A Guided Tour to Concepts and Developments in Quantum Field Theory on Curved Spacetime

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Göttingen Conference
AQFT-The First 50 Years

29-31 July 2009

Historical development of QFT in CST, outline

Modern stage, Construction and Application

1995...today

* microlocal spectrum condition
* local general covariance
* QEIs
* local thermal equilibrium
* perturbative renorm. & construction of interacting models
* spin & statistics, PCT
* structural results & axiomatics

Conceptual & Math consolidation

1978...1995

* Hadamard states
* renormalization of stress-energy
* algebraic formulation
* scattering on spacetimes with horizons

Particle creation by black holes, etc

Hawking, Fulling, Unruh, Wald

1973...78

Particle creation in expanding universe

Takahasi-Umezawa 1957

Parker 1969

Simple model: linear scalar field

- $(\mathcal{M}, g)$ 4-dim, globally hyperbolic spacetime
- $K = (\nabla^\mu \nabla_\nu + m^2)$ scalar Klein-Gordon operator on $(\mathcal{M}, g)$
- G_+, G_- advanced/retarded fundamental solutions (Green-functions) for K

  $\mathscr{A}(\mathcal{M}, g)$ is defined as the star-algebra with unit $\mathbf{1}$ and generated by

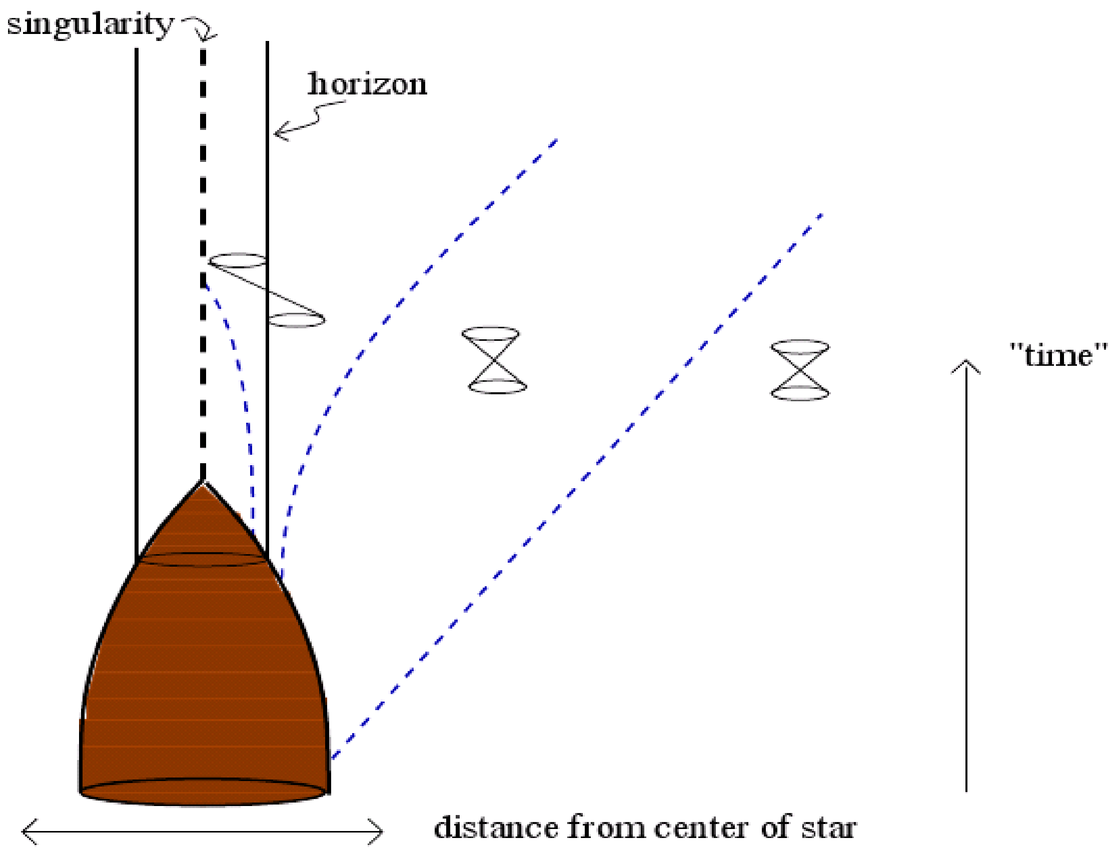
  $\Phi(f)$, $f \in C_0^\infty(\mathcal{M})$, with relations

$$
\begin{aligned}
f \mapsto \Phi(f) & \quad \text{is linear} \\
\Phi(f)^* &= \Phi(\overline{f}) \\
\Phi(Kf) &= 0 \\
[\Phi(f_1), \Phi(f_2)] &= i(G_+(f_1, f_2) - G_-(f_1, f_2))\mathbf{1}
\end{aligned}
$$

