Yet More Ado About Nothing:
The Remarkable Relativistic Vacuum State

What is the vacuum in modern science? Roughly speaking, it is that which is left over after all which can “possibly” be removed has been removed. The vacuum is therefore an idealization which is only approximately realized in the laboratory and in nature. But it is a most useful idealization and a surprisingly rich concept. Among other roles, it serves as a physically distinguished reference state with respect to which other physical states can be defined and referred.

The Mathematical Framework of AQFT

The operationally primary objects are the observables (equivalence classes of measuring apparata) of the quantum system under investigation and the states (equivalence classes of preparation apparata) in which the system is prepared. These determine (in principle) the basic data of AQFT:

- An (isotonous) net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset M}$ of unital C*–algebras generated by all observables measurable in the spacetime regions $\mathcal{O} \subset M$ (d spacetime dimensional Minkowski space).
- A state ω on the quasilocal observable algebra $\mathcal{A}$ generated by $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset M}$.

Nets of von Neumann Algebras

Let $\mathcal{R}_\omega(\mathcal{O}) \doteq \pi_\omega(\mathcal{A}(\mathcal{O}))''$ and $\mathcal{R}_\omega \doteq \pi_\omega(\mathcal{A})''$. Under different sets of general conditions (Driessler; Fredenhagen; Buchholz, D'Antoni & Fredenhagen *etc*), the algebras $\mathcal{R}_\omega(\mathcal{O})$ are mutually isomorphic for a large class of regions $\mathcal{O}$.

The primary encoding of information is located in the inclusions

$$\ldots \mathcal{R}_\omega(\mathcal{O}_1) \subset \mathcal{R}_\omega(\mathcal{O}_2) \subset \mathcal{R}_\omega(\mathcal{O}_3) \ldots$$

in the net $\{\mathcal{R}_\omega(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{R}}$ and not in the algebras themselves.

Vacuum State

- **Vacuum state** : A translation invariant state ω on a covariant net whose corresponding GNS–representation satisfies the spectrum condition: the joint spectrum of the self–adjoint generators of the strongly continuous unitary representation $U_\omega(\mathbb{R}^d)$ of the translation subgroup of $\mathcal{P}_+^\uparrow$ lies in the closed forward light cone. The corresponding GNS representation is a vacuum representation.

Note: Though this is the standard definition, there are crucial elements which are not expressed solely in terms of the initial net and state: the action of the translation group on the space–time and on the observable algebras and the stability condition which is the spectrum condition. (Indeed, even Minkowski space and the translation group themselves.)

Examples of This Structure Exist!

Concrete examples have been rigorously constructed by various means!

(Araki; Glimm & Jaffe; Brunetti, Guido & Longo; Lechner *etc*.)

Associated Vacuum Representations

Moreover, general conditions are known under which to a quantum field model without a vacuum state can be (under certain conditions uniquely) associated a vacuum representation which is physically equivalent and locally unitarily equivalent to it. These ideas go back to Borchers, Haag and Schroer: Consider $\langle \Phi, A(x)\Phi \rangle$ for suitable states Φ and sufficiently many observables A as x tends to spacelike infinity. Although the subsequent discovery of soliton states and topological charges excluded the existence of such limits in general, under certain conditions

$$\langle \Omega, A\Omega \rangle \doteq \lim_{x \to \infty} \langle \Phi, A(x)\Phi \rangle$$

defines a vacuum state on the given net. Hence, the mathematical existence of a vacuum state is often assured even in models which are not initially provided with one.

Examples of such conditions are:

- Φ is a vector in a "massive particle representation." (Buchholz & Fredenhagen)
- There is a sufficiently large set $\mathcal{D}$ of local observables such that for some $r \in [1, d-1)$

$$\sup_{x_0} \int \, d^{d-1}\vec{x} \quad \| \, [A^*, A(x_0, \vec{x})] \, \Phi \|^r$$

  for all $\Phi \in \mathcal{H}$ and $A \in \mathcal{D}$. (Buchholz & Wanzenberg)

- A strengthened nuclearity condition, satisfied *e.g.* by the free massless field. (Dybalski)

Immediate Consequences of the Definition

Theorem 1 (Reeh & Schlieder; Araki)**.** *In any vacuum representation satisfying locality and the condition* $\mathcal{R}_\omega = \bigvee_{x \in M} \mathcal{R}_\omega(\mathcal{O} + x)$*, all* $\mathcal{O}$*, the implementing vector* Ω_ω *is cyclic and separating for* $\mathcal{R}_\omega(\mathcal{O})$*, for all* $\mathcal{O}$*.*

Any (vector) state can be arbitrarily well approximated by a local perturbation of the vacuum state.

