Flows of conformal field theories

S. Hollands

UNIVERSITÄT LEIPZIG

based on 1710.05601

“Quantum Physics Meets Mathematics”
on occasion of 70th birthday of K. Fredenhagen
Hamburg, 8.12.2017

European Research Council
Established by the European Commission

Operator product expansion: Statement

The Operator Product Expansion (OPE) states that for any sufficiently regular quantum state Ψ and any set of local fields $\mathcal{O}_A$, one should have, for points x_i that are space-like to each other,

Operator product expansion (formal)

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \rangle_\Psi \sim \sum_B \underbrace{\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \rangle_\Psi$$

where $\sim$ means for example that the difference between both sides is of order $\sim dist^D$ when $dist \to 0$, where $dist$ is the maximum $\max(dist(x_i, x_j))$ and D increases as more operators $\mathcal{O}_B$ are included in the sum.

Operator product expansion: History

- 1969 Wilson proposes **O**perator **P**roduct **E**xpansion (OPE)
- 1970 Zimmermann shows OPE consistent with renormalized perturbation theory [Brandeis lectures 1970]
- 1972 Wilson-Zimmermann: general arguments in favor of OPE
- In 1970s, various groups [Polyakov 1974, Mack 1977, Gatto et al. 1973, Schroer et al. 1974] realize that the OPE simplifies in CFTs and associativity constraints can be turned into "conformal bootstrap".
- In 1980s, OPE used to study conformal field theories in $d = 2$ [Belavin et al. 1984,...].
- Borcherds and others propose to formalize their ideas in the framework of Vertex Operator Algebras [Borcherds 1988] $\Rightarrow$ Mathematics
- Recently numerical bootstrap: new approach to CFTs in $d > 2$ [Rychkov, Simmons-Duffin, El-Showk, Paulos, Penedones, Rattazzi and others 2010s],...
- This talk: Variational principle and OPE, marginal flows.

Formulating QFT via operator product expansion

An axiomatic, intrinsically *"generally covariant" formulation of QFT* can be given via algebraic methods, e.g. by formulating QFT via OPE [Hollands & Wald 2012, Hollands 2010]. A quantum field theory consists of:

- An abstract countable list of quantum fields $\{\mathcal{O}_A\}$, where A is an abstract label (incl. tensor/spinor indices),
- A list of real non-negative numbers $\{\Delta_A\}$, called the "dimensions",
- A list of OPE coefficients $\mathcal{C}_{A_1 \dots A_N}^{B}(x_1, \dots, x_N)$ with a number of properties.
- A permutation (anti-) symmetry property related to the bosonic/fermionic nature of operators.
- A rule how the OPE coefficients change under a change in the coupling parameter(s) (Hochschild cohomology [Hollands 2010], or "action principle" [Holland & Hollands 2012], new)

Formulating QFT via operator product expansion

The most important properties are:

Properties of OPE

- The OPE coefficients satisfy a scaling law related to the dimensions $\{\Delta_A\}$
- The OPE coefficients should be generally covariant functionals of the spacetime metric g within each sufficiently small neighborhood.
- The OPE should satisfy an associativity law.
- The singularities of the OPE coefficients are characterized by microlocal spectrum condition.

Scaling behavior

If I fix $p = x_N$ as a reference point and mutually distinct non-zero vectors $\xi_i \in T_pM$, then:

$$(x_1, \ldots, x_N) = (\mathrm{Exp}_p(\epsilon\xi_1), \ldots, \mathrm{Exp}_p(\epsilon\xi_{N-1}), p)$$

describes 1-parameter family of configurations coalescing as $\epsilon \to 0$. The scaling property means that, for this configuration

$$\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N) = o(\epsilon^{-\Delta_{A_1} \cdots - \Delta_{A_N} + \Delta_B})$$

when $\epsilon \to 0$.

Rationality/nuclearity

If N_Δ is the number of fields with dimension Δ, then $\sum_\Delta N_\Delta q^\Delta$ should exist for small $0 < q < 1$.

Associativity

This condition states that if all points are spacelike, and if

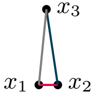
$$\xi \equiv \frac{\max_{1 \le i \le M}(dist(x_i, x_M))}{\min_{M < j \le N}(dist(x_j, x_M))} < 1$$

then

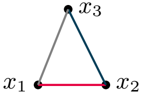
Associativity law

$$\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N) = \sum_C \mathcal{C}^C_{A_1 \ldots A_M}(x_1, \ldots, x_M) \mathcal{C}^B_{C \ldots A_M}(x_M, \ldots, x_M)$$

in the sense of an absolutely convergent (!) sum.



for $\xi \ll 1$



for $\xi \approx 1$

Microlocal spectrum condition

This is a condition on the wave front set of each coefficient which characterizes the nature of the singularities from the point of view of "local momentum space". It uses the concept of the "wave front set" of a distribution and was proposed in a similar context by [Brunetti, Fredenhagen, Köhler 1999].

