Axiomatic Quantum Field Theory in Curved Spacetime

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Quantum Field Theory in Curved Spacetime

Quantum field theory in curved spacetime (QFTCS) is a theory wherein matter is treated fully in accord with the principles of quantum field theory, but gravity is treated classically in accord with general relativity. It is not expected to be an exact theory of nature, but it should provide a good approximate description in circumstances where the quantum effects of gravity itself do not play a dominant role. Despite its classical treatment of gravity, QFTCS has provided us with some of the deepest insights we presently have into the nature of quantum gravity.

What Are the Essential Elements of Quantum Field Theory?

Quantum field theory (QFT) as usually formulated contains many elements that are very special to Minkowski spacetime. But we know from general relativity that spacetime is not flat, and, indeed there are very interesting QFT phenomena that occur in contexts (such as in the early universe and near black holes) where spacetime cannot even be approximated as nearly flat.

It is a relatively simple matter to generalize classical field theory from flat to curved spacetime. That is because there is a clean separation between the field equations and the solutions. The field equations can be straightforwardly generalized to curved spacetime in an entirely local and covariant manner. Solutions to the field equations need not generalize from flat to curved spacetime, but this doesn't matter for the formulation of the theory.

In QFT, "states" are the analogs of "solutions" in classical field theory. However, properties of states—in particular, the existence of a Poincare invariant vacuum state—are deeply embedded in the usual formulations of QFT in Minkowski spacetime. For this reason and a number of other reasons, it is highly nontrivial to generalize the formulation of QFT from flat to curved spacetime.

Wightman Axioms in Minkowski Spacetime

- The states of the theory are unit rays in a Hilbert space, $\mathcal{H}$, that carries a unitary representation of the Poincare group.
- The 4-momentum (defined by the action of the Poincare group on the Hilbert space) is positive, i.e., its spectrum is contained within the closed future light cone ("spectrum condition").
- There exists a unique, Poincare invariant state ("the vacuum").
- The quantum fields are operator-valued distributions defined on a dense domain $\mathcal{D} \subset \mathcal{H}$ that is both

Poincare invariant and invariant under the action of the fields and their adjoints.

- The fields transform in a covariant manner under the action of Poincare transformations.
- At spacelike separations, quantum fields either commute or anticommute.

Difficulties with Extending the Wightman Axioms to Curved Spacetime

- A generic curved spacetime will not possess any symmetries at all, so one certainly cannot require "Poincare invariance/covariance" or invariance under any other type of spacetime symmetry.
- There exist unitarily inequivalent Hilbert space constructions of free quantum fields in spacetimes with a noncompact Cauchy surface and (in the absence of symmetries of the spacetime) none appears "preferred".
- In a generic curved spacetime, there is no "preferred" choice of a vacuum state.

- There is no analog of the spectrum condition in curved spacetime that can be formulated in terms of the "total energy-momentum" of the quantum field.

Thus, of all of the Wightman axioms, only the last one (commutativity or anticommutativity at spacelike separations) generalizes straightforwardly to curved spacetime.

Total Energy in Curved Spacetime

The stress energy tensor, T_{ab}, of a classical field in curved spacetime is well defined. Have local energy-momentum conservation in the sense that $\nabla^a T_{ab} = 0$. If t^a is a vector field on spacetime representing time translations and Σ is a Cauchy surface, can define the total energy, E, of the field at "time" Σ by

$$E = \int_\Sigma T_{ab} t^a n^b d\Sigma \ .$$

Classically, $T_{ab} t^a n^b \geq 0$ (dominant energy condition) so, classically, $E \geq 0$, but unless t^a is a Killing field (i.e., the spacetime is stationary), E will not be conserved.

In QFT, expect $T_{ab}(f^{ab})$ to be well defined, and expect $\nabla^a T_{ab} = 0$. However, on account of the lack of global conservation, in the absence of time translation symmetry, cannot expect E to be well defined at a "sharp" moment of time. Furthermore, since T_{ab} does not satisfy the dominant energy condition in QFT, cannot expect even a "time smeared" version of E to be positive in a curved spacetime. Thus, it does not appear possible to generalize the spectrum condition to curved spacetime in terms of the positivity of a quantity representing "total energy".