Thus, in principle, in a laboratory on earth one can, by artfully manipulating vacuum fluctuations, construct a house on the backside of the moon.

There are no local particle counters. Indeed, every nonzero local projection has nonzero vacuum expectation.

If C is a particle counter, then since there are no particles in the vacuum, one must have $\langle \Omega, C\Omega \rangle = 0$. If $C \in \mathcal{R}_\omega(\mathcal{O})$, then $C = 0$.

The vacuum is entangled across the pair $(\mathcal{R}_\omega(\mathcal{O}_1), \mathcal{R}_\omega(\mathcal{O}_2))$ for any spacelike separated $\mathcal{O}_1, \mathcal{O}_2$. (Halvorson & Clifton)

A state is entangled across $(\mathcal{R}_\omega(\mathcal{O}_1), \mathcal{R}_\omega(\mathcal{O}_2))$ if it is **not** (a limit of) a mixture of product states:

$$\langle \Phi, A_1 A_2 \Phi \rangle = \langle \Phi, A_1 \Phi \rangle \langle \Phi, A_2 \Phi \rangle$$

for all $A_1 \in \mathcal{R}_\omega(\mathcal{O}_1)$ and $A_2 \in \mathcal{R}_\omega(\mathcal{O}_2)$.

Indeed, the vacuum is 1–distillable across $(\mathcal{R}_\omega(\mathcal{O}_1), \mathcal{R}_\omega(\mathcal{O}_2))$. (Verch & Werner)

In the vacuum state, Bell's inequalities are maximally violated across the pair $(\mathcal{R}_\omega(\mathcal{O}_1), \mathcal{R}_\omega(\mathcal{O}_2))$ for any spacelike separated tangent $\mathcal{O}_1, \mathcal{O}_2$. Hence, the vacuum is maximally entangled. (S. & Werner)

Bell's inequality (CHSH form):

$$\frac{1}{2}\langle \Phi, (A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2)\Phi \rangle \leq 1$$

for all $A_i \in \mathcal{R}_\omega(\mathcal{O}_1)$, $B_j \in \mathcal{R}_\omega(\mathcal{O}_2)$, $\|A_i\|, \|B_j\| \leq 1$.

If fact, for such regions $\mathcal{O}_1, \mathcal{O}_2$,

$$\beta(\phi, \mathcal{R}_\omega(\mathcal{O}_1), \mathcal{R}_\omega(\mathcal{O}_2)) = \sqrt{2}\,,$$

for all states ϕ, including the vacuum.

However, **all** of the above assertions are also true of any states analytic for the energy. What, then, distinguishes the vacuum state?

Tomita–Takesaki Theory

Given a von Neumann algebra $\mathcal{M}$ with a cyclic and separating vector Ω, the modular theory of Tomita and Takesaki yields a unique antiunitary involution J and positive Δ such that $J\Omega = \Omega = \Delta\Omega$,

$$J\mathcal{M}J = \mathcal{M}' \quad , \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

for all $t \in \mathbb{R}$.

Hence, by the Reeh–Schlieder Theorem, in a vacuum representation one has the modular objects $J_{\mathcal{O}}, \Delta_{\mathcal{O}}$ corresponding to $(\mathcal{R}_\omega(\mathcal{O}), \Omega_\omega)$. **Crucial: The modular objects are completely determined by the algebra and state, *i.e.* by the observables and preparation of the quantum system.**

