat spacetime.

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For $f$ supported only on this length scale $\tau_0$, $\xi = 0$, for even number of dimensions $n = 2m$ and if we restrict to a class of Hadamard states for which the field's magnitude is bounded

$$\int d\tau f^2(\tau) R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu \geq -\frac{8\pi S_{2m-2}}{2m(2\pi)^{2m}}||f^{(m)}||^2 - \frac{8\pi\mu^2\phi^2_{\max}}{2m-2}||f||^2 .$$

# Partition of unity

To discuss averages over long timescales we will use a partition of unity. We define bump functions $\phi_n$, where $\phi$ is supported on $(-\tau_0, \tau_0)$ we obtain a sum of integrals, each of which can be bounded

$$\int_{-\infty}^{\infty} R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f^2(\tau)d\tau \geq -\frac{4\pi S_{2m-2}}{m(2\pi)^{2m}}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}[(f\phi_n)^{(m)}]^2 d\tau - \frac{8\pi\mu^2\phi_{\max}^2}{2(m-1)}\|f\|^2 .$$

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$$\int_{-\infty}^{\infty} R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f^2(\tau)d\tau \geq -Q_m||f^{(m)}||^2 - Q_0||f||^2 .$$

- The $Q_m$ and $Q_0$ depend on $\phi_{\max}$, the mass, the number of dimensions and the maximum value of the bump function and its derivatives.
- It is exactly the form of the weakened energy condition for the Hawking-type singularity theorem.

# Conclusions and future directions

- Proved singularity theorems with weakened energy conditions using an alternative method that gives us information about the timescale of creation of the focal point
- Derived a QSEI for the non-minimally coupled scalar field and proved a singularity theorem with an energy condition derived by a QEI obeyed by the minimally coupled quantum scalar field

# Conclusions and future directions

- Proved singularity theorems with weakened energy conditions using an alternative method that gives us information about the timescale of creation of the focal point
- Derived a QSEI for the non-minimally coupled scalar field and proved a singularity theorem with an energy condition derived by a QEI obeyed by the minimally coupled quantum scalar field
- Prove an absolute (Hadamard renormalised) QSEI for spacetimes with curvature and verify that it satisfies the hypothesis of a singularity theorem (work in progress)
- Examine solutions of the semiclassical Einstein equation for cosmological spacetimes (work in progress with D. Siemssen)
- Penrose singularity theorem?