Nonexistence of a "Preferred Vacuum State" and Notion of "Particles"

For a free field in Minkowski spacetime, the notion of "particles" and "vacuum" is intimately tied to the notion of "positive frequency solutions", which, in turn relies on the existence of a time translation symmetry. These notions of a (unique) "vacuum state" and "particles" can be straightforwardly generalized to (globally) stationary curved spacetimes, but not to general curved spacetimes.

For a free field on a general curved spacetime, one has the general notion of a quasi-free Hadamard state (i.e., vacuum) and a corresponding notion of "particles". However, these notions are highly non-unique—and, indeed, for spacetimes with a non-compact Cauchy surface different choices of quasi-free Hadamard states give rise, in general, to unitarily inequivalent Hilbert space constructions of the theory.

In my view, the quest for a "preferred vacuum state" in quantum field theory in curved spacetime is much like the quest for a "preferred coordinate system" in classical general relativity. In 90+ years of experience with classical general relativity, we have learned that it is fruitless to seek a preferred coordinate system for general spacetimes, and that the theory is best formulated geometrically, wherein one does not have to specify a choice of coordinate system to formulate the theory.

Similarly, it is my view that in 40+ years of experience with quantum field theory in curved spacetime, we have learned that it is fruitless to seek a preferred vacuum state for general spacetimes, and that the theory is best formulated in terms of the algebra of local field observables, wherein one does not have to specify a choice of state (or representation) to formulate the theory.

Overcoming These Difficulties

- The difficulties that arise from the existence of unitarily inequivalent Hilbert space constructions of quantum field theory in curved spacetime can be overcome by formulating the theory via the algebraic framework. The algebraic approach also fits in very well with the viewpoint naturally arising in quantum field theory in curved spacetime that the fundamental observables in QFT are the local quantum fields themselves.
- The difficulties that arise from the lack of an appropriate notion of the total energy of the quantum field can be overcome by replacing the

spectrum condition by a “microlocal spectrum condition” that restricts the singularity structure of the expectation values of the correlation functions of the local quantum fields.

- Many aspects of the requirement of Poincare invariance of the quantum fields can be replaced by the requirement that the quantum fields be locally and covariantly constructed out of the metric.