But the OPE also makes sense on Riemannian spaces, where the condition is simply that $\mathcal{C}_{A_1 \dots A_N}^{B}(x_1, \dots, x_N)$ is smooth for $x_i \neq x_j$.

Assumption from now on:

From now on, I will even assume to be in flat Euclidean space of dimension $d > 1$, and I will assume that the OPE coefficients are invariant under Euclidean group.

Action principle

In many examples, we have 1-parameter family of QFTs depending on "coupling" g. Then OPE coefficients are functions of g. Even though our definition of QFT is abstract, we can think of g typically as a coupling parameter in the underlying classical action (if any).

Action principle

There is a kind of "action principle" for OPE coefficients if we "deform" class. action S by

$$S \to S + g \int \mathcal{O} \, \mathrm{d}^d x$$

Terminology

Renormalization "slang":

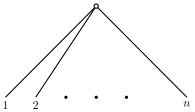
- If $\Delta_{\mathcal{O}} < d$, one speaks of "relevant" perturbation (e.g. mass term).
- If $\Delta_{O} = d$, one speaks of "marginal" perturbation.
- If deformed theory remains conformal for finite g one speaks of a "strictly marginal" perturbation. Main part of this talk.

Action principle

To write down the action principle, use graphical notation. I draw an OPE coefficient

$$\mathcal{C}_{A_1 \dots A_n}^{B}(x_1, \dots, x_n)$$

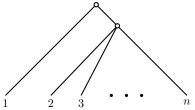
as



I draw a concatenation of OPE coefficients

$$\mathcal{C}_{A_1 C}^{B}(x_1, x_n)\mathcal{C}_{A_2 \dots A_n}^{C}(x_2, \dots, x_n)$$

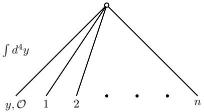
as



Attention: None of these diagrams is a "Feynman graph"!

Action principle

I also write



where

- $\mathcal{O}$ denotes the “deformation”
- $\int \mathrm{d}^d y =$ integral over $\{|y - x_n| < L\}$.
- $L =$ length scale that is part of the definition of the theory.

Action principle

There is a kind of "action principle" for OPE coefficients if we "deform $S \to S + g \int \mathcal{O}$" [Holland & SH 2014]:

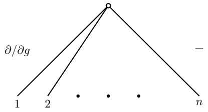


Figure: Functional equation, left side. The tree represents a coefficient $\mathcal{C}^{B}_{A_1 \dots A_n}(x_1, \dots, x_n)$

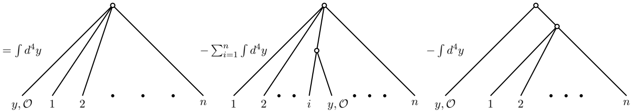


Figure: Functional equation, right side. The composite trees represent concatenations of coefficients, e.g. the rightmost tree means $\sum_C \mathcal{C}^{C}_{A_1 \dots A_n}(x_1, \dots, x_n) \mathcal{C}^{B}_{\mathcal{O}C}(y, x_n)$

Action principle

Theorem

In perturbation theory, $d = 4$, to any order in g [Holland & SH 2013, 2014]:

$$\partial_g \mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N) = \int_{|y - x_N| < L} \mathrm{d}^4 y \Big[\mathcal{C}^B_{\mathcal{O} A_1 \dots A_N}(y, x_1, \dots, x_N) - \sum_{i=1}^{N} \sum_{\Delta_C \leq \Delta_i} \mathcal{C}^C_{\mathcal{O} A_i}(y, x_i)\, \mathcal{C}^B_{A_1 \dots \widehat{A_i}\, C \dots A_N}(x_1, \dots, x_N) - \sum_{\Delta_C < \Delta_B} \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{C}^B_{\mathcal{O} C}(y, x_N) \Big]$$

when $\mathcal{O}$ is a marginal/relevant operator.

- Can compute OPE coefficients to any perturbation order by iteration.
- But equation should also hold non-perturbatively!
- $L \to \hat{L}$ equivalent to

$$\mathcal{O}_A \to \hat{\mathcal{O}}_A = \sum Z^B_A(g, \tau) \cdot \mathcal{O}_B \tag{0.1}$$

and $g \to \hat{g} = \hat{g}(g, \tau)$. $\Rightarrow$ RG equations! ($\tau = \log L/\hat{L} =$ RG "time").

