Perspectives on local covariance

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- The category of locally covariant theories
  Can QFT be analysed fruitfully at the functorial level?
- Physical content of local covariance
  Does it enforce the 'same physics in all spacetimes'?

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  Can QFT be analysed fruitfully at the functorial level?
- Physical content of local covariance
  Does it enforce the 'same physics in all spacetimes'?

Slogan: the Same Physics in All Spacetimes = SPASs.

Work in progress; partly in collaboration with R Verch.

Verstehen Sie SPASs?

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- If a theory represents the same physics in all spacetimes, it should represent the same physics in the same spacetime.
  Seek a generally covariant formulation.

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- If a theory represents the same physics in all spacetimes, it should represent the same physics in the same spacetime. Seek a generally covariant formulation.
- If two theories both represent SPASs and are equivalent in one spacetime, then they should be equivalent in all spacetimes.

Example:

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(\nabla^a\phi\nabla_a\phi - \xi R\phi^2\right)$$

Although the equation of motion and solution space are independent of $\xi \in \mathbb{R}$ in Ricci-flat spacetimes, these theories are distinguished by the stress-energy tensor.

However,

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(\nabla^a\phi\nabla_a\phi - \zeta(R)\phi^2\right)$$

with ζ smooth and vanishing in a neighbourhood of 0, cannot be distinguished from the $\zeta \equiv 0$ model in Ricci-flat spacetimes.

- SPASs can be ensured by restricting to actions depending analytically on the metric.
- Not a great problem when constructing models from known Lagrangians – use good taste and experience!
- More problematic in axiomatic settings.

In this talk we consider a restricted version of SPASs, only comparing theories by embedding one as a subtheory of the other.

Working definition:

A class of theories $\mathfrak{T}$ has the SPASs property if no proper subtheory $\mathscr{T}'$ of $\mathscr{T}$ in $\mathfrak{T}$ can fully account for the physics of $\mathscr{T}$ in any single spacetime.

Locally covariant QFT BFV: Brunetti, Fredenhagen, Verch (2003)

Define a locally covariant QFT to be a functor $\mathscr{A}$ from a category of spacetimes Man to a category of $*$-algebras Alg

Man

- **Objects** Globally hyperbolic spacetimes with orientation and time orientation.
- **Morphisms** Hyperbolic embeddings, i.e., isometric embeddings with causally convex image that preserve the (time)-orientation.

Alg

- **Objects** Unital $*$-algebras (or C^*, to taste...)
- **Morphisms** Unit-preserving $*$-monomorphisms

Unpacking the definition:

- To each spacetime $\mathbf{M}$ there is an algebra $\mathscr{A}(\mathbf{M})$ of observables
- To each hyperbolic embedding $\mathbf{M} \xrightarrow{\psi} \mathbf{N}$ there is

$$\mathscr{A}(\mathbf{M}) \xrightarrow{\mathscr{A}(\psi)} \mathscr{A}(\mathbf{N})$$

embedding the observables on $\mathbf{M}$ among the observables on $\mathbf{N}$ with

$$\mathscr{A}(\psi \circ \varphi) = \mathscr{A}(\psi) \circ \mathscr{A}(\varphi) \qquad \text{(covariance)}$$
$$\mathscr{A}(\mathrm{id}_{\mathbf{M}}) = \mathrm{id}_{\mathscr{A}(\mathbf{M})}$$

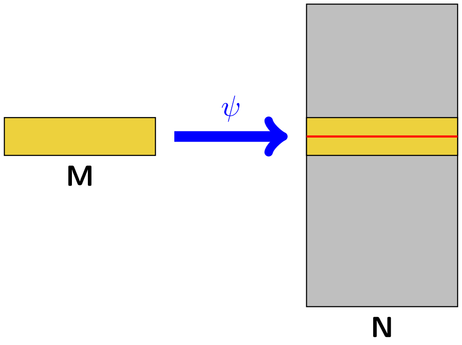
Successes of the locally covariant approach

- Spin and statistics – Verch
- Perturbation theory – Brunetti & Fredenhagen;
  Hollands & Wald
- Existence of a covariant stress-energy tensor – BFV
- Superselection theory – Ruzzi; Brunetti & Ruzzi
- Quantum (Energy) Inequalities – CJF & Pfenning; Marecki
- Reeh–Schlieder theorem – Sanders

