Constructing quantum field theories with Fedosov quantization Sketch of (another) programme, with G. Collini

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Outline

Rudolph Haag has proposed many years ago that it should be possible–and useful–to think of quantum field theory as being defined by some sort of algebraic structure. This viewpoint has been confirmed, and expanded, since then by many investigations, bringing to light, and making contact with, not least, several interesting mathematical structures. It is fair to say that his ideas/investigations have laid the foundation to a "school" of thought.

Different attitudes towards ‘schools’ and ‘dogmas’

1. “Instruct your children to follow all our laws and teachings.” [Moses, Dtn 32, 46-47]
2. “My doctrine is not a doctrine but just a vision. I have not given you any set rules, nor system.” [Lord Gautama Siddharta Buddha (?) or Bhagwan Shri Rajneesh (?)]
3. “I think that a man’s duty is to find where the truth is, or, if he cannot, at least take the best possible human doctrine and hardest to prove ...” [Plato]; Attidude of **this talk.**

I would like to suggest a general approach quantizing field theories based on the method of "deformation quantization", especially a construction due to Fedosov.

'Traditional method of quantization'

1. **Kinematical level:** Start with phase space $\mathscr{S}$, such as $\mathscr{S} = \{(x, p) \in \mathbb{R}^2\}$, together with Poisson bracket, such as $\{q, p\} = 1$. Quantization: Promote q, p, and more general observables $F(q, p)$ to operators on Hilbert space $L^2(\mathbb{R}, dq)$

$$q \to \hat{q}\,, \quad p \to \hat{p} = \frac{\hbar}{i}\frac{d}{dx}\,, \quad F(q,p) \to \hat{F} = ???$$

2. **Dynamical level:** In particular, promote classical Hamiltonian H to self-adjoint operator, $\hat{H}$, and define time evolution as

$$\hat{F}(t) = \mathrm{e}^{it\hat{H}}\hat{F}\mathrm{e}^{-it\hat{H}}\ .$$

But not all phase spaces $\mathscr{S}$ admit global coordinates. What are reasonable rules for defining $\hat{F}$ ('ordering ambiguity')?

Variant of this programme: ‘deformation quantization’. More ‘modest’, because one constructs the observables in a ‘weaker sense’. One assumes to be given in this programme a phase space $\mathscr{S}$ equipped with an associative Poisson bracket $\{\ .\ ,\ .\ \}$, which may or may not come from a ‘symplectic form’ σ (above example: $\sigma = dq \wedge dp$).

‘Deformation quantization’

1. $C^\infty(\mathscr{S})$ already is a Poisson algebra under anti-symmetric bracket $\{F, G\}$.
2. Introduce an associative product $\star_\hbar : C^\infty(\mathscr{S}) \otimes C^\infty(\mathscr{S}) \to C^\infty(\mathscr{S})$ with properties

$$1)\ F \star_\hbar G = FG + O(\hbar)\ ,\ \ 2)\ \frac{1}{i\hbar}(F \star_\hbar G - G \star_\hbar F) = \{F, G\} + O(\hbar)$$

(+ technical properties.)

No Hilbert space, no ordering ambiguity (we do not change the functions F, but the product).

It is a well-defined mathematical question whether all (finite-dimensional) Poisson manifolds $(\mathscr{S}, \{\ .\ ,\ .\ \})$ admit a deformation quantization.

1) **Simplest case:** In the case $\mathscr{S} = \mathbb{R}^{2n}$ with

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}\right)$$

a closed expression can be given for the $\star_\hbar$-product ('Moyal quantization')

$$F \star_\hbar G = \sum_{N=0}^{\infty} \frac{(i\hbar)^N}{N!} \sigma^{\mu_1\nu_1} \cdots \sigma^{\mu_N\nu_N} \partial_{\mu_1} \ldots \partial_{\mu_N} F \ \partial_{\nu_1} \ldots \partial_{\nu_N} G \ .$$

Here $\sigma = \sigma_{\mu\nu} dx^\mu \wedge dx^\nu \equiv dq^i \wedge dp_i$ in terms of global coordinates $(x^\mu) = (q^i, p_j)$ on $\mathbb{R}^{2n}$.