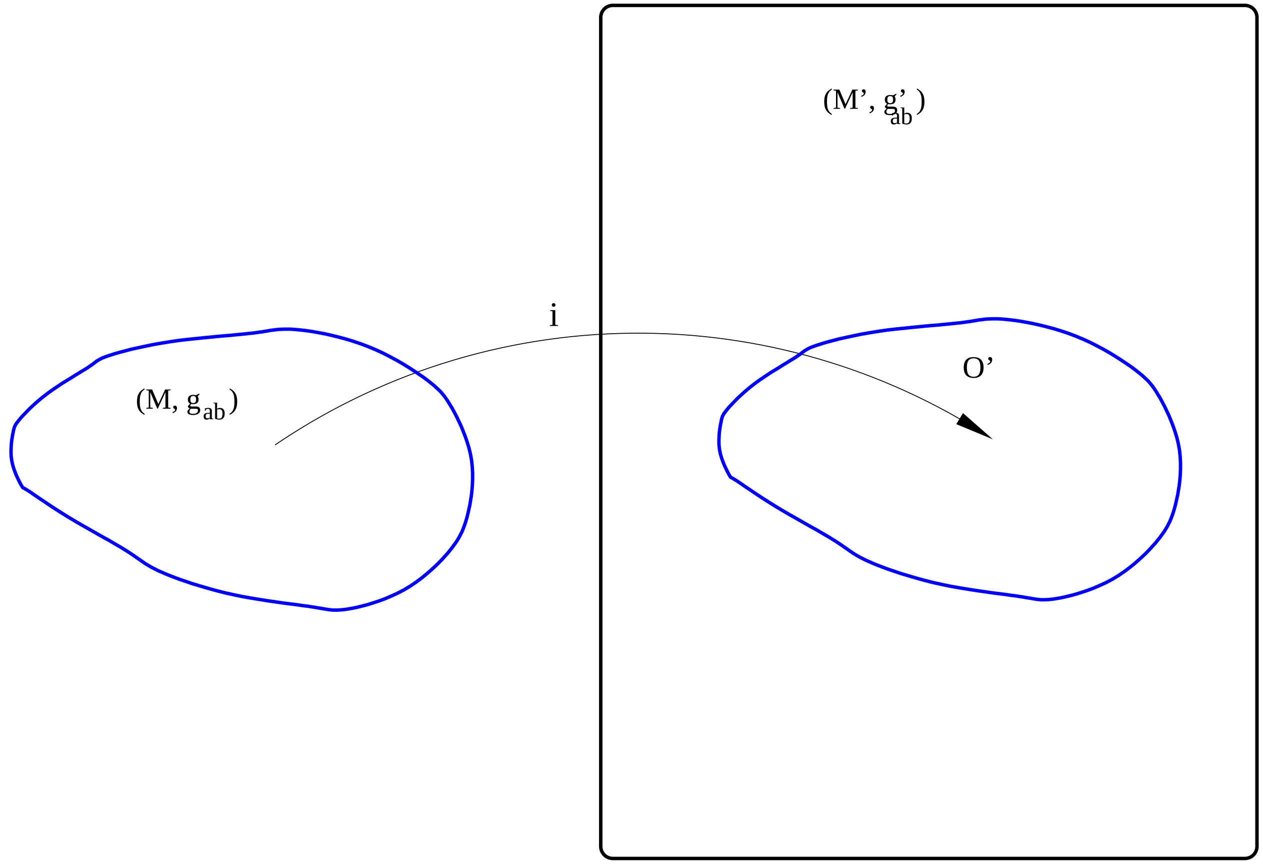
Microlocal Spectrum Condition

Microlocal analysis provides a refined characterization of the singularities of a distribution by examining the decay properties of the Fourier transform of the distribution (after it has been localized near point x). It therefore provides a notion of the singular points and directions (x, k) of a distribution, α, called the the *wavefront set*, denoted $\mathrm{WF}(\alpha)$. It provides an ideal way of characterizing the singular behavior of the distributions $\omega[\Phi_1(x_1) \ldots \Phi_n(x_n)]$ as being of a "locally positive frequency character" even in situations where there is no natural global notion of "positive frequency" (i.e., no global notion of Fourier transform).

Local and Covariant Fields

We wish to impose the requirement that quantum fields Φ in an arbitrarily small neighborhood of a point x "be locally and covariantly constructed out of the spacetime geometry" in that neighborhood. In order to formulate this requirement, it is essential that quantum field theory in curved spacetime be formulated for *all* (globally hyperbolic) curved spacetimes—so that we can formulate the notion that "nothing happens" to the fields near x when we vary the metric in an arbitrary manner away from point x.

Suppose that we have a causality preserving isometric embedding $i : M \to \mathcal{O}' \subset M'$ of a spacetime (M, g_{ab}) into a region $\mathcal{O}'$, of a spacetime (M', g'_{ab}).



We require that this embedding induce a natural isomorphism of the quantum field algebra $\mathcal{A}(M)$ of the spacetime (M, g_{ab}) and the subalgebra of the quantum field algebra $\mathcal{A}(M')$ associated with region $\mathcal{O}'$. We further demand that under this isomorphism, each quantum field $\Phi(f)$ on M be taken into the corresponding quantum field $\Phi(i^* f)$ in $\mathcal{O}'$.

What does this have to do with Poincare covariance? We can isometrically embed all of Minkowski spacetime into itself by a Poincare transformation. The above condition provides us with an action of the Poincare group on the field algebra of Minkowski spacetime and also requires each quantum field in Minkowski spacetime to transform covariantly under Poincare transformations. The above condition contains much of the essential content of Poincare invariance, but is applicable to arbitrary curved spacetimes without symmetries.

Generalizing the Wightman Axioms to Curved Spacetime

- The states of the theory are unit rays in a Hilbert space, $\mathcal{H}$, that carries a unitary representation of the Poincare group. Replaced by formulating theory via algebraic approach and local and covariant field condition.

- The 4-momentum (defined by the action of the Poincare group on the Hilbert space) is positive, i.e., its spectrum is contained within the closed future light cone ("spectrum condition"). Replaced by microlocal spectrum condition.

- There exists a unique, Poincare invariant state ("the vacuum").

- The quantum fields are operator-valued distributions defined on a dense domain $\mathcal{D} \subset \mathcal{H}$ that is both Poincare invariant and invariant under the action of the fields and their adjoints. Replaced by GNS construction in algebraic approach and local and covariant field condition.

