Perturbative Quantum Field Theory and Vertex Algebras

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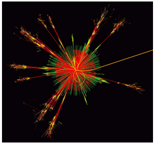

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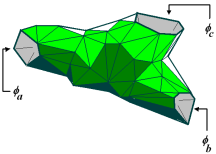

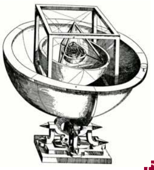

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Outline

Different approaches to QFT:

- Path-integral: $Z[j] = \int d\phi \exp(-iS/\hbar + \langle j, \phi \rangle)$. Intuitive, easy to remember, relation to statistical mechanics ($t \to i\tau$), "classical mathematics" tools.
- S-matrix: Clear-cut relation to scattering experiments, perturbative formulation, graphical representation.
- Algebraic approaches: View algebraic relations between quantized fields as the essential information. *This talk:* Encode relations into OPE/consistency conditions.


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Main tool in my approach: OPE

General formula: [Wilson, Zimmermann 1969, ..., S.H. 2006]

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_\Psi \sim \sum_{\phi_b} \underbrace{C^b_{a_1\dots a_n}(x_1,\dots,x_n)}_{\text{OPE}-\text{coefficients}\leftrightarrow\text{structure``constants''}} \langle \phi_b(x_n)\rangle_\Psi$$

- **Physical idea:** Separate the short distance regime of theory (large "energies") from the energy scale of the state (small) $E^4 \sim \langle\rho\rangle_\Psi$.
- **Application:** OPE-coefficients may be calculated within perturbation theory (Yang-Mills-type theories) $\longrightarrow$ applications deep inelastic scattering in QCD.

Axiomatization of QFT

I propose to **axiomatize** quantum field theory as a collection of fields (vectors in an abstract vector space V) and operator product coefficients $C(x_1, \dots, x_n) : V \otimes \cdots \otimes V \to V$, each of with is an analytic function on $(\mathbb{R}^D)^n \setminus \{\text{diagonals}\}$, subject to

- Covariance
- Local (anti-) commutativity
- Analyticity (Euclidean framework)
- Consistency (Associativity)
- Hermitian adjoint

Consequences:

- New intrinsic formulation of perturbation theory
- Constructive tool

- Considering the product of quantum fields at three different spacetime points, associativity of the field operators, $\phi_a(x_1)\,(\phi_b(x_2)\phi_c(x_3)) = (\phi_a(x_1)\phi_b(x_2))\,\phi_c(x_3)$, yields the *consistency condition*

$$\sum_c C_{ac}^{e}(x_1,x_3)C_{bd}^{c}(x_2,x_3) = \sum_c C_{ab}^{c}(x_1,x_2)C_{cd}^{e}(x_2,x_3)$$

  on domain $D_3 = \{r_{12} < r_{23} < r_{13}\}$.
- Idea: Elevate the OPE to an axiom of QFT, i.e. define a QFT by a set of coefficients $C_{ab}^{c}(x,y)$ satisfying the consistency condition (among other axioms)

Mathematical formulation of the consistency condition:

Postulate that

$$C(x_2, x_3)\Big(C(x_1, x_2) \otimes id\Big) = C(x_1, x_3)\Big(id \otimes C(x_2, x_3)\Big) ,$$

Here, we view $C(x_1, x_2)$ abstractly as a mapping $V \otimes V \to V$ ("index-free notation"), where V is the space of all composite fields of the given theory. The above equation is valid in the sense of analytic functions on domain $D_3 = \{r_{12} < r_{23} < r_{13}\}$.

Key Idea: The mappings $C(x_1, x_2, \dots)$ *define* (and hence *determine*) the quantum field theory!

Coherence theorem: *All "higher order" C's and consistency conditions follow from this one.* (Analogy $(AB)C = A(BC)$ implies "higher associativity" conditions such as $(AB)(CD) = (A(BC))D$ etc. in ordinary algebra).

Perturbation theory

Suppose we have a family of QFT’s depending on parameter:

- Coupling parameter: λ.
- ’t Hooft limit: $\epsilon = 1/N$.
- Classical limit: $\hbar$-expansion.

Expand OPE-coefficients:

$$C_i(x_1, x_2) := \frac{d^i}{d\lambda^i} C(x_1, x_2; \lambda)\Big|_{\lambda=0} .$$

Then C_i should satisfy *order by order* version of consistency condition. Lowest order condition *determines* higher order ones.