2) **Symplectic case:** In this case Poisson bracket comes from a closed, non-degenerate, 2-form σ on $\mathscr{S}$. Defined by

$$\{F, G\} = \sigma^{-1}(dF, dG)\ ,$$

which generalizes formula in the 'simplest case'. *Locally* same as 'simplest case' (Darboux coordinates), but not necessarily *globally*. Solution [Fedosov], ... $\to$ **This talk.**

3) **General case:** Poisson bracket does not come from a non-degenerate symplectic form. Solution given by [Kontsevich] (generalization: "formality conjecture"). This will *not be needed* in this talk.

In 'simplest case' 1), 'non-perturbative' solution to deformation problem is also available, similarly in case 2) for compact Kähler manifolds $(\mathscr{S}, \sigma, g)$, similarly 3). In general, only "formal solution"–i.e. formal power series in $\hbar$–available.

Deformation quantization in QFT

We would like to apply the philosophy of deformation quantization to field theories arising from a classical Lagrangian. $\Rightarrow$ 'Symplectic case', but *infinite dimensional* $\mathscr{S}$! The simplest, but very instructive, **example** is *linear* KG-theory (on Minkowski space)

$$(\Box - m^2)u = 0 \ .$$

In this case, $\mathscr{S}$, the underlying symplectic space is *linear*

$$\begin{aligned} \mathscr{S} &= \{\text{smooth sol. } u, \text{ compact supp. on } \Sigma\} \\ &= \{\text{pairs } (q,p) \in C_0^\infty \times C_0^\infty \text{ where } q = u|_\Sigma, p = \partial_0 u|_\Sigma\} \end{aligned}$$

The symplectic form $\sigma : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ is:

$$\sigma(u_1, u_2) = \int_\Sigma (q_1 p_2 - p_1 q_2) \ d^3x \ .$$

The *correct* deformation quantization ($\Leftrightarrow$ traditional quantization) is as follows [Dütsch & Fredenhagen]:

1. Choose 'observables' [replacing $C^\infty(\mathscr{S})$] as F's of form

$$F = \int f(x_1, \ldots, x_n)\varphi(x_1)\cdots\varphi(x_n)\ ,$$

   $\varphi(x)[u] \equiv u(x)$ is the *evaluation functional* on $\mathscr{S}$, and f is *any* distribution on $(\mathbb{R}^4)^n$ having $\mathrm{WF}(f) \cap (V_+^n \cup V_-^n) = \emptyset$.

2. $\star_\hbar$-product:

$$F \star_\hbar G = \sum_{N=0}^{\infty} \frac{\hbar^N}{N!} \int W(x_1, y_1)\cdots W(x_N, y_N) \frac{\delta^N}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)} F \ \frac{\delta^N}{\delta\varphi(y_1)\ldots\delta\varphi(y_n)} G\ ,$$

   $W =$ 'Wightman function'='2-pt function'.

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Explicitly

$$W(x,y) = \int_{V_+} d^4p\ \delta^4(p^2 - m^2)\ \mathrm{e}^{ip(x-y)}\ .$$

1. **Relation to 'traditional quantization':** It is a version of traditional quantization if we replace

$$F \to \hat{F} = \int f(x_1, \ldots, x_n)\ :\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n):$$

where $\hat{\varphi}(x)$ = standard field operator (creation + annihilation operators), $:\ :$:= normal ordering. Then $\star_\hbar$-product = Wick's theorem.

2. **Relation to 'simplest case':** Note that $\mathscr{S}$ is linear. Formally *equivalent* to Moyal quantization if we write $W^{jk} = i\sigma^{jk} + g^{jk}$, and *replace* in formula for $\star_\hbar$-product

$$i\hbar\sigma^{kl} \to \hbar(i\sigma^{kl} + g^{kl})\ .$$

Note: g^{kl} is a *metric* on $\mathscr{S}$ (really $T^*\mathscr{S}$), and $J^k{}_l = g_{lj}\sigma^{jk}$ is a *complex structure*.

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How can we generalize this methods to more interesting theories like

$$\Box\phi - V'(\phi) = 0 \quad ?$$

Let $\mathscr{S}$ be the space of solutions. Given $\phi \in \mathscr{S}$, consider solution to the *linearized* equations

$$(\Box - V''(\phi))u = 0$$

around ϕ as an element in the *tangent space* $T_\phi\mathscr{S}$. For each background solution define a corresponding 2-point function $W_\phi(x, y)$ having $\mathrm{WF}(W_\phi) \subset V_+ \times V_-$, which defines a quasi-free, pure state for the linearized theory in the background. Let

$$\mathscr{W}_\phi = \{\text{algebra of non-lin. functions } F : T_\phi\mathscr{S} \to \mathbb{R}, \text{ under } \star_\hbar\}$$

Note that this defines a bundle

$$\mathscr{W} = \bigcup_{\phi\in\mathscr{S}} \mathscr{W}_\phi \ ,$$

of *-algebras over $\mathscr{S}$, which is itself an algebra!