Hawking Effect: An observer kept (by acceleration) at constant distance to the black hole will register a thermal equilibrium state (at large times) having the temperature

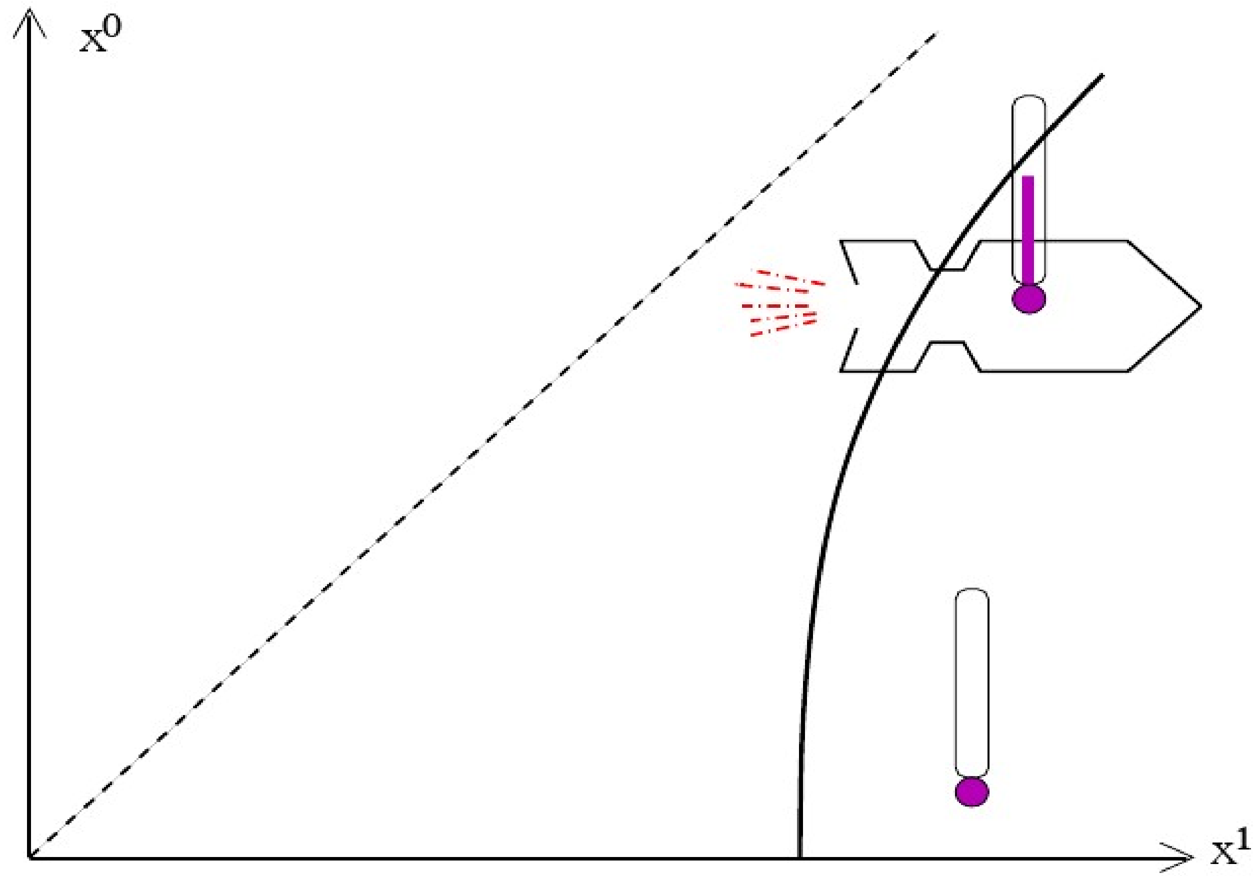
$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G M}, \quad M = \text{mass of black hole},$$

if the quantum field state at early times (before the stellar collapse to a black hole) was a vacuum state.