- $\mathcal{W}$: The set of wedges: After choosing a coordinatization of M, define the right wedge $W_R = \{x = (x_0, x_1, x_2, x_3) \in M \mid x_1 > |x_0|\}$ and the set of wedges $\mathcal{W} = \{\lambda W_R \mid \lambda \in \mathcal{P}_+^\uparrow\}$. ($\mathcal{W}$ is independent of the choice of coordinatization.)
- θ_W : $\theta_R \in \mathcal{P}_+$ is the reflection through the edge $\{(0, 0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$ of the wedge W_R. θ_W is the corresponding "reflection" about the edge of W ($\theta_W = \lambda \theta_R \lambda^{-1}$, for $W = \lambda W_R$).
- $\lambda_W(t)$: $\{\lambda_W(t) \mid t \in \mathbb{R}\} \subset \mathcal{P}_+^\uparrow$ is the one-parameter subgroup of Lorentz boosts leaving W invariant.

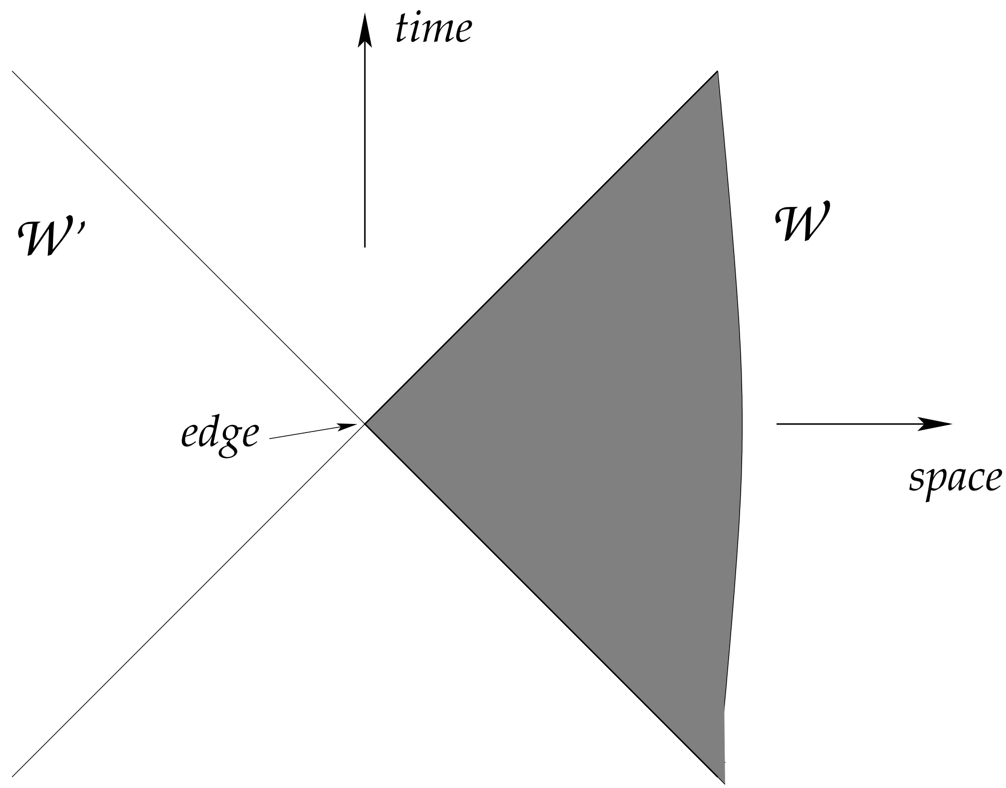

Figure 1: A wedge $\mathcal{W}$, its causal complement $\mathcal{W}'$ and their common edge

Bisognano–Wichmann Theorem

Theorem 2 (Bisognano & Wichmann)**.** *Given a net of von Neumann algebras* $\{\mathcal{R}\omega(\mathcal{O})\}$ *locally associated with a quantum field satisfying the Wightman axioms (i.e. in a vacuum representation), one has*

$$J_{W_R} = \Theta U_\pi \, , \, \Delta_W^{it} = U(\lambda_W(2\pi t))$$

where Θ *is the PCT-operator associated to the Wightman field and* U_π *implements the rotation through the angle* π *about the* 1*-axis. Hence,*

$$J_W \mathcal{R}_\omega(\mathcal{O}) J_W = \mathcal{R}_\omega(\theta_W \mathcal{O}) \, , \, \Delta_W^{it} \mathcal{R}_\omega(\mathcal{O}) \Delta_W^{-it} = \mathcal{R}_\omega(\lambda_W(2\pi t)\mathcal{O}) \, ,$$

for all $W \in \mathcal{W}$ *and* $\mathcal{O} \subset M$*.*

Consequences

- (Buchholz & S.) If $\mathcal{J}$ is the group generated by $\{J_W \mid W \in \mathcal{W}\}$, then $\mathcal{J} = U(\mathcal{P}_+)$. (Modular involutions encode the isometries of M and the dynamics of the quantum field.)