Two key tools

$\psi : \mathbf{M} \to \mathbf{N}$ is Cauchy if $\psi(\mathbf{M})$ contains a Cauchy surface of $\mathbf{N}$.
A locally covariant theory $\mathscr{A}$ satisfies the time-slice axiom if

$$\psi \text{ is Cauchy} \implies \mathscr{A}(\psi) \text{ is an isomorphism}$$



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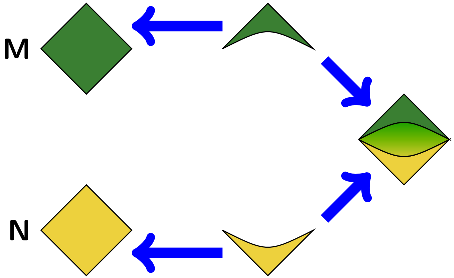
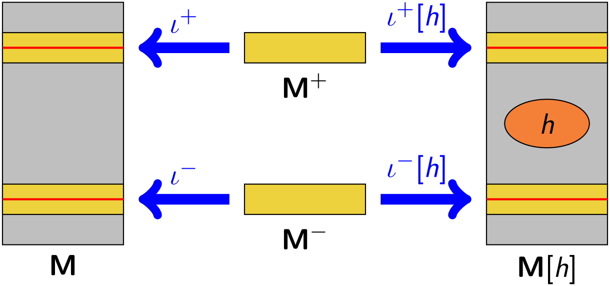
Any two spacetimes with homeomorphic Cauchy surfaces are linked by Cauchy morphisms. Fulling, Narcowich & Wald 1981



Relative Cauchy evolution and the stress-energy tensor



If $\mathscr{A}$ obeys timeslice, any metric perturbation h preserving global hyperbolicity defines an automorphism of $\mathscr{A}(\mathbf{M})$,

$$\mathrm{rce}_{\mathbf{M}}[h] = \mathscr{A}(\iota^-) \circ \mathscr{A}(\iota^-[h])^{-1} \circ \mathscr{A}(\iota^+[h]) \circ \mathscr{A}(\iota^+)^{-1}$$

whose functional derivative gives a stress-energy tensor:

$$[\mathrm{T}_{\mathbf{M}}(f), A] = 2i \left. \frac{d}{ds} \mathrm{rce}_{\mathbf{M}}[h(s)] A \right|_{s=0} \qquad f = \left. \frac{dh(s)}{ds} \right|_{s=0} .$$

The category of locally covariant theories

LCT	(= Fun(Man, Alg))
Objects	functors from Man to Alg
Morphisms	natural transformations

A natural transformation $\zeta : \mathscr{A} \stackrel{.}{\to} \mathscr{B}$ between theories $\mathscr{A}$ and $\mathscr{B}$ assigns to each $\mathbf{M}$ a morphism $\mathscr{A}(\mathbf{M}) \xrightarrow{\zeta_{\mathbf{M}}} \mathscr{B}(\mathbf{M})$ so that for each hyperbolic embedding ψ,

$$\zeta_{\mathbf{N}} \circ \mathscr{A}(\psi) = \mathscr{B}(\psi) \circ \zeta_{\mathbf{M}}$$

$$\begin{array}{ccc} \mathscr{A}(\mathbf{M}) & \xrightarrow{\zeta_{\mathbf{M}}} & \mathscr{B}(\mathbf{M}) \\ \downarrow \mathscr{A}(\psi) & & \downarrow \mathscr{B}(\psi) \\ \mathscr{A}(\mathbf{N}) & \xrightarrow{\zeta_{\mathbf{N}}} & \mathscr{B}(\mathbf{N}) \end{array}$$

Interpretation: ζ embeds $\mathscr{A}$ as a sub-theory of $\mathscr{B}$. If every $\zeta_{\mathbf{M}}$ is an isomorphism, ζ is an equivalence of $\mathscr{A}$ and $\mathscr{B}$.