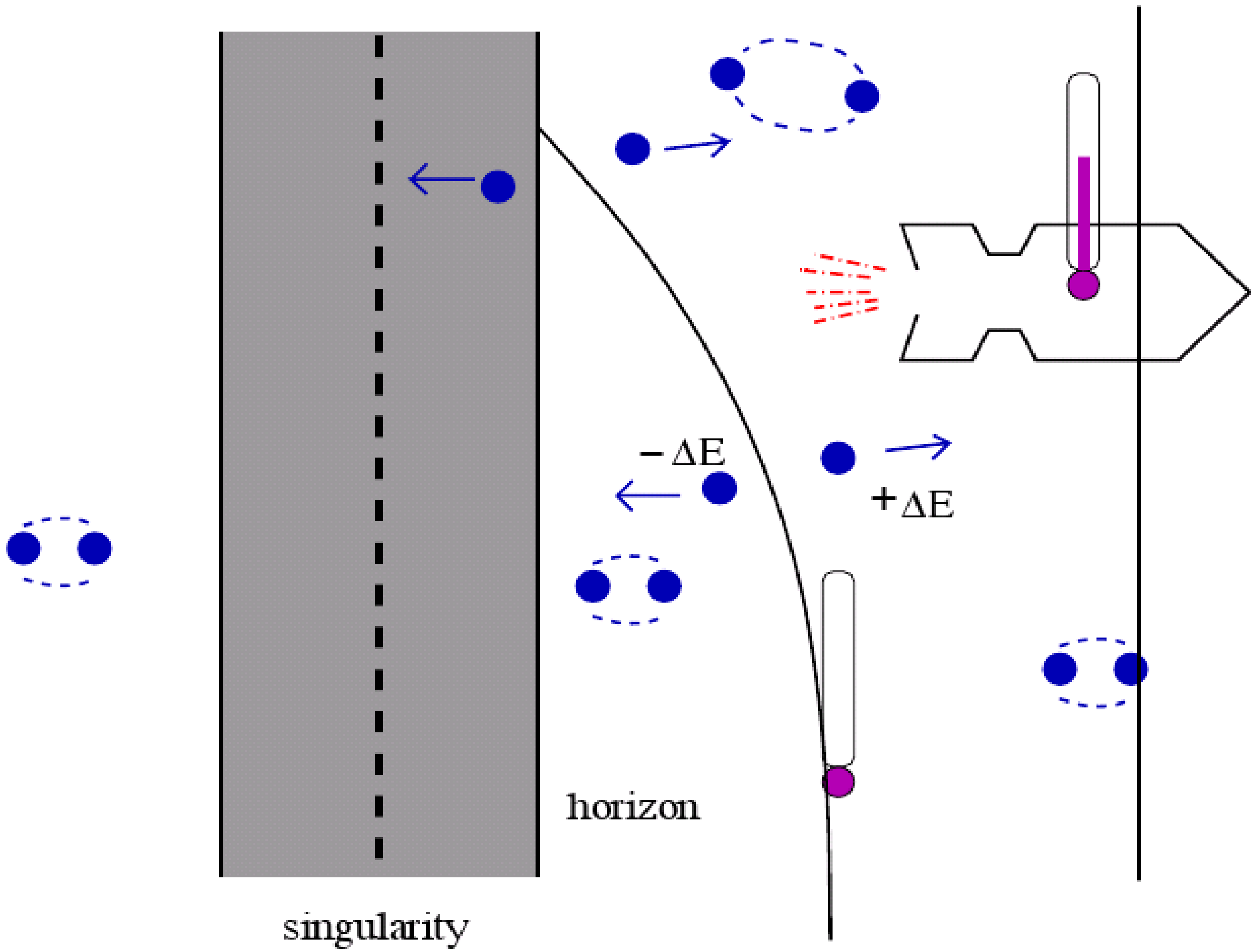


The Fulling-Unruh Effect: An observer moving with constant acceleration registers the vacuum state – defined with respect to a Lorentz frame – as a thermal equilibrium state having the temperature

$$T_a = \frac{\hbar a}{2\pi k_B c}, \quad a = \text{proper acceleration}$$



Hawking Effect and analogy to the Fulling-Unruh Effect



Interlude: Hawking effect and Bisognano-Wichmann

Let $f \to F(f)$ be a Wightman quantum field on Minkowski spacetime

$$\mathscr{A}(\mathcal{W}) = \text{vN alg generated by } F(f), \text{ supp } f \subset \mathcal{W}$$