- (Schroer) If the quantum field satisfies asymptotic completeness and $J_{W_R}^{(0)}$ represents the modular involution corresponding to $(\mathcal{R}^{(0)}(W_R), \Omega)$, then

$$S = J_{W_R} J_{W_R}^{(0)} ,$$

where S is the scattering matrix for the field theory.

Further Consequences

- (Sewell) The vacuum state is a thermal equilibrium state at temperature $T = a\hbar/(2\pi k_B c)$ (in the observer's proper time) for every uniformly accelerated observer. (With a one G acceleration, $T = 4 \times 10^{-20}$ K.)
- (Buchholz & S.) $J_W J_{\mathcal{O}} J_W = J_{\theta_W \mathcal{O}}$, for all $W \in \mathcal{W}$ and $\mathcal{O}$.

  Algebraic relations among the modular objects are determined by and, in turn, encode information about the inclusions $\ldots \mathcal{R}_\omega(\mathcal{O}_1) \subset \mathcal{R}_\omega(\mathcal{O}_2) \ldots$.

Do such results hold only for the vacuum state?

Recall:

In the definition of the vacuum, the action of the translation group on the space–time and on the observable algebras, as well as the spectrum condition, are not expressed in terms of the operationally intrinsic states and observables.

Is there an intrinsic characterization of vacuum states?

The role of the vacuum state in Minkowski space theories has proven to be so central that when theorists tried to formulate quantum field theory in space–times other than Minkowski space, they tried to find analogous states in these new settings, thereby running into some serious conceptual and mathematical problems.

Some representative problems:

- What could replace the large isometry group of Minkowski space in the definition of “vacuum state”, in light of the fact that the isometry group of a generic space–time is trivial?
- In the definition of “vacuum state” the spectrum condition serves as a stability condition; what could replace it even in such highly symmetric space–times as de Sitter space, where the isometry group, though large, does not contain any translations?

Finding intrinsic characterizations of the Minkowski space vacuum has lead to answers to such questions (and shall lead to more in the future).

Condition of Geometric Modular Action

Definition 0.1 (Buchholz & S.)**.** *A state* ω *on a net* $\{\mathcal{R}_\omega(\mathcal{O})\}$ *satisfies the* Condition of Geometric Modular Action *if the vector* Ω_ω *is cyclic and separating for* $\mathcal{R}_\omega(W)$*,* $W \in \mathcal{W}$*, and if the modular conjugation* J_W *corresponding to* $(\mathcal{R}_\omega(W), \Omega_\omega)$ *satisfies*

$$J_W \{\mathcal{R}_\omega(\widetilde{W}) \mid \widetilde{W} \in \mathcal{W}\} J_W \subset \{\mathcal{R}_\omega(\widetilde{W}) \mid \widetilde{W} \in \mathcal{W}\} .$$

for all $W \in \mathcal{W}$*.*

Note: This condition is a consequence of the theorem of Bisognano and Wichmann.

Theorem 3 (Buchholz, Dreyer, Florig & S.). *If a state ω on a net $\{\mathcal{R}_\omega(\mathcal{O})\}$ satisfies the Condition of Geometric Modular Action and some weak technical conditions (all expressed solely in terms of the state and net), then $J_W \mathcal{R}_\omega(\mathcal{O}) J_W = \mathcal{R}_\omega(\theta_W \mathcal{O})$, all $\mathcal{O}$, $W \in \mathcal{W}$. Moreover, $\mathcal{J}$ provides a canonical strongly continuous (anti)unitary representation of $\mathcal{P}_+$ under which $\{\mathcal{R}_\omega(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{R}}$ transforms covariantly and which leaves Ω invariant. In addition, the net