$\Longrightarrow$ Conditions have formulation in terms of *Hochschild cohomology*.

Idea:

Express perturbative consistency condition in term of differential. Let

$$f_n(x_1, \dots, x_n) : V \otimes \dots \otimes V \to V, \quad (x_1, \dots, x_n) \in D_n \,.$$

We next introduce a boundary operator b on such objects by the formula

$$(bf_n)(x_1, \dots, x_{n+1}) := C_0(x_1, x_{n+1})(id \otimes f_n(x_2, \dots, x_n))$$
$$+ \sum_{i=1}^{n} (-1)^i f_n(x_1, \dots, \widehat{x}_i, \dots, x_{n+1})(id^{i-1} \otimes C_0(x_i, x_{i+1}) \otimes id^{n-i})$$
$$+ (-1)^n \, C_0(x_n, x_{n+1})(f_n(x_1, \dots, x_n) \otimes id) \,.$$

A calculation reveals $b^2 = 0$.

1. The first order consistency condition states that C_1 must satisfy $bC_1 = 0$.
2. If C_1 arises from a field redefinition (a map $z : V \to V$), then this means that $C_1 = bz_1$.

$$\Longrightarrow \{\text{1st order perturbations } C_1\} \cong H^2(b) = \ker b / \operatorname{ran} b$$

3. At i-th order, we get a condition of the form $bC_i = w_i$, where $bw_i = 0$, which we want to solve for C_i (with w_i defined by lower order perturbations).

$$\Longrightarrow i\text{th order obstruction } w_i \in H^3(b) = \ker b / \operatorname{ran} b$$

Gauge theories

For gauge theories (e.g. Yang-Mills) need a further modification: *BRST symmetry*
(e.g. Yang-Mills: $sA = du - i\lambda[A, u], su = \lambda/2i\,[u, u]$, ...)

BRST-transformation defines map $s(\lambda) : V \to V$. Must satisfy compatibility condition (γ is $\mathbb{Z}_2$-grading for bosons/fermions)

$$sC(x_1, x_2) = C(x_1, x_2)\big(s \otimes id + \gamma \otimes s\big)\,.$$

Expand:

$$s_i := \frac{d^i}{d\lambda^i} s(\lambda)\bigg|_{\lambda=0}.$$

Then s_i, C_i should satisfy *order by order* version of compatibility condition.

$\Longrightarrow$ Conditions can be reformulated in terms of modified Hochschild cohomology: Define new differential B by

$$\begin{aligned} & (Bf_n)(x_1, \dots, x_n) \\ := \quad & sf_n(x_1, \dots, x_n) - \sum_{i=1}^{n} f_n(x_1, \dots, x_n)(\gamma^{i-1} \otimes s \otimes id^{n-i}) \,. \end{aligned}$$

Then one can prove

$$B^2 = 0 = \{b, B\} \,,$$

so $\delta = b + B$ defines new differential. We can then discuss associativity and BRST condition simultaneously for C_i, s_i in terms of δ.

Connection to *Vertex algebras* arises as follows:

We view this set of coefficients as matrix elements of operators $Y(x, \phi_a)$ on the space V spanned by the fields ϕ_a:

$$C_{ab}^{c}(x) = \langle \phi_c \, | Y(\phi_a, x) | \, \phi_b \rangle$$

This is very useful to construct the OPE in non-trivial perturbative QFT's! (rest of this talk). From now: $\phi_a \to a$.

OPE vertex algebras

Axioms imply that Y satisfy axioms of a "vertex algebra" :

An *OPE vertex algebra* is a 4-tuple $(V, Y, \nabla^\mu, |0\rangle)$, where V is a vector space, $\nabla^\mu \in \mathrm{End}(V)$ a derivation, $\mu = 1, ..., D$, $|0\rangle \in V$, and $Y : V \to \text{End}(V) \otimes \mathcal{O}(\mathbb{R}^D \setminus \{\text{diagonals}\})$, linear in V, satisfying:

- Vacuum: $Y(x, 1) = \mathbf{1}_V, \ \nabla^\mu |0\rangle = 0, \ Y(x, a)|0\rangle = a + O(x)$
- Compatible derivations: $Y(x, \nabla^\mu a) = \partial^\mu Y(x, a)$
- Euclidean invariance
- Consistency condition: $Y(x, a)Y(y, b) = Y(y, Y(x - y, a)b)$ for $|x| > |y| > |x - y|$
- Quasisymmetry: $Y(x, a)b = \exp(x \cdot \nabla)Y(-x, b)a$
- Scaling degree: $\mathrm{sd}_{x=0} Y(x, a) \leq \dim(a)$

How to construct OPE vertex algebras?