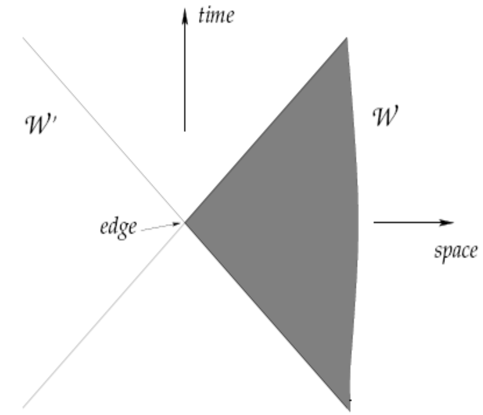
$$S : \mathscr{A}(\mathcal{W})\Omega \to \mathscr{A}(\mathcal{W})\Omega, \; S(A\Omega) = A^*\Omega$$

$$S = J\Delta^{1/2} \text{ Tomita-Takesaki}$$

$U(\,.\,)$ = repr of Poincaré grp, spectrum condition

Ω = vacuum vector

$\mathcal{W}$ = right wedge region



Bisognano-Wichmann (1973)

$$J = P_1 CT$$

$$\Delta^{it} = U(\Lambda_{2\pi t}) \quad \Rightarrow \langle \Omega | \,.\, | \Omega \rangle \text{ is KMS for } U(\Lambda_\tau)$$

$$U(\Lambda_\tau) = \begin{pmatrix} \cosh(\tau) & \sinh(\tau) & 0 & 0 \\ \sinh(\tau) & \cosh(\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Generalized to static black-holes by

Sewell (1982),

Haag-Narnhofer-Stein (1984)

Summers-R.V. (1996)

Some lessons from “particle creation” effects

Concepts of “particle” or “ground state” in curved spacetime are strongly dependent on certain global or asymptotic “reference” systems,

i.e. “observers” in the sense of distinguished states or observables.

The (local) interpretation of physical processes in generic curved spacetimes should not be based on the concept of “particle” or “vacuum” but on (relations between) local field quantities, i.e. local observables.

Of special interest is $\langle \mathbf{T}_{\mu\nu}(x) \rangle$,

the expectation value of the stress- energy-tensor in a physical state, since it appears on the right hand side of the the semiclassical Einstein equations,

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = -\frac{8\pi \mathrm{G}}{c^2} \left(T_{\mu\nu}(x) + \langle \mathbf{T}_{\mu\nu}(x) \rangle \right)$$

Stress-Energy and Hadamard states

What properties should $\langle \mathrm{T}_{\mu\nu}(x) \rangle$ have?

Wald 1977:

* $\langle \mathrm{T}_{\mu\nu}(x) \rangle$ should depend covariantly on the spacetime metric
* divergence-free: $\nabla^{\mu} \langle \mathrm{T}_{\mu\nu}(x) \rangle = 0$

Theorem:

(1) The difference of different descriptions for defining $\langle \mathrm{T}_{\mu\nu}(x) \rangle$ depends locally on spacetime metric and is divergence-free

(2) The point-splitting + Hadamard singularity subtraction yields a defn for $\langle \mathrm{T}_{\mu\nu}(x) \rangle$ with the required properties

Simple model: linear scalar field

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\end{aligned}
$$

C* version by Dimock (1980)

For a state $\omega \equiv \langle\,.\,\rangle_\omega : \mathscr{A}(\mathcal{M}, g) \to \mathbb{C}$

- $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \Omega_\omega)$
  Wightman-GNS-representation

- $\Phi_\omega(f) = \pi_\omega(\Phi(f))$,
  simply $\Phi(f) = \Phi_\omega(f)$ in selected repr

Hadamard condition on 2-point correlation:

smooth,
metric-dependent
part

$$\langle\psi|\Phi(x)\Phi(y)|\psi\rangle = \frac{U(x,y)}{\sigma(x,y)} + V(x,y)\ln(\sigma(x,y)) + W(x,y)$$

$\sigma(x,y)$ = squared geodesic distance between x and y

For the stress-energy-tensor of a classical field $\varphi(x)$,

$$T_{\mu\nu}(x) = \lim_{y \to x} \mathcal{P}_{\mu\nu}(x, y, \nabla_x, \nabla_y)[\varphi(x)\varphi(y)]$$

$\mathcal{P}_{\mu\nu}(x, y, \nabla_x, \nabla_y)$

is a PDE

Consequently, define

$$\begin{aligned} \langle\psi|\mathbf{T}^{ren}_{\mu\nu}(x)|\psi\rangle &= \lim_{y\to x} \mathcal{P}_{\mu\nu}(x, y, \nabla_x, \nabla_y)[\langle\psi|\Phi(x)\Phi(y)|\psi\rangle - \text{metric-dependent singular part}] \\ &= \lim_{y\to x} \mathcal{P}_{\mu\nu}(x, y, \nabla_x, \nabla_y)[W(x, y)] \end{aligned}$$

- $\langle\psi|\mathbf{T}_{\mu\nu}(x)|\psi\rangle = \langle\psi|\mathbf{T}^{ren}_{\mu\nu}(x)|\psi\rangle - Q(x)g_{\mu\nu}(x)$
  is divergence-free, with
  $Q(x)$ state-independent, constructed locally from the spacetime metric

Hadamard states = states with 2-point correlation of Hadamard form

Since 1978, a better understanding was successively reached how to characterize the relevant Hilbert space representations of quantum fields in curved spacetimes (for linear quantum fields):

- **Hadamard states** (resp., Hadamard representations) allow a systematic definition of $\langle \mathbf{T}_{\mu\nu}(x) \rangle$ (Wald 1978)

- Hawking effect appears in a natural manner in Hadamard representations (Haag, Narnhofer u. Stein 1984; Fredenhagen u. Haag 1990; Kay u. Wald 1991)

- Hadamard states define a unique Hilbert space representation (Verch 1994)

Microlocal Spectrum Condition (μSC)

An important step was the introduction of the

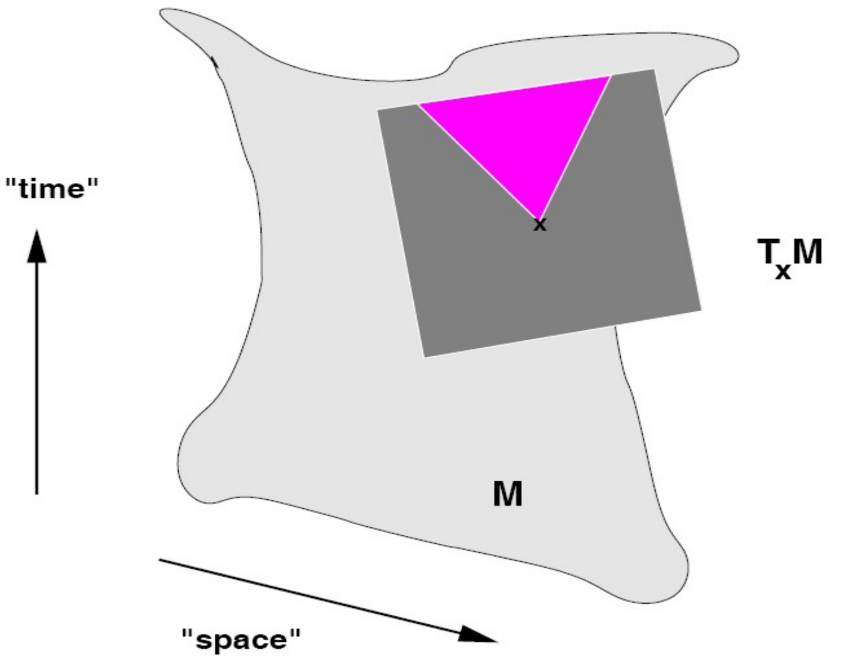
microlocal spectrum condition by Radzikowski 1996; Brunetti, Fredenhagen and Köhler 1996.

The microlocal spectrum condition for a state vector $|\psi\rangle$ requires that

$$\widehat{(\chi\Phi)}(k)|\psi\rangle \sim \frac{1}{|k|^N} \quad \forall\ N\ (|k| \to \infty)\,,$$

if $k \in T_x^*M$ is <u>not</u> contained in the (dual) forward light cone of $x \in M$, with test-fuction χ concentrated around x.

(It says that $WF(f \mapsto \Phi(f)|\psi\rangle)$ is contained in the forward light cone bundle)



The microlocal spectrum condition was shown to be equivalent to the Hadamard condition by Radzikowski 1996:

$\langle\psi|\Phi(x)\Phi(y)|\psi\rangle$ is Hadamard $\iff$ $|\psi\rangle$ fulfills μSC

The μSC is more general than the Hadamard condition since it can be generalized to nonlinear quantum fields,

by imposing fall-off conditions on expressions of the form

$$\widehat{(\chi_1\Phi)}(k_1)\cdots\widehat{(\chi_n\Phi)}(k_n)|\psi\rangle$$

for $|k_1|+\cdots+|k_n|\to\infty$ outside of certain conic sets

The μSC can be seen as a short distance/high energy remnant of the spectrum condition in combination with the equivalence principle

Quantum Energy Inequalities (QEIs)

In 1978, L. Ford introduced another condition for “admissible” Hilbert space representations of quantum fields on curved spacetimes:

They should satisfy **quantum energy inequalities**:

- ☐ for every timelike curve γ
- ☐ for every positive C^∞ weight function f

there should be a bound of the form

$$\min_{|\psi\rangle} \int_\gamma d\tau\, f(\tau) \langle \psi | \mathbf{T}_{00}(\tau) | \psi \rangle \geq -c_{\gamma,f} > -\infty$$

Interpretation: When averaging over finite time, it is impossible to extract an arbitrary amount of energy from any state.

Note: The classical **pointwise** weak energy energy condition

$T_{00}(x) = T_{\mu\nu}(x)t^\mu t^\nu \geq 0$ for all timelike vectors t^μ at $x \in M$

is **violated** in quantum field theory (also on Minkowski spacetime); it holds that

$$\min_{|\psi\rangle} \langle\psi|\mathbf{T}_{00}(x)|\psi\rangle = -\infty \quad !$$

Thus, the QEIs impose a nontrivial constraint on Hilbert space representations to be admissible.

Relations between the conditions:

For Hilbert space representations of linear quantum fields (Klein-Gordon, Dirac, Maxwell) on generic spacetime manifolds, it could be shown (Fewster 2000; Fewster and Verch 2001; Fewster and Pfenning 2003) that

$$\mu SC \quad \Longrightarrow QEIs$$

For linear quantum fields on static spacetimes, it was found that the following conditions on their Hilbert space representations are equivalent (Fewster and Verch 2002):

$$\begin{aligned} & \boldsymbol{\mu SC} && \text{“microscopic condition”} \\ \Longleftrightarrow\quad & \boldsymbol{QEIs} && \text{“mesoscopic condition”} \\ \Longleftrightarrow\quad & \text{existence of thermal equilibrium states (passive states)} \\ & \quad \text{“macroscopic condition”} \end{aligned}$$

This shows that μSC and QEIs can be viewed as equivalent characterizations of quantum field states (or Hilbert space representations) which are dynamically stable — they also coincide with the usual characterizations of the “correct” Hilbert space representations when the spacetime admits time-symmetries.

Ruling out "Designer Spacetimes"

Since energy densities of quantum fields (in physical states) need not be positive, there is the possibility to obtain spacetimes with pathological causal behaviour (e.g. closed timelike curves, wormholes...) as solutions to the semiclassical Einstein equations.
(And this could be hinting at very unusual effects in quantum gravity.)

But microlocal spectrum condition and QEIs put strong limits to this a priori possibility.

$\ominus$ **time machines** (Kay, Radzikowski and Wald 1997)

For quantum field theories in Hilbert space representations fulfilling μSC and existence of a causal dynamical law, spacetimes with Cauchy-horizons are excluded as solutions to the equations of semiclassical gravity.

$\ominus$ **warp drive** (Pfenning and Ford 1997)

For quantum fields in Hilbert space representations fulfilling QEIs, extreme amounts of negative energy (comparable to the total energy of the luminous universe) would have to be concentrated in microscopic domains of space.

$\ominus$ **wormholes** (Ford and Roman 1995)

Again, for quantum fields in Hilbert space representations fulfilling QEIs, extreme amounts of negative energy (comparable to the total energy of the luminous universe) would have to be concentrated in microscopic domains of space in order to sustain macroscopic wormholes. (The situation for microscopic wormholes is not completely clarified.)

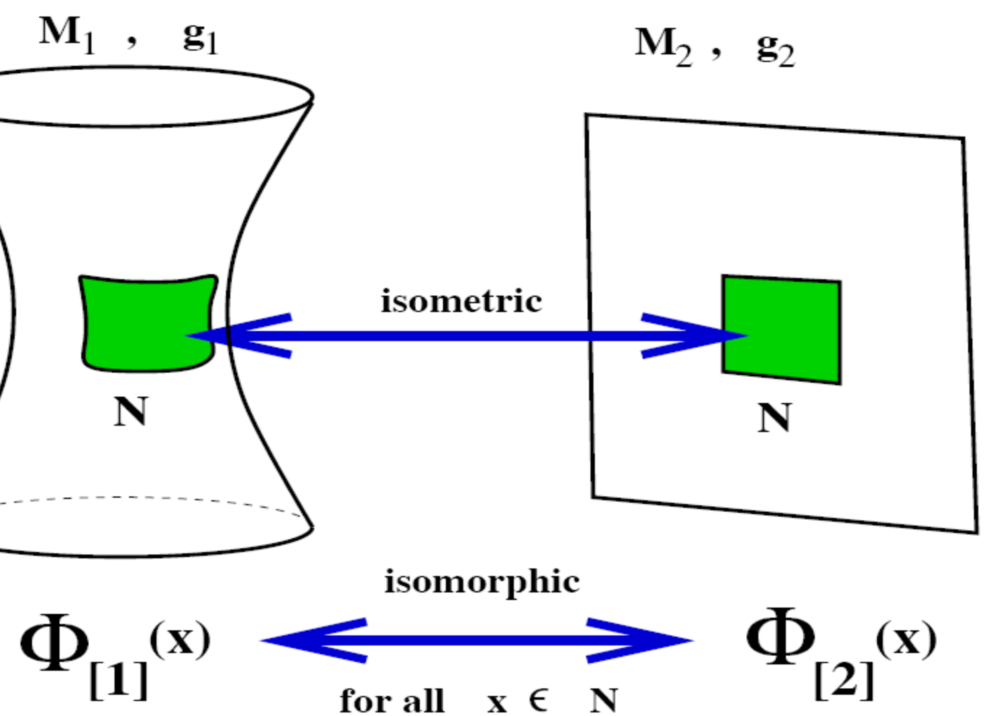
Recent progress on this question by Fewster-Roman 2005, Fewster-Smith 2008