Examples:

- Given any $\mathscr{A}$, define $\mathscr{A}^{\otimes k}$ by

$$\mathscr{A}^{\otimes k}(\mathbf{M}) = \mathscr{A}(\mathbf{M})^{\otimes k}, \qquad \mathscr{A}^{\otimes k}(\psi) = \mathscr{A}(\psi)^{\otimes k}$$

  i.e., k independent copies of $\mathscr{A}$. Then

$$\eta^{k,l}_{\mathbf{M}} : \mathscr{A}^{\otimes k}(\mathbf{M}) \to \mathscr{A}^{\otimes l}(\mathbf{M})$$
$$A \mapsto A \otimes \mathbf{1}^{\otimes (l-k)}_{\mathscr{A}(\mathbf{M})}$$

  defines a natural $\eta^{k,l} : \mathscr{A}^{\otimes k} \stackrel{.}{\to} \mathscr{A}^{\otimes l}$ for $k \leq l$.
  Naturally, $\eta^{k,m} = \eta^{l,m} \circ \eta^{k,l}$ if $k \leq l \leq m$.

- Theories with distinct mass spectra in Minkowski space are inequivalent.

Some immediate questions:

- Is there any operational content to the morphisms of LCT? Can two morphisms be distinguished on the basis of their action in a single spacetime?
- How large can the set of morphisms between two theories be? Can the hom-sets be computed in concrete cases?

Suppose $\zeta, \zeta' : \mathscr{A} \stackrel{.}{\to} \mathscr{B}$ and $\zeta_{\mathbf{M}} = \zeta'_{\mathbf{M}}$ for some $\mathbf{M}$. Then:

- if $\mathbf{L} \xrightarrow{\psi} \mathbf{M}$ then $\zeta_{\mathbf{L}} = \zeta'_{\mathbf{L}}$
- if $\mathbf{M} \xrightarrow{\varphi} \mathbf{N}$ is Cauchy and $\mathscr{A}$ obeys timeslice then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$
- if $\mathscr{A}$ obeys timeslice and $\mathbf{M}$ and $\mathbf{N}$ have homeomorphic Cauchy surfaces then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$

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$$\mathscr{B}(\psi) \circ \zeta_{\mathbf{L}} = \zeta_{\mathbf{M}} \circ \mathscr{A}(\psi) = \zeta'_{\mathbf{M}} \circ \mathscr{A}(\psi) = \mathscr{B}(\psi) \circ \zeta'_{\mathbf{L}}$$

  and $\mathscr{B}(\psi)$ is monic.
- if $\mathbf{M} \xrightarrow{\varphi} \mathbf{N}$ is Cauchy and $\mathscr{A}$ obeys timeslice then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$
- if $\mathscr{A}$ obeys timeslice and $\mathbf{M}$ and $\mathbf{N}$ have homeomorphic Cauchy surfaces then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$

Suppose $\zeta, \zeta' : \mathscr{A} \stackrel{.}{\to} \mathscr{B}$ and $\zeta_{\mathbf{M}} = \zeta'_{\mathbf{M}}$ for some **M**. Then:

- if $\mathbf{L} \xrightarrow{\psi} \mathbf{M}$ then $\zeta_{\mathbf{L}} = \zeta'_{\mathbf{L}}$

$$\mathscr{B}(\psi) \circ \zeta_{\mathbf{L}} = \zeta_{\mathbf{M}} \circ \mathscr{A}(\psi) = \zeta'_{\mathbf{M}} \circ \mathscr{A}(\psi) = \mathscr{B}(\psi) \circ \zeta'_{\mathbf{L}}$$

  and $\mathscr{B}(\psi)$ is monic.

- if $\mathbf{M} \xrightarrow{\varphi} \mathbf{N}$ is Cauchy and $\mathscr{A}$ obeys timeslice then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$

$$\zeta_{\mathbf{N}} \circ \mathscr{A}(\varphi) = \mathscr{B}(\varphi) \circ \zeta_{\mathbf{M}} = \mathscr{B}(\varphi) \circ \zeta'_{\mathbf{M}} = \zeta'_{\mathbf{N}} \circ \mathscr{A}(\varphi)$$

  and $\mathscr{A}(\varphi)$ is an isomorphism, hence epic.