Further structure on $\mathscr{S}$:

1. Decompose into real and imaginary parts $W_\phi = i\Delta_\phi + G_\phi$, Then Δ_ϕ is causal propagator for linear operator $\Box - V''(\phi)$.
2. The object
$$\Delta_\phi = \int \Delta_\phi(x,y)\frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)} \in T_\phi\mathscr{S} \wedge T_\phi\mathscr{S}$$
defines a non-degenerate *skew symmetric tensor field* on $\mathscr{S}$ (= inverse symplectic form).
3. The similar object defined from G_ϕ defines a Riemannian *metric* on $\mathscr{S}$.
4. The composition of $\Delta^{-1} = \sigma$ and G gives a tensor field J of type $(1,1)$ on $\mathscr{S}$, which is an *almost complex structure*.
5. Using G, we can define (a) the *Levi-Civita connection* ∇, (b) *Riemann tensor*, R.

Aim:

The **aim** is to define a $\star_\hbar$-product on a suitable set of observables $F \in \mathrm{Func}(\mathscr{S})$ which should (at least) contain "composite fields" of the type

$$F(\phi) = \int f(x) \prod \partial^{w_i} \phi(x) \ , \quad f \in C_0^\infty(\mathbb{R}^4) \ ,$$

The $\star_\hbar$-product should be defined (at least) in sense of formal power series.

Already know: $\mathrm{Sect}(\mathscr{W})$ is algebra under "fibrewise" product

$$(F \cdot_\hbar G)(\phi) \equiv F_\phi \star_\hbar G_\phi \ ,$$

since $F_\phi \in \mathscr{W}_\phi$ for any $\phi \in \mathscr{S}$, and since $\mathscr{W}_\phi$ is already an algebra (the "free field Wick polynomial algebra" of the linearized theory around ϕ) with product $\star_\hbar$.

But: This does not give desired deformation quantization of $\mathrm{Func}(\mathscr{S})$, since $\mathrm{Sect}(\mathscr{S}, \mathscr{W})$ is a *different* (much larger) space!

How to relate $\mathrm{Func}(\mathscr{S})$ with $\mathrm{Sect}(\mathscr{S}, \mathscr{W})$?

Basic construction of $\star_\hbar$-product

1. Define a *flat* connection $DF = \nabla F + [A, F]_{\cdot_\hbar}$ for $F \in \mathrm{Sect}(\mathscr{W})$, where A is a suitable $\mathscr{W}$-valued 1-form on $\mathscr{S}$ satisfying $\nabla A + [A, A]_{\cdot_\hbar} = 0$.
2. Set up a one-one correspondence $\tau : \mathrm{Func}(\mathscr{S}) \to \mathrm{Sect}^0(\mathscr{W})$ between smooth functions and *flat* sections.
3. Define $(F \star_\hbar G)(\phi) := \tau^{-1}(\tau(F) \cdot_\hbar \tau(G))$.

Note that it is not clear that such a flat connection A exists, nor how to define it, but this works in *finite dimensions* [Fedosov], [Schlichenmacher et al.], [Waldmann],...

Many **new issues** due to *infinite-dimensional* nature of $\mathscr{S}$ in field theory!

First main theorem

The first main result we look for is:

Theorem 1

(approximate version) There exists a Fedosov connection A as a formal power series in $\hbar$ with the property

$$\text{order } \hbar^n \text{ part of } A = p_n(\nabla^{n-2}R, \nabla^{n-1}J) \ .$$

p_n is a polynomial. There is a one-one correspondence $\mathrm{Func}(\mathscr{S}) \to \mathrm{Sect}^0(\mathscr{W})$. $\hat{F} = \tau(F)$ has the form of a "perturbation series"

$$\hat{F} = F_0 + \hbar F_1 + \hbar^2 F_2 + \dots,$$

with $F_0 = F \cdot id_{\mathscr{W}}$.

This hoped-for theorem would give the desired "deformation quantization": Replace F by $\hat{F}$ and take the $\cdot_\hbar$ product in the bundle $\mathscr{W}$.