Examples

- In ϕ^4 theory ($d = 4$), i.e. $\mathcal{O} = -\phi^4$ (marginal pert.), one can compute OPE coefficients order by order in g. Renormalization and associativity "automatic" [Holland & SH 2014].
- For local gauge theories (e.g. YM-theory), there holds a similar action principle, supplemented by an "evolution equation" for the BRST-operator (as a function of g) [Fröb & Holland 2016]
- For relevant pert., antecedent in [Guida & Magnoli 1996]
- For exactly marginal pert. of conformal field theories, simplification of equation to ODE for conformal data (this talk) [SH 2017, Behan 2017].

CFT basics

In the rest of the talk, I want to show how this works more concretely in conformal QFTs (CFTs) in $d > 1$ dimensions.

CFT structure

- In a CFT, one can single out particular fields called "primaries" which have a particularly simple "transformation law" under conformal transformations. These primaries are called $\{\mathcal{O}_i\}$ from now. All other fields are "descendants" and are simply derivatives of primary fields.
- For primaries, the vacuum N-point functions transform "covariantly" under a conf. transf. $\gamma \in \mathrm{SO}(d+1,1)$

$$\langle \mathcal{O}_1(\gamma x_1) \dots \mathcal{O}_N(\gamma x_N) \rangle = \prod_{j=1}^{N} \Omega(x_j)^{\Delta_j} \bigotimes_{j=1}^{N} D_j(R(x_j)) \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) \rangle,$$

where $R(x,\gamma) \in \mathrm{SO}(d)$ and the conformal factor $\Omega(x,\gamma)$ are defined by $\partial\gamma(x)/\partial x = \Omega(x)R(x)$ and $D_j =$ finite dim. irrep.

CFT basics ctd

CFT structure

- $\Rightarrow$ The 3-point function must have form

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = |x_{12}|^{-\Delta_i-\Delta_j+\Delta_k}|x_{23}|^{-\Delta_j-\Delta_k+\Delta_i}|x_{13}|^{-\Delta_k-\Delta_i+\Delta_j} \times \sum_\alpha \lambda^\alpha_{ijk}\, t^\alpha_{ijk}(x_1,x_2,x_2),$$

  where $t^\alpha_{ijk}(x_1,x_2,x_3)$ runs through a basis of conformally covariant tensor structures (see e.g. [Kravchuk et al. 2016, Osborne 1999, Weinberg 2010,...] for details).
- The higher N-point functions are also restricted.

The OPE between two primary fields in CFT has a special form that is entirely determined by certain data:

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \lambda^{\alpha}_{ijk} |x_{12}|^{-\Delta_i-\Delta_j+\Delta_k}\, \mathcal{P}^{\alpha}_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2)$$

where the $\mathcal{P}$s are (pseudo-) differential operators determined completely by group theoretical considerations [Schroer & Swieca 1974],....

CFT=conformal data

In view of associativity, all OPE coefficients, hence the entire CFT is uniquely determined by conformal data λ_{ijk} and Δ_k! [Gatto et al., Mack, Migdal, Polyakov, Belavin et al., ... 1970-80s, conformal bootstrap 2010s: Rychkov, Simons-Duffin, Rattazzi, Paulos, Penedones,...]

If we have a family of CFTs, the conformal data are functions of g, $\lambda_{ijk}(g)$ and $\Delta_k(g)$. We are interested in the dependence on g.

Action principle in CFT: flow equation

The action principle implies the following dynamical system for the conformal data [SH 2017]

Main formula of this talk

$$\frac{\mathrm{d}}{\mathrm{d}g}\Delta_i = \sum_\alpha \mathcal{D}_i^\alpha \lambda_{\mathcal{O}ii}^\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}g}\lambda_{jkl}^\mu = \sum_{m\alpha\beta} \left({}^a\mathcal{T}_{jklm}^{\alpha\beta\mu}\, \lambda_{\mathcal{O}jm}^\alpha \lambda_{klm}^\beta + {}^b\mathcal{T}_{jklm}^{\alpha\beta\mu}\, \lambda_{jkm}^\alpha \lambda_{\mathcal{O}lm}^\beta + {}^c\mathcal{T}_{jklm}^{\alpha\beta\mu}\, \lambda_{\mathcal{O}km}^\alpha \lambda_{jlm}^\beta \right),$$

Here:

- The $\mathcal{T}, \mathcal{D}$s are numbers that are entirely determined by the representation theory of $\mathrm{SO}(d+1,1)$.
- The flow equation holds as long as spectrum of dimensions is non-degenerate $\Delta_i - \Delta_j \notin \mathbb{Z}$.