- How to characterize a QFT? E.g. by a classical field equation:
$$\Box\varphi = \lambda\varphi^3$$
- This yields some restrictions on the OPE coefficients and thus on the vertex operators:
$$\Box Y(x,\varphi) = \lambda Y(x,\varphi^3)$$
- We want to exploit these relations and develop an iterative construction scheme

Construction of OPE vertex algebras

Perturbative construction of the QFT associated to the scalar field satisfying the field equation $\Box\varphi = \lambda\varphi^3$:

Construct (formal) power series of vertex operators

$$Y(x,a) = \sum_{i=0}^{\infty} \lambda^i Y_i(x,a) \quad \text{satisfying} \tag{1}$$

a) the field equation,

$$\Box Y(x,\varphi) = \lambda Y(x,\varphi^3) \quad \Leftrightarrow \quad \Box Y_i(x,\varphi) = Y_{i-1}(x,\varphi^3)$$

b) the consistency condition,

$$\sum_{k=0}^{i} Y_k(x,a) Y_{i-k}(x,b) = \sum_{k=0}^{i} Y_k(y, Y_{i-k}(x-y,a)b)$$

Computing higher order vertex operators

- Start with the vertex operators of the free field (0-th order vertex operators)
- Invert the field equation to get to the next order: $Y_1(x, \varphi) = \Box^{-1} Y_0(x, \varphi^3)$
- Use the consistency condition in the limit $x \to y$ to find the first order vertex operators with non-linear vector arguments:

$$Y_1(x, \varphi^2) = \lim_{y \to x} \Big\{ Y_1(y, \varphi) Y_0(x, \varphi) - \sum_a \langle a | Y_1(y - x, \varphi) | \varphi \rangle \, Y_0(x, a) + (0 \leftrightarrow 1) \Big\}$$

or, more generally,

$$Y_i(x, a \cdot b) = \lim_{y \to x} \sum_{j=0}^{i} Y_j(y, a) Y_{i-j}(x, b) - \text{“counterterms”}$$

The Euclidean free field

Consider the Euclidean free field $\varphi(x)$ in $D \geq 3$ dimensions (Schwinger two point-function $G(x, y) = |x - y|^{2-D}$). We define the corresponding OPE vertex algebra:

- V=unital, free commutative ring generated by $\mathbf{1}$, φ and its symmetric trace free derivatives,

$$\partial^{l,m}\varphi = c_l \mathrm{S}_{l,m}(\partial)\varphi,$$

  where $\mathrm{S}_{l,m}(\hat{x})$ are the spherical harmonics in D dimensions and c_l is some normalisation constant.
- We introduce creation and annihilition operators on V,

$$\mathbf{b}^+_{l,m}|\mathbf{1}\rangle = \partial^{l,m}\varphi, \quad \mathbf{b}_{l,m}|\mathbf{1}\rangle = 0, \quad \left[\mathbf{b}_{l,m}, \mathbf{b}^+_{l',m'}\right] = 1$$

Y can be read off the OPE of the free field normal ordered products

$$Y(x,\varphi) = \text{const.}\, r^{-(D-2)/2} \sum_{l=0}^{\infty} \sum_{m=1}^{N(l,D)} \frac{1}{\sqrt{\omega(D,l)}} \times$$
$$\left[r^{l+(D-2)/2} \mathrm{S}_{l,m}(\hat{x})\, \mathbf{b}^{+}_{l,m} + r^{-l-(D-2)/2} \overline{\mathrm{S}_{l,m}(\hat{x})}\, \mathbf{b}_{l,m} \right]$$

- $r = |x|$,
- $\omega(l,D) = 2l + D - 2$
- $N(l,D) =$ number of linearly independent spherical harmonics $\mathrm{S}_{l,m}(\hat{x})$ of degree l in D dimensions

1. Calculate the first order vertex operator $Y_1(x, \varphi)$:

$$Y_1(x, \varphi) = \Box^{-1} Y_0(x, \varphi^3)$$

2. Use the consistency condition to find $Y_1(x, \varphi^2)$ and $Y_1(x, \varphi^3)$
3. Go to 2nd order by $Y_2(x, \varphi) = \Box^{-1} Y_1(x, \varphi^3)$, and so on
4. vertex operators $Y_j(x, \varphi^p)$, $p > 3$ can also be calculated by using the consistency condition