"Fedosov-" vs "ordinary" perturbation theory

Let $F \in \mathrm{Func}(\mathscr{S})$ be polynomial, local function of the field ϕ,

$$F(x) = F[\phi(x), \partial\phi(x), \dots, \partial^n\phi(x)] \equiv F[\phi(x)] \ ,$$

"Conventional" perturbation theory

Fock-space formula for interacting field for F:

$$= \sum_{N=0}^{\infty} (i/\hbar)^N \int_{x^0 > y_1^0 > \dots > y_N^0} [\dots [: F[\hat{\varphi}(x)] :, : V[\hat{\varphi}(y_1)] :], \dots, : V[\hat{\varphi}(x_N)] :] \ ,$$

$=$ operator on Fock-space $=$ "retarded product".

This formula is sometimes called "Haag's series" [Haag], [Fredenhagen & Duetsch].) This formula, like other formulas in perturbation theory, requires renormalization. If we take a VEV, we obtain (formally) ordinary "Gell-Mann-Low" formula.

In order to relate "conventional" and "Fedosov-type" construction, we *re-interprete* the conventional perturbative formula as an element of the algebra bundle $\mathscr{W}$. Recall that for $\phi \in \mathscr{S}$, one can define a "retarded product" as a map (for each $N \in \mathbb{N}_0$)

$$R_\phi : \mathrm{Func}(\mathscr{S}) \otimes \bigvee^{N} \mathrm{Func}(C^\infty(\mathbb{R}^4)) \to \mathscr{W}_\phi$$

denoted by $R_\phi(F, G^{\otimes N})$. Retarded products are explicitly constructed using Epstein-Glaser renormalization in [Fredenhagen & Duetsch], [Hollands & Wald].

Remark: Actually Func has to be replaced by an "extension", i.e. the cohomology of a certain chain complex

$$\mathrm{Func}_\hbar(\mathscr{S}) := H^*(s_\hbar : \mathrm{Polyvec}^{n+1}(\mathscr{S}) \to \mathrm{Polyvec}^{n}(\mathscr{S})) ,$$

where $s_\hbar = s_0 + \hbar s_1 + \hbar^2 s_2 + \ldots$ and $s_0 =$ 'Koszul-Tate differential'.

Fedosov vs. traditional perturbation theory

Using ‘retarded products’, define map
$\rho : \mathrm{Func}_\hbar(\mathscr{S}) \to \mathrm{Sect}(\mathscr{S}, \mathscr{W})$ by [with $F_\phi \equiv F(\phi + \varphi)$]:

$$\rho(F) = \sum_{N=0}^{\infty} \frac{(i/\hbar)^N}{N!} R_\phi \left(F_\phi; (\int V \ d^4x)^{\otimes N} \right)$$

Then it is hoped for that the following theorem can be proved

Theorem 2

Let $\tau : \mathrm{Func}_\hbar(\mathscr{S}) \to \mathrm{Sect}^0(\mathscr{S}, \mathscr{W})$ be the map which associates with an F a flat section $\hat{F} = \tau(F)$, $0 = \nabla \hat{F} + [A, \hat{F}]_{\cdot_\hbar}$. Then there there exists a unitary section $U \in \mathrm{Sect}(\mathscr{S}, \mathscr{W})$ (meaning $U \cdot_\hbar U^* = id$) such that

$$\rho(F) = U \cdot_\hbar \tau(F) \cdot_\hbar U^*$$

i.e. Fedosov quantization (i.e. τ) and ordinary perturbation theory (i.e. ρ) are equivalent.

For theories with a richer symmetry structure (supersymmetric field theories), the Fedosov approach makes natural several very interesting mathematical constructions. E.g. $N = 4$ Super-Yang-Mills theory:

$$\begin{aligned} I &= \mathrm{trace}\Big\{ -\frac{1}{2}\, F^2 + \frac{1}{4}(DG^+)^2 + \frac{1}{4}(DG^-)^2 \\ &\quad + \frac{e^2}{32}[G^+, G^+] + \frac{e^2}{32}[G^-, G^-] + \frac{e^2}{16}[G^+, G^-] \\ &\quad + i\bar{\psi}D\psi + ie\ \bar{\psi}[\psi, G^-] + ie\ \bar{\psi}\Gamma[\psi, G^+]\Big\} + \text{BRST closed} \end{aligned}$$