- if $\mathscr{A}$ obeys timeslice and **M** and **N** have homeomorphic Cauchy surfaces then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$

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- if $\mathbf{M} \xrightarrow{\varphi} \mathbf{N}$ is Cauchy and $\mathscr{A}$ obeys timeslice then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$

$$\zeta_{\mathbf{N}} \circ \mathscr{A}(\varphi) = \mathscr{B}(\varphi) \circ \zeta_{\mathbf{M}} = \mathscr{B}(\varphi) \circ \zeta'_{\mathbf{M}} = \zeta'_{\mathbf{N}} \circ \mathscr{A}(\varphi)$$

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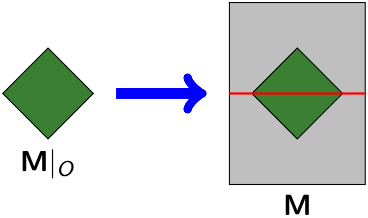
- if $\mathscr{A}$ obeys timeslice and $\mathbf{M}$ and $\mathbf{N}$ have homeomorphic Cauchy surfaces then $\zeta_{\mathbf{N}} = \zeta'_{\mathbf{N}}$
  Using spacetime deformation, there are Cauchy morphisms

$$\mathbf{M} \leftarrow \mathbf{M}' \to \mathbf{M}'' \leftarrow \mathbf{M}''' \to \mathbf{N}$$

A diamond O of $\mathbf{M}$ is the domain of determinacy of an open ball w.r.t. local coordinates in a Cauchy surface of $\mathbf{M}$.



Consider O as a spacetime $\mathbf{M}|_O$, with inclusion $\iota_O : \mathbf{M}|_O \to \mathbf{M}$.

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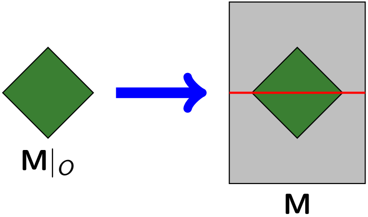

Consider O as a spacetime $\mathbf{M}|_O$, with inclusion $\iota_O : \mathbf{M}|_O \to \mathbf{M}$. Then

$$\zeta_{\mathbf{M}} = \zeta'_{\mathbf{M}} \quad \Longrightarrow \quad \zeta_{\mathbf{M}|_O} = \zeta'_{\mathbf{M}|_O}$$

But all diamonds have homeomorphic Cauchy surfaces:

$$\zeta_{\mathbf{M}} = \zeta'_{\mathbf{M}} \quad \Longrightarrow \quad \zeta_{\widetilde{\mathbf{M}}|_{\widetilde{O}}} = \zeta'_{\widetilde{\mathbf{M}}|_{\widetilde{O}}}$$

where $\widetilde{O}$ is a diamond of any spacetime $\widetilde{\mathbf{M}}$.

If $\mathscr{A}$ also obeys additivity in the form

$$\mathscr{A}(\widetilde{\mathbf{M}}) = \bigvee_{O} \mathscr{A}(\iota_O)(\mathscr{A}(\widetilde{\mathbf{M}}|_O))$$

where O ranges over the diamonds of $\widetilde{\mathbf{M}}$, then

$$\zeta_{\mathbf{M}} = \zeta'_{\mathbf{M}} \quad \Longrightarrow \quad \zeta = \zeta'$$

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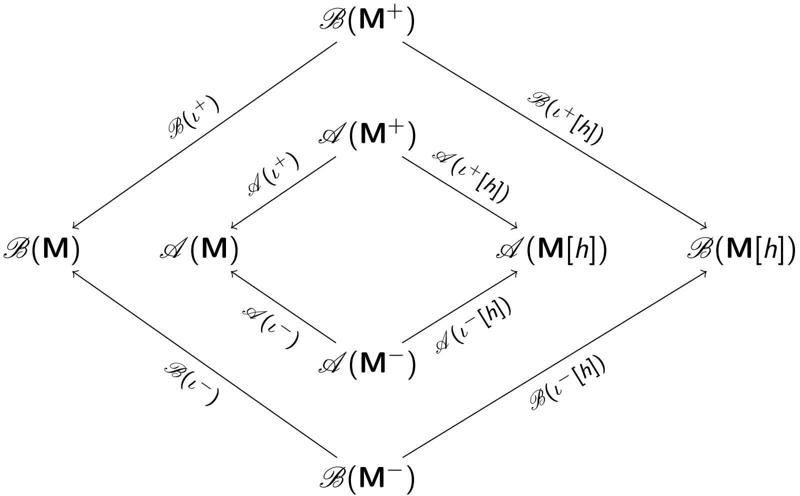
This result is supplemented by a strong constraint from the r.c.e.:

$$\mathrm{rce}_{\mathbf{M}}^{(\mathscr{B})}[h] \circ \zeta_{\mathbf{M}} = \zeta_{\mathbf{M}} \circ \mathrm{rce}_{\mathbf{M}}^{(\mathscr{A})}[h]$$

for all hyperbolic perturbations h of $\mathbf{M}$.