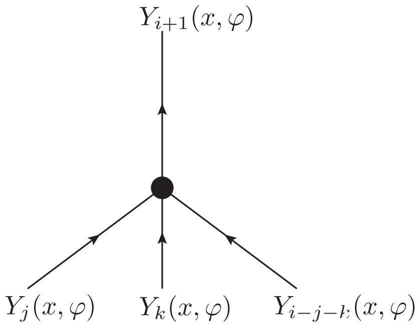
Formula for the iteration step :

$$
\begin{aligned}
Y_{i+1}(x,\varphi) &= \Box^{-1} Y_i(x,\varphi^3) \\
&= \Box^{-1} \lim_{y\to x} \left[\sum_{j=0}^{i} Y_j(y,\varphi) Y_{i-j}(x,\varphi^2) - \text{counterterms} \right] \\
&= \Box^{-1} \lim_{y_1\to x} \left[\sum_{j=0}^{i} Y_j(y_1,\varphi) \lim_{y_2\to x} \left[\sum_{k=0}^{i-j} Y_k(y_1,\varphi) Y_{i-j-k}(x,\varphi) \right.\right. \\
&\qquad \left.\left. - \text{more counterterms} \right] - \text{counterterms} \right]
\end{aligned}
$$

Dropping the counterterms and limits for the moment, this reads

$$
Y_{i+1}(x,\varphi) = \Box^{-1} \sum_{j=0}^{i} \sum_{k=0}^{i-j} Y_j(x,\varphi) Y_k(x,\varphi) Y_{i-j-k}(x,\varphi)
$$

This suggests a graphical representation by trees:

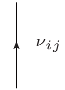


Diagrammatic rules for writing down an integral expression for $Y_n(x, \varphi)$:

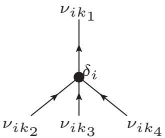
- Draw all trees with n 4-valent vertices (labeled by $1, ..., n$)
- With the vertex i, associate a number $\delta_i \in \mathbb{C} \setminus \mathbb{Z}$ and a unit vector $\hat{x}_i$
- With the line between vertices i and j, associate a "momentum" $\nu_{ij} \in \mathbb{C} \setminus \mathbb{Z}$
- Label the leaves by numbers $1, ..., n_L$. With the leaf j, associate numbers $l_j, m_j \in \mathbb{N}$ ($m_j \leq N(l_j, D)$)

"Feynman rules" for vertex operators

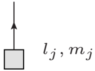
Now to each tree, we apply the following graphical rules:



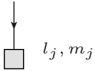
$$\to \frac{\pi}{\sin \pi\nu_{ij}} \mathrm{P}(-\hat{x}_i \cdot \hat{x}_j, \nu_{ij}, D)$$



$$\to r^{2+\delta_i} \delta(2 + \nu_{ik_1} + \nu_{ik_2} + \nu_{ik_3} - \nu_{ik_4} + \delta_i)$$



$$\to K_D \omega(l_j)^{-1/2} \mathrm{S}_{l_j,m_j}(\hat{x}) r^{l_j} \mathbf{b}^+_{l_j,m_j}$$

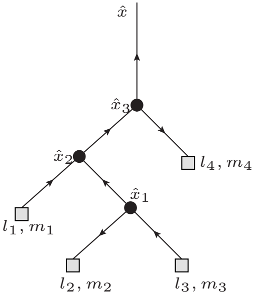


$$\to K_D \omega(l_j)^{-1/2} \overline{\mathrm{S}_{l_j,m_j}(\hat{x})} r^{-l_j-D+2} \mathbf{b}_{l_j,m_j}$$

Write down all these factors and

- integrate over all $\hat{x}_i \to \int_{S^{D-1}} d\hat{x}_i$
- integrate over all $\nu_{ij} \to \int_{\mathbb{C}} d\nu_{ij}$
- integrate over all $\delta_i \to \frac{1}{2\pi i} \oint \frac{d\delta_i}{\delta_i}$
- take the sum over all l_j, m_j (Here, the expression becomes ill-defined $\to$ consideration of counterterms/renormalization necessary)