All fields in adjoint rep some simple Lie-algebra $\mathfrak{g}$. A classical solution $\phi = (A, G^+, G^-)$ is by definition a solution with vanishing fermion fields ψ. Analog of algebra $\mathscr{W}_\phi$:

1. $\mathscr{W}_\phi$ is tensor product of bosonic and fermionic fields
2. $\mathscr{W}_\phi$ contains additional "ghost fields"
3. $\mathscr{W}_\phi$ has graded derivation q_ϕ (BRST) with $q_\phi \circ q_\phi = 0$

Additionally, assume ϕ is "supersymmetric" or "BPS". This means that some of supersymmetry variations of the full theory annihilates ϕ.

EXAMPLE:

Certain kinds of classical "monopole configurations" e.g. [E. Weinberg]. Such solutions span a *finite-dimensional smooth submanifold* $\mathscr{M}_k \subset \mathscr{S}$ of dimension k (related to number of monopoles)

$\Rightarrow$ The linearized theory around ϕ also has a corresponding supersymmetry!

1. If ϕ supersymmetric, then $\mathscr{W}_\phi$ has graded derivation Q_ϕ such that $Q_\phi \circ q_\phi + q_\phi \circ Q_\phi = 0$, and such that $Q_\phi^2 =$ generator of time-translations in linearized theory.
2. We can assume $\nabla Q_\phi + [A_\phi, Q_\phi]_{\cdot_\hbar} = 0$, where A is Fedosov connection.

Now consider GNS-representation of $\mathscr{W}_\phi$ of H_ϕ induced by quasifree state ω_ϕ (associated with Wightman function $W(\phi, x, y)$), with vacuum vector Ω_ϕ annihilated by Q_ϕ. This gives a bundle of Hilbert spaces

$$H = \bigcup_{\phi \in \mathscr{M}_k} H_\phi$$

Let Z be the following operator:

$$Z : \Omega^*(\mathscr{M}_k; H) \to \Omega^*(\mathscr{M}_k; H) \ , \quad Z := \gamma D + iQ \ ,$$

where $D = \nabla + A$ Fedosov connection, γ even-odd automorphism.

Can define cocycle in the "(B, b)"-complex [Cuntz, Connes, Quillen] in the algebra $A_D = \mathrm{Sect}^0(\mathscr{M}_k, \mathscr{W})$:

$$\Phi_\omega(a_0, \ldots, a_{2n}) := \int_{0<t_1<\ldots<t_{2n}<1} dt_1 \ldots dt_{2n} \cdot$$
$$\cdot \int_{\mathscr{M}_k} \omega \wedge \mathrm{trace}\Big(\gamma \ a_0[Z, a_1(t_1))][Z, a_2(t_2)^\gamma] \cdots [Z, a_{2n}(t_{2n})^\gamma] \ e^{Z^2}\Big)$$

for any *closed* $\mathbb{C}$-valued form ω.

This construction can give rise to:

1. A Chern-type character $Ch : K_*(A_D) \to \mathbb{Z}$ by a general construction of Connes
2. Let $Ch(H)$ be the ordinary Chern character of the bundle H, and let $\widehat{A}(\mathscr{M}_k)$ denote the $\widehat{A}$-roof genus. The connection between the character map $\mathrm{Ch}(H_E, \nabla + \nabla^*, \gamma_E) : K_0(A) \to \mathbb{Z}$ derived from spectral triple $H_E := \Omega^*(\mathscr{M}_k, H)$, and cocycle Φ_ω is that

$$\mathrm{Ch}(H_E, D_E, \gamma_E)(e) = \mathrm{trace} \ \circ \ \Phi_\omega(e, e, ..., e + 1/2) \ ,$$

with

$$\omega = \widehat{A}(\mathscr{M}_k) \wedge \mathrm{Ch}(H) \ .$$

Conclusions

In this talk I have outlined the general idea for a new approach to the **construction** of quantum field theories from **classical field theories**. The output of the construction is an algebra of observables, as Haag proposed. This algebra has the structure of a space of **flat sections in a bundle of algebras** over the space of classical solutions. Each fibre is a canonical, CCR/CAR type algebra, corresponding to the linearized theory around the given classical solution. The non-trivial dynamical content is encoded in the **flatness condition** of the sections. Interesting connections with other parts of mathematics such as cyclic cohmology, topology of moduli spaces.