$$\mathrm{rce}_{\mathbf{M}}^{(\mathscr{B})}[h] \circ \zeta_{\mathbf{M}} = \zeta_{\mathbf{M}} \circ \mathrm{rce}_{\mathbf{M}}^{(\mathscr{A})}[h]$$



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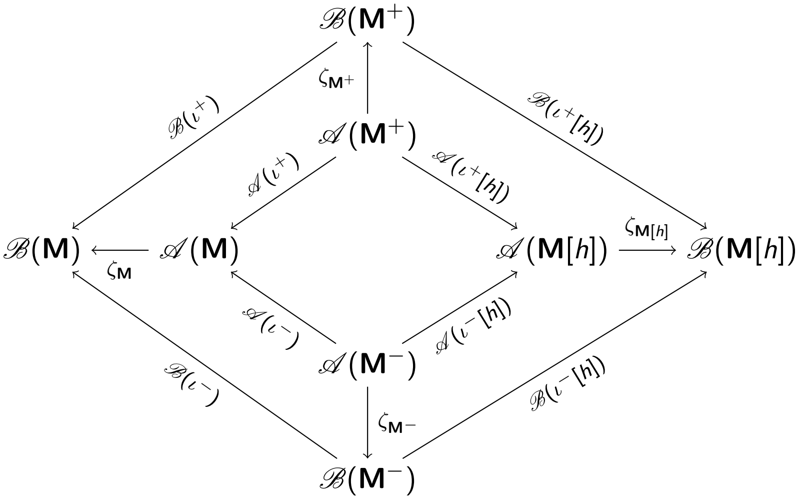

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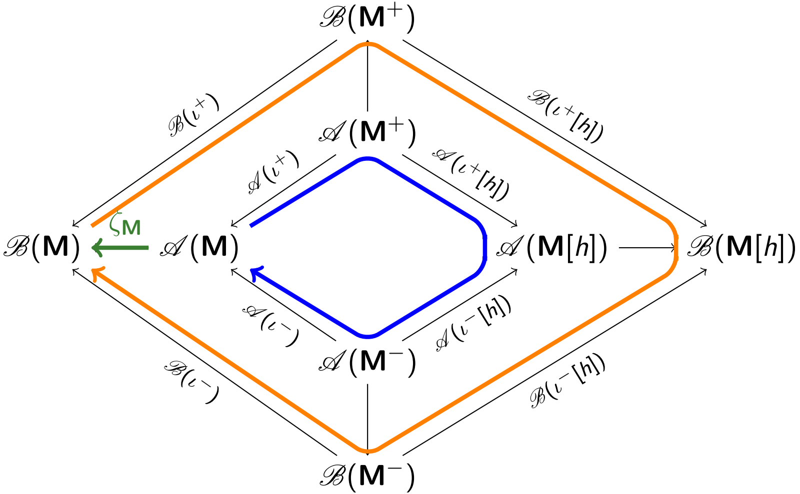

Summarising:

Theorem

If $\mathscr{A}$ and $\mathscr{B}$ obey timeslice and $\mathscr{A}$ is additive then

- $\zeta : \mathscr{A} \stackrel{.}{\to} \mathscr{B}$ *is uniquely determined by any of its components*
- $rce_{\mathbf{M}}^{(\mathscr{B})}[h] \circ \zeta_{\mathbf{M}} = \zeta_{\mathbf{M}} \circ rce_{\mathbf{M}}^{(\mathscr{A})}[h]$
- *if the stress-energy tensors exist as derivations*

$$[\mathrm{T}_{\mathbf{M}}^{(\mathscr{B})}(f), \zeta_{\mathbf{M}} A] = \zeta_{\mathbf{M}}[\mathrm{T}_{\mathbf{M}}^{(\mathscr{A})}(f), A]$$

for all $A \in \mathscr{A}(\mathbf{M})$ and symmetric C_0^∞-tensors f.

Computation in concrete examples now becomes possible.