Example

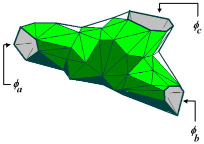


$\longrightarrow$

$$\left(\frac{1}{2\pi i}\right)^3 K_D^4 \oint \frac{d\delta_3}{\delta_3} \oint \frac{d\delta_2}{\delta_2} \oint \frac{d\delta_1}{\delta_1} \times$$
$$\int_{S^{D-1}} d\hat{x}_3 \int_{S^{D-1}} d\hat{x}_2 \int_{S^{D-1}} d\hat{x}_1 \times$$
$$\frac{\pi}{\sin \pi(l_3 - l_2 - D + 4 + \delta_1)} \times$$
$$\mathrm{P}(-\hat{x}_1 \cdot \hat{x}_2, l_3 - l_2 - D + 4 + \delta_1, D) \times$$
$$\frac{\pi}{\sin \pi(l_1 + l_3 - l_2 - D + 6 + \delta_1 + \delta_2)} \times$$
$$\mathrm{P}(-\hat{x}_2 \cdot \hat{x}_3, l_1 + l_3 - l_2 - D + 6 + \delta_1 + \delta_2, D)$$
$$\frac{\pi}{\sin \pi(l_1 + l_3 - l_2 - l_4 - 2D + 10 + \sum \delta_i)} \mathrm{P}(-\hat{x} \cdot \hat{x}_3,$$
$$, l_1 + l_3 - l_2 - l_4 - 2D + 10 + \sum \delta_i, D) \times$$
$$\mathrm{S}_{l_1, m_1}(\hat{x}) \overline{\mathrm{S}_{l_2, m_2}(\hat{x})} \mathrm{S}_{l_3, m_3}(\hat{x}) \overline{\mathrm{S}_{l_4, m_4}(\hat{x})} \times$$
$$\mathbf{b}^+_{l_1, m_1} \mathbf{b}_{l_2, m_2} \mathbf{b}^+_{l_3, m_3} \mathbf{b}_{l_4, m_4} r^{l_1 + l_3 - l_2 - l_4 - 2D + 10 + \sum}$$

Alternative representations of Y:

1. In terms of trees and Legendre functions (above)
2. In terms of sums of generalized hypergeometric type (connections theory of special functions)
3. In terms of $6j$-symbols and foamlike structures



Surfaces are decorated by “spins” and $6j$-symbols for $SO(D)$

$$\langle \vec{p}_-, \vec{l}_- | Y_i(G, \varphi, x) | \vec{p}_+, \vec{l}_+ \rangle = \sum_{l_e \in \mathbb{Z}:\, e \in G \setminus T} \; \sum_{k_e \in \mathbb{Z}^D:\, e \in T} \mathrm{Res}_{\delta_v \in T}$$

$$\times \prod_{e \in T} \frac{\Gamma(-l_e/2 - \delta_e/2 + |k_e|/2)\Gamma(l_e/2 + \delta_e + D/2 - 1 + |k_e|/2)}{k_e!}$$

$$\times \prod_{v \in T} \frac{\prod_\mu \Gamma(\sum_{e \,\mathrm{on}\, v} k_{e,\mu} + 1/2)}{\Gamma(D/2 + \sum_{e \,\mathrm{on}\, v} |k_e|)} \prod_{e \in G \setminus T} \frac{\Gamma(l_e - j_e + D/2 - 1)}{j_e!(l_e - 2j_e)!}$$

$$\times \prod_{e \,\mathrm{in}} \frac{\Gamma(l_{+e} - j_e + D/2 - 1)}{j_e!(l_{+e} - 2j_e)!} \prod_{e \,\mathrm{out}} \frac{\Gamma(l_{-e} - j_e + D/2 - 1)}{j_e!(l_{-e} - 2j_e)!}$$

$$\times \; \hat{x}^{k_0} \, r^{\sum_{e \,\mathrm{in}} l_{+e} - \sum_{e \,\mathrm{out}} l_{-e} + \sum_{v \in T}(2 + \delta_v)} \, (-2)^{\sum_e |k_e|} \prod_{e \,\mathrm{in}} p_{+e}^{k_e} \prod_{e \,\mathrm{out}} p_{-e}^{k_e} \, .$$

Conclusions

1. The OPE can be used to give a general definition of QFT independent of Lagrangians or special states (such as vacuum).
2. One can impose powerful consistency conditions on the OPE. These incorporate algebraic content of QFT.
3. Perturbations can be characterized intrinsically
4. Consistency conditions together with field equations give rise to new and efficient scheme for pert. calculations.
5. Renormalization not needed.