- The fields transform in a covariant manner under the action of Poincare transformations. Replaced by local and covariant field condition.

- At spacelike separations, quantum fields either commute or anticommute. OK in the first place.

What is the appropriate replacement in curved spacetime of the requirement that there exist a Poincare invariant state in Minkowski spacetime?

The Operator Product Expansion

An *operator product expansion* (OPE) is a short-distance asymptotic formula for products of fields near point x in terms of fields defined at x. For example, for a free Klein-Gordon field in curved spacetime, we have

$$\phi(x)\phi(y) = H(x,y)\mathbf{1} + \phi^2(x) + \ldots$$

where H is a locally and covariantly constructed Hadamard distribution and "$\ldots$" has higher scaling degree than the other terms (i.e., it goes to zero more rapidly in the coincidence limit). An OPE exists for free fields in curved spacetime and Hollands has shown that it holds order-by-order in perturbation theory for renormalizable interacting fields in curved spacetime. However, one would not expect an OPE to exist for field theories that are not renormalizable or are not asymptotically free. We believe that the existence of an operator product expansion satisfying certain properties should be elevated to the status of a fundamental property of quantum fields. The requirement that such an operator product expansion exist appears to provide a suitable replacement for the requirement of existence of a Poincare invariant state! In particular, the distributional coefficients of the identity element in OPE expansions play much of the role played by "vacuum expectation values" in Minkowski spacetime quantum field theory.

Our Present Viewpoint on QFT

The background structure, $\mathcal{M}$, of quantum field theory (in curved spacetime) is the spacetime (M, g_{ab}), together with choices of time orientation, spacetime orientation, and spin structure. For each $\mathcal{M}$, we have an algebra $\mathcal{A}(\mathcal{M})$ of local field observables. In traditional algebraic approaches to QFT, $\mathcal{A}(\mathcal{M})$ would contain the full information about the QFT. However, in our approach, $\mathcal{A}(\mathcal{M})$, is essentially nothing more than the free *-algebra generated by the list of quantum fields $\phi^{(i)}(f)$ (including "composite fields"), though it may be factored by relations that arise from the OPE (see below).

All of the nontrivial information about the QFT is contained in its OPE, i.e., formulae of the form

$$\phi^{(i_1)}(x_1)\cdots\phi^{(i_n)}(x_n) \sim \sum_{(j)} C_{(j)}^{(i_1)\dots(i_n)}(x_1,\dots,x_n;y)\,\phi^{(j)}(y)$$

for all $i_1,\dots,i_n$, which hold as asymptotic relations as $\{x_1,\dots,x_n\} \to y$. The distributions $C_{(j)}^{(i_1)\dots(i_n)}(x_1,\dots,x_n;y)$ are required to satisfy a list of axioms:

- Locality and Covariance
- Identity element
- Compatibility with the $\star$-operation
- Causality (commutativity/anti-commutativity at spacelike separations)

- Dimension and Scaling Degree
- “Associativity”
- Spectrum condition
- Analytic dependence upon the metric

Dimension and Scaling Degree

Consider the OPE for $\phi^{(i)}(x)\phi^{*(i)}(x')$ (with $\phi^{(i)} \neq \mathbb{1}$)

$$\phi^{(i)}(x)\phi^{(i)*}(x') \sim C^{(i)(i^*)}_{(\mathbb{1})}(x,x')\mathbb{1} + \sum_{(j)\neq\mathbb{1}} C^{(i)(i^*)}_{(j)}(x,x';y)\,\phi^{(j)}(y)$$

We require that the scaling degree of $C^{(i)(i^*)}_{(\mathbb{1})}$ be positive, s.d. $\left\{C^{(i)(i^*)}_{\mathbb{1}}\right\} > 0$. We define

$$2\,\mathrm{dim}\phi^{(i)} = \mathrm{s.d.}\left\{C^{(i)(i^*)}_{\mathbb{1}}\right\}.$$

We require

$$\mathrm{s.d.}\left\{C^{(i)(j)}_{(k)}\right\} \leq \mathrm{dim}\phi^{(i)} + \mathrm{dim}\phi^{(j)} - \mathrm{dim}\phi^{(k)}.$$

The Algebra $\mathcal{A}(\mathcal{M})$ and State Space $\mathcal{S}(\mathcal{M})$