Example: For $\mathscr{A}$ = real Klein–Gordon field of mass m,

$$\mathrm{End}(\mathscr{A}^{\otimes N}) = \mathrm{Aut}(\mathscr{A}^{\otimes N})$$

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$$(\zeta_R \quad \Phi)_{\mathbf{M}}(f) = \Phi_{\mathbf{M}}(Rf)$$

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$$(\zeta_{R,\alpha}\Phi)_{\mathbf{M}}(f) = \Phi_{\mathbf{M}}(Rf) + \left(\int d\mathrm{vol}_{\mathbf{M}}\, \alpha^T f\right) \mathbf{1}_{\mathscr{A}(\mathbf{M})}$$

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Interpretation & consequences:

- $\mathrm{Aut}(\mathscr{A})$ is the global gauge group of 'field functor' $\mathscr{A}$
- $\mathrm{Aut}(\mathscr{A})$ is trivial for an 'observable functor'.
- Linear fields of the theory appear in $\mathrm{Aut}(\mathscr{A})$-multiplets.
- Superselection theory at the functorial level?
  (Complementary to Ruzzi/Brunetti–Ruzzi results)

Fun with functors

Return to the question of whether local covariance implies SPASs.

Any functor $\varphi : \mathsf{Man} \to \mathsf{LCT}$, i.e.,

$$\varphi \in \mathsf{Fun}(\mathsf{Man}, \mathsf{Fun}(\mathsf{Man}, \mathsf{Alg}))$$

is a locally covariant choice of locally covariant theory.

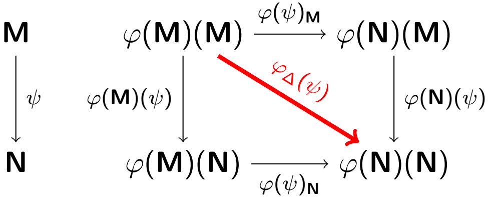
- Each $\varphi(\mathbf{M})$ is a theory $\varphi(\mathbf{M}) \in \mathsf{LCT}$
- Each hyperbolic embedding ψ corresponds to an embedding $\varphi(\psi)$ of $\varphi(\mathbf{M})$ as a sub-theory of $\varphi(\mathbf{N})$.

We use φ to define a diagonal theory $\varphi_\Delta \in \mathsf{LCT}$.

Diagonal theories

Given $\varphi \in \mathrm{Fun}(\mathrm{Man}, \mathrm{LCT})$, define, for spacetime $\mathbf{M}$ and hyperbolic embedding $\psi : \mathbf{M} \to \mathbf{N}$

$$\varphi_\Delta(\mathbf{M}) = \varphi(\mathbf{M})(\mathbf{M}) \qquad \varphi_\Delta(\psi) = \varphi(\psi)_{\mathbf{N}} \circ \varphi(\mathbf{M})(\psi)$$



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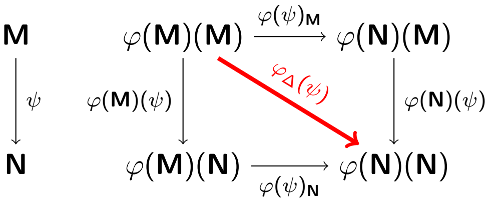

φ_Δ is a functor and therefore defines a locally covariant theory!

Example:

- Write $\Sigma(\mathbf{M})$ = Cauchy surface of $\mathbf{M}$/homeomorphisms
- Let $\mu : \mathsf{Man} \to \mathbb{N} = \{1, 2, \ldots\}$ be a topological invariant of Cauchy surfaces s.t. $\mu(\mathbf{M}) = 1$ if $\Sigma(\mathbf{M})$ is noncompact.
- Set

$$\varphi(\mathbf{M}) = \mathscr{A}^{\otimes\mu(\mathbf{M})} \qquad \varphi(\mathbf{M} \xrightarrow{\psi} \mathbf{N}) = \eta^{\mu(\mathbf{M}),\mu(\mathbf{N})}$$

Then $\varphi \in \mathrm{Fun}(\mathsf{Man}, \mathsf{LCT})$.

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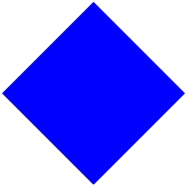
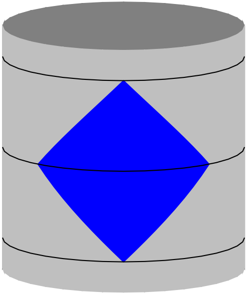
Then $\varphi \in \mathsf{Fun}(\mathsf{Man}, \mathsf{LCT})$.