Action principle in CFT: $d = 2$

$d = 2$ is not really special in this approach, but technically simpler.

1. Identify $\mathbb{R}^2$ with $\mathbb{C}$.
2. Conformal group $\mathrm{SO}(3, 1)$.
3. Tensor structure of operator $\rightarrow$ spin $s_{\mathcal{O}} \in \mathbb{Z}/2$. Introduce: $h + \bar{h} = \Delta, h - \bar{h} = s$.
4. Anharmonic ratios (any $d > 1$),

$$u = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{13}|^2 |x_{24}|^2}, \quad v = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2}$$

   Introduce: Dolan-Osborn variables $u = \bar{z}z, v = (1 - \bar{z})(1 - z)$

5. Introduce: Fundamental domain $\mathcal{F} = \mathbb{C}/S_3$ under group action $S_3 = \{z \mapsto 1/z, z/(z-1), (z-1)/z\}$.

Action principle in CFT: $d = 2$

In $d = 2$, tensor structures of operators are unique, no need for index α labelling tensor structures. Then

Main formula [SH 2017]

$$
{}^{a}\mathcal{T}_{jklm} = \int_{\mathcal{F}} \mathrm{d}^2 z \; \frac{(z-1)^{h_{kl}-2}}{z^{1-h_j-h_m}} \frac{(\bar{z}-1)^{\bar{h}_{kl}-2}}{\bar{z}^{1-\bar{h}_j-\bar{h}_m}} \times
$$
$$
{}_2F_1(h_m+h_j-1, h_m+h_{kl}, 2h_m; z)\; {}_2F_1(\bar{h}_m+\bar{h}_j-1, \bar{h}_m+\bar{h}_{kl}, 2\bar{h}_m; \bar{z})
$$

Analogous formulas exist for ${}^{b}\mathcal{T}_{jklm}, {}^{c}\mathcal{T}_{jklm}$. In $d = 1$ formulas even simpler and independently obtained by [Behan 2017]

Action principle in CFT: $d = 2$

Since tensor structures are trivial, dynamical system for the conformal data simplifies to [SH 2017]

Dynamical system f. conformal data, $d = 2$

$$\frac{\mathrm{d}}{\mathrm{d}g}\Delta_i = 4\pi\lambda_{\mathcal{O}ii}$$

$$\frac{\mathrm{d}}{\mathrm{d}g}\lambda_{jkl} = \sum_m \left({}^a\mathcal{T}_{jklm}\ \lambda_{\mathcal{O}jm}\lambda_{klm} + {}^b\mathcal{T}_{jklm}\ \lambda_{jkm}\lambda_{\mathcal{O}lm} + {}^c\mathcal{T}_{jklm}\ \lambda_{\mathcal{O}km}\lambda_{jlm} \right),$$

Here:

- The $\mathcal{T}$s are numbers defined before.
- The flow equation holds as long as spectrum of dimensions is non-degenerate $\Delta_i - \Delta_j \notin \mathbb{Z}$.

Action principle in CFT: general $d > 1$

- The case of general dimension d is qualitatively not different.
- However, in practice one needs (a) classification of conformally covariant 3-point and 4-point tensor structures (possible see e.g. [Kravchuk et al. 2016, Osborne 1999, Weinberg 2010,...]) (b) spinning "conformal blocks" (technically difficult, see e.g. [Costa et al. 2011, Costa et al. 2016, Kravchuk 2017, Schomerus & Karateev 2017, ...]. Needed: Representation/invariant theory of $\mathrm{SO}(d+1,1)$.
- Examples of marginal flows in $d = 4$: $\mathcal{N} = 2, 4$ Super-YM theories. Degeneracies in spectrum!
- Dynamical system gives in principle a way to construct these theories non-perturbatively starting from the underlying free field theory ($g = 0$)! (Newton iteration)

Conclusions & Outlook

Main points

1. QFT on curved manifolds is best formulated in terms of algebraic relations + states
2. The OPE satisfies an action principle giving the dependence on coupling(s) g (manifold of QFTs)
3. This action principle leads to a dynamical system for conformal data in CFTs
4. This dynamical system can in principle be used to construct $d = 4$ $\mathcal{N} = 2, 4$ Super-YM theories non-perturbatively at finite N if degeneracy problem can be solved.
   Perhaps large N/integrability techniques useful to lift degeneracy near $g = 0$ [Beisert et al.] for $\mathcal{N} = 4$-theory.