Let

$$\mathcal{O} = \sum_{(i)} \alpha_{(i)} \phi^{(i)}$$

where the sum is finite and each $\alpha_{(i)}$ is a differential operator. From the OPE for $\phi^{(i)}(x)\phi^{(j)}(x')$ we can obtain an OPE for $\mathcal{O}(x)\mathcal{O}^*(x')$ of the form

$$\mathcal{O}(x)\mathcal{O}^*(x') = \sum_{(j)} \tilde{C}_{(j)}(x, x'; y)\, \phi^{(j)}(y)$$

If the scaling degree of each $\tilde{C}_{(j)}$ is negative in all spacetimes (so that, in particular, $\tilde{C}_{(\mathbb{1})} = 0$), then we set $\mathcal{O} = 0$. The algebra $\mathcal{A}(\mathcal{M})$ is taken to be the free

*-algebra generated by the smeared fields $\phi^{(i)}(f)$ factored by the relations $\mathcal{O} = 0$ (if there are any such relations) and factored by spacelike commutativity/anti-commutativity relations.

The state space $\mathcal{S}(\mathcal{M})$ is taken to be all positive linear functions on $\mathcal{A}(\mathcal{M})$ that satisfy the microlocal spectrum condition and satisfy all of the OPE relations.

Both $\mathcal{A}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$ are uniquely determined by the OPE.

Some Key Results

A spin-statistics theorem has been proven. It takes the same form as in Minkowski spacetime axiomatic quantum field theory, i.e., commutation at spacelike separations is inconsistent for half-integral spin fields, and anti-commutation at spacelike separations is inconsistent for integer spin fields.

A PCT theorem has been proven. It takes a rather different form as compared with Minkowski spacetime axiomatic quantum field theory, since P and T do not correspond to isometries:

PCT Theorem: Let $\mathcal{M}$ be a background structure, i.e., a spacetime (M, g_{ab}), together with choices of time orientation and spacetime orientation. Let $\bar{\mathcal{M}}$ denote the same spacetime (M, g_{ab}) with the same choice of spacetime orientation, but the opposite choice of time orientation (and a corresponding choice of spin structure). Then $\mathcal{A}(\mathcal{M})$ and $\mathcal{A}(\bar{\mathcal{M}})$ are naturally (anti-linearly) *-isomorphic, and the dual action of this isomorphism yields an isomorphism of $\mathcal{S}(\bar{\mathcal{M}})$ and $\mathcal{S}(\mathcal{M})$.

Outlook and Grand Hopes

The attempt to generalize the axiomatic formulation QFT to curved spacetime—where no symmetries are present and no "preferred vacuum state" exists—has led us to a viewpoint wherein the existence of an OPE is elevated to a fundamental status, and the OPE itself contains all of the nontrivial information about the quantum field theory. It thereby gives a perspective that is much closer in nature to classical field theory, in that the entire content of the theory is expressed by local relations satisfied by the fields. Symmetries and "preferred states" play no role whatsoever in the formulation of the theory.

It is possible that the well known difficulties with the perturbative construction of interacting quantum field theory trace back to the non-analytic dependence of the vacuum state on the coupling parameters. This is illustrated by the perturbative construction of a massive Klein-Gordon field about $m^2 = 0$. The VEV

$$\langle 0|\varphi(x_1)\varphi(x_2)|0\rangle =$$
$$\frac{1}{4\pi^2}\Big(\frac{1}{\Delta x^2 + i0t} + m^2\, j[m^2\Delta x^2]\, \log[m^2(\Delta x^2 + i0t)] +$$
$$+m^2\, h[m^2\Delta x^2]\Big)$$

fails to be analytic in m^2, but the OPE coefficient

$$C(x_1, x_2; y) = \frac{1}{4\pi^2}\left(\frac{1}{\Delta x^2 + i0t} + m^2\, j[m^2 \Delta x^2]\, \log[\mu^2(\Delta x^2 + i0t)] + \right. \left. + m^2\, h[m^2 \Delta x^2]\right)$$

is analytic in m^2. Very recently, Hollands and Olbermann have made considerable progress toward developing perturbative rules to enable the direct computation of OPE coefficients. It would be interesting to re-examine the convergence of perturbation theory within this framework.