Local covariant QFT (R.V. 2001, Hollands-Wald 2001, Brunetti-Fredenhagen-R.V. 2003)

(1) To every spacetime (M, g), a quantum field is assigned:

$$(M, g) \longrightarrow \Phi_{[M,g]}(x) \quad \text{quantum field on } (M, g)$$

(2) If two spacetimes have isometric subregions, then the Hilbert space representations of the corresponding quantum fields (restricted to the subregions) have to be isomorphic.



Local covariant QFT has a functorial structure

Definition 2.1. *(i)* *A* **locally covariant quantum field theory** *is a covariant functor* $\mathscr{A}$ *between the two categories* $\mathfrak{Man}$ *and* $\mathfrak{Alg}$, *i.e., writing* α_ψ *for* $\mathscr{A}(\psi)$, *in typical diagramatic form:*

$$\begin{tikzcd}
(M, \boldsymbol{g}) \arrow[r, "\psi"] \arrow[d, "\mathscr{A}"'] & (M', \boldsymbol{g}') \arrow[d, "\mathscr{A}"] \\
\mathscr{A}(M, \boldsymbol{g}) \arrow[r, "\alpha_\psi"] & \mathscr{A}(M', \boldsymbol{g}')
\end{tikzcd}$$

together with the covariance properties

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi} \,, \quad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathscr{A}(M, \boldsymbol{g})} \,,$$

for all morphisms $\psi \in \hom_{\mathfrak{Man}}((M_1, \boldsymbol{g}_1), (M_2, \boldsymbol{g}_2))$, *all morphisms* $\psi' \in \hom_{\mathfrak{Man}}((M_2, \boldsymbol{g}_2), (M_3, \boldsymbol{g}_3))$ *and all* $(M, \boldsymbol{g}) \in \mathrm{Obj}(\mathfrak{Man})$.

$\mathfrak{Man}$ Category of 4-dim globally hyperbolic spacetimes

$\mathfrak{Alg}$ Category of unital *-algebras

Covariant quantum field = natural transformation

Definition 2.4. *A* **locally covariant quantum field** Φ *is a natural transformation between the functors* $\mathscr{D}$ *and* $\mathscr{A}$, *i.e. for any object* $(M, \mathbf{g})$ *in* $\mathfrak{Man}$ *there exists a morphism* $\Phi_{(M,\mathbf{g})} : \mathscr{D}(M, \mathbf{g}) \to \mathscr{A}(M, \mathbf{g})$ *in* $\mathfrak{Top}$ *such that for each given morphism* $\psi \in \hom_{\mathfrak{Man}}((M_1, \mathbf{g}_1), (M_2, \mathbf{g}_2))$ *the following diagram*

$$\begin{tikzcd} \mathscr{D}(M_1, \mathbf{g}_1) \arrow[r, "\Phi_{(M_1,g_1)}"] \arrow[d, "\psi_*"'] & \mathscr{A}(M_1, \mathbf{g}_1) \arrow[d, "\alpha_\psi"] \\ \mathscr{D}(M_2, \mathbf{g}_2) \arrow[r, "\Phi_{(M_2,g_2)}"'] & \mathscr{A}(M_2, \mathbf{g}_2) \end{tikzcd}$$

commutes.

The commutativity of the diagram means, explicitly, that

$$\alpha_\psi \circ \Phi_{(M_1,g_1)} = \Phi_{(M_2,g_2)} \circ \psi_*,$$

i.e., the requirement of covariance for fields.

The stress-energy-tensor of a linear quantum field defined before is a local, covariant quantum field in this sense.

Structural results making use of local general covariance

Let

$$(M, g) \longrightarrow \Phi_{[M,g]}$$

be a quantum field on curved spacetimes fulfilling local general covariance.

(I) **Spin and Statistics** (Verch 2001):

Suppose that the quantum field fulfills the Wightman axioms on Minkowski spacetime and obeys a causal dynamical law. Then

$\Phi_{[M,g]}$ has the correct relation between spin and statistics on each (M, g) :

— if $\Phi_{[M,g]}$ has integer spin, it is bosonic

— if $\Phi_{[M,g]}$ has half-integer spin, it is fermionic.

Structural results making use of local general covariance

(II) **PCT** (Hollands 2003):

Suppose that the quantum field fulfills (a strong form of) μSC and admits an operator product expansion around each point in spacetime.

Then for each given spacetime there is an anti-linear operator relating the operator product expansion of the quantum field on the given spacetime with the operator product expansion of the conjugate-charged quantum field on the same spacetime, but with the reversed spacetime-orientation.

Linear quantum fields of fixed type (e.g Dirac, Proca..) in μSC representations are examples for local generally covariant quantum fields

Perturbative Construction of Interacting Models

In order to study interacting quantum fields — here, the scalar field with $P(\Phi)_4$ self-interaction — one starts with the free scalar Klein-Gordon field

$$\Phi = \Phi_{[M,g]} \text{ on the spacetime } (M,g)$$

in a μSC Hilbert space representation and then tries to define

- **normal ordered products**

$$\mathcal{N}_n(\Phi(x_1)\cdots\Phi(x_n))\,, \quad \text{and}$$

- **time ordered products**

$$\mathcal{T}_n(\Phi(x_1)\cdots\Phi(x_n))$$

of the field operators to all orders n, as well as time-ordered products of normal ordered products, and so on.

Infinite Renormalization (Brunetti and Fredenhagen 2000):

Brunetti and Fredenhagen generalized the Stueckelberg-Shirkov-Epstein-Glaser approach to renormalizing selfinteracting quantum fields to curved spacetime.

Using μSC, they showed that the $\mathcal{N}_n$, $\mathcal{T}_n \ldots$ can be defined inductively by a consistent prescription extracting finite parts of their singularities at coinciding spacetime points.

The $\mathcal{N}_n$, $\mathcal{T}_n \ldots$ are then defined up to smooth parts (renormalization ambiguity). The renormalizability criteria are the same as the power-counting criteria on Minkowski spacetime.

Reduction of the Renormalization Ambiguity and General Covariance (Hollands and Wald 2001, 2002):

Hollands and Wald showed that the normal ordering and time ordering prescriptions can be implemented such that

$$(M,g) \longrightarrow \mathcal{N}_{n[M,g]}\,, \quad (M,g) \longrightarrow \mathcal{T}_{n[M,g]}\,, \quad \text{etc}$$

fulfill the principle of local general covariance.

Then the remaining ambiguity in the definition of these quantities is up to only finitely many parameters for each order of perturbation theory (i.e., for each n):

$$\tilde{\mathcal{T}}_n(x_1, \dots, x_n) = \mathcal{T}_n(x_1, \dots, x_n) + \mathcal{P}_n(x_1, \dots, x_n)$$

where $\mathcal{P}_n$ is a (known) polynomial in the $\mathcal{N}_k$ and curvature quantities.

Recently, Hollands (2008) has extended this construction to the case of the Yang-Mills-model in curved spacetime.

Topics not treated

* (Local, covariant) superselection theory on curved spacetime (Guido, Longo, Roberts, R.V.; Brunetti, Ruzzi)

* vN-algebraic structure of Hadamard representations (R.V.; Hollands, D'Antoni)

* Scattering theory (Dimock, Kay; Bachelot)

* Asymptotia, holography (Rehren; Dappiaggi, Moretti, Pinamonti)

* "Geometric modular action" on curved spacetimes; special representations on de Sitter spacetimes (Buchholz, Summers, Mund et al; Bros, Epstein, Moschella)

* New reference states on FRW cosmological spacetimes and new results on cosmological particle creation (Olbermann; R.V., Degner)

* Local thermal equilibrium states (Buchholz et al, Schlemmer, R.V.)

* Applications of results and techniques of QFT in CST to cosmology (Hollands, Wald; Fredenhagen, Dappiaggi, Pinamonti... + more to come in the timelike future!)

The next 50 years

QFT in CST reaches a mature state; further development will make it possible to calculate effects relevant to cosmology from first principles.

Thus, it is likely that its importance will be growing.

A lot depends on the appropiate blend of conceptual clarity and applicability.

QFT in CST once was a very fashionable topic...it may or may not come into fashion again – at any rate it will continue to be very closely interlaced with algebraic quantum field theory, to the benefit of both domains of research.

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In order to be irreplaceable one must always be different

Coco Chanel