Key point: If $\Sigma(\mathbf{M})$ is compact and $\exists\ \mathbf{M} \to \mathbf{N}$ then $\Sigma(\mathbf{M}) = \Sigma(\mathbf{N})$. Thus $\mu(\mathbf{M}) \le \mu(\mathbf{N})$ whenever $\mathbf{M} \to \mathbf{N}$; functorial properties follow immediately from properties of $\eta^{k,l}$.

E.g. $$\mu(\mathbf{M}) = \begin{cases} 1 & \Sigma(\mathbf{M}) \text{ noncompact} \\ 2 & \text{otherwise} \end{cases}$$



$\varphi_\Delta(\mathbf{D}) = \mathscr{A}(\mathbf{D})$



$\varphi_\Delta(\mathbf{C}) = \mathscr{A}^{\otimes 2}(\mathbf{C})$

The subtheory embeddings $\mathscr{A} \stackrel{.}{\longrightarrow} \varphi_\Delta \stackrel{.}{\longrightarrow} \mathscr{A}^{\otimes 2}$ are isomorphisms in some spacetimes but not in others. SPASs fails in LCT.

Properties of φ_Δ

1. φ_Δ is more than one copy of $\mathscr{A}$, but less than two!

$$\mathscr{A} \stackrel{\cdot}{\longrightarrow} \varphi_\Delta \stackrel{\cdot}{\longrightarrow} \mathscr{A}^{\otimes 2}$$

2. φ_Δ shares the timeslice and causality properties with the underlying theory $\mathscr{A}$.

3. For any diamond O of any spacetime, $\varphi_\Delta(\mathbf{M}|_O) = \mathscr{A}(\mathbf{M}|_O)$.

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2. φ_Δ shares the timeslice and causality properties with the underlying theory $\mathscr{A}$.

3. For any diamond O of any spacetime, $\varphi_\Delta(\mathbf{M}|_O) = \mathscr{A}(\mathbf{M}|_O)$.
   - In particular φ_Δ is not additive.
   - The local algebras are insensitive to the ambient algebra
   - Is φ_Δ really just one copy of $\mathscr{A}$ in disguise?

Relative Cauchy evolution again

The intertwining rule $\mathrm{rce}_{\mathbf{M}}^{(\mathscr{B})}[h] \circ \zeta_{\mathbf{M}} = \zeta_{\mathbf{M}} \circ \mathrm{rce}_{\mathbf{M}}^{(\mathscr{A})}[h]$ implies

$$\mathrm{rce}_{\mathbf{M}}^{(\varphi_\Delta)}[h] = (\mathrm{rce}_{\mathbf{M}}^{(\varphi)}[h])_{\mathbf{M}} \circ \mathrm{rce}_{\mathbf{M}}^{(\varphi(\mathbf{M}))}[h]$$

where $\mathrm{rce}_{\mathbf{M}}^{(\varphi)}[h] \in \mathrm{Aut}(\varphi(\mathbf{M}))$ is the r.c.e. of φ in LCT!
If $\mathrm{rce}^{(\varphi)}$ is trivial and stress-energy tensors exist:

$$[\mathrm{T}_{\mathbf{M}}^{(\varphi_\Delta)}(f), A] = [\mathrm{T}_{\mathbf{M}}^{(\varphi(\mathbf{M}))}(f), A]$$

E.g., in the example above:

$$\mathrm{T}_{\mathbf{D}}^{(\varphi_\Delta)}(f) = \mathrm{T}_{\mathbf{D}}^{(\mathscr{A})}(f); \qquad \mathrm{T}_{\mathbf{C}}^{(\varphi_\Delta)}(f) = \mathrm{T}_{\mathbf{C}}^{(\mathscr{A}\otimes\mathscr{A})}(f)$$

R.c.e. detects ambient degrees of freedom missed by local algebras.

Intrinsic local algebras

For any compact set K define

$$\mathscr{A}^\bullet(\mathbf{M}; K) = \{A \in \mathscr{A}(\mathbf{M}) : \mathrm{rce}_{\mathbf{M}}[h]A = A \quad \text{for all } h \in H(\mathbf{M}; K^\perp)\}$$

i.e., the subalgebra insensitive to spacetime geometry in the causal complement $K^\perp$.

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i.e., the subalgebra insensitive to spacetime geometry in the causal complement $K^\perp$.

For any open O, define

$$\mathscr{A}^{\mathrm{int}}(\mathbf{M}; O) = \bigvee_{K \subset O} \mathscr{A}^\bullet(\mathbf{M}; K)$$

running over compact K with a diamond neighbourhoods.

Intrinsic local algebras

For any compact set K define

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i.e., the subalgebra insensitive to spacetime geometry in the causal complement $K^{\perp}$.

For any open O, define

$$\mathscr{A}^{\mathrm{int}}(\mathbf{M};O) = \bigvee_{K\subset O} \mathscr{A}^{\bullet}(\mathbf{M};K)$$

running over compact K with a diamond neighbourhoods.

Say that $\mathscr{A}$ is strongly local if

$$\mathscr{A}^{\mathrm{int}}(\mathbf{M};O) = \mathscr{A}^{\mathrm{ext}}(\mathbf{M};O) \stackrel{\mathrm{def}}{=} \mathscr{A}(\iota_O)(\mathscr{A}(\mathbf{M}|_O))$$

for all open globally hyperbolic subsets O of $\mathbf{M}$.

Consequences of strong locality

- Additivity: if $\mathscr{A}$ is strongly local then

$$\mathscr{A}(\mathbf{M}) = \bigvee_O \mathscr{A}^{\rm int}(\mathbf{M}; O) = \bigvee_O \mathscr{A}^{\rm ext}(\mathbf{M}; O)$$

  for O running over the diamonds of $\mathbf{M}$.
- SPASs: Suppose
  - $\mathscr{A}$ and $\mathscr{B}$ are strongly local,
  - $\zeta : \mathscr{A} \stackrel{.}{\to} \mathscr{B}$ and
  - $\zeta_{\mathbf{M}}$ is an isomorphism for some $\mathbf{M}$

  Then ζ is an equivalence.

The category of strongly local theories has the SPASs property.

Strongly local diagonal theories

Suppose φ_Δ is a diagonal theory (with rce^φ trivial) such that φ_Δ and every $\varphi(\mathbf{M})$ are strongly local.

Then

- $\varphi(\mathbf{M}) \cong \varphi(\mathbf{N})$ for all $\mathbf{M}, \mathbf{N} \in \mathsf{Man}$
- $\varphi_\Delta \cong \mathscr{A}$ for $\mathscr{A}$ with

$$\mathscr{A}(\psi) = \eta(\psi)_{\mathbf{N}} \circ \varphi(\mathbf{M}_0)(\psi)$$

  for every $\psi : \mathbf{M} \to \mathbf{N}$, where $\eta \in \mathsf{Fun}(\mathsf{Man}, \mathrm{Aut}(\varphi(\mathbf{M}_0)))$ and $\mathbf{M}_0 \in \mathsf{Man}$ is arbitrary.
- If $\mathrm{Aut}(\varphi(\mathbf{M}_0))$ is trivial, then $\varphi_\Delta \cong \varphi(\mathbf{M}_0)$.

Example The Klein–Gordon theory of mass $m > 0$ is strongly local:

$$\mathscr{A}^\bullet(\mathbf{M}; K) = \text{subalgebra of } \mathscr{A}(\mathbf{M}) \text{ generated by } \Phi_{\mathbf{M}}(f) \text{ with } \operatorname{supp} f \subset K$$

$$\mathscr{A}^{\mathrm{int}}(\mathbf{M}; O) = \text{subalgebra of } \mathscr{A}(\mathbf{M}) \text{ generated by } \Phi_{\mathbf{M}}(f) \text{ with } \operatorname{supp} f \subset O$$

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Complication: if $m = 0$ and $\Sigma(\mathbf{M})$ is compact then

$$\mathscr{A}^{\rm int}(\mathbf{M}; O) = \mathscr{A}^{\rm ext}(\mathbf{M}; O) \bigvee \langle A_0 \rangle$$

where A_0 is the 'pure gauge' generator induced by the classical solution $\phi \equiv 1$.
Related to other pathologies (e.g., absence of static ground state).

Summary and outlook

Summary

Local covariance does not enforce SPASs, but does open new ways of analysing QFT.

- Global gauge group
- Diagonal theories
- Strong locality
- Key role of the r.c.e. and stress-tensor.

Outlook

- Analysis at the functorial level:
  QFT in CST without the spacetimes?
- Superselection theory?
- Cohomology of Man

Traditionally: QFT in CST is 'hard' because of the absence of global symmetries available in Minkowksi space.

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But perhaps: QFT in Minkowski space is ‘hard’ because of the absence of the stress-energy tensor available in locally covariant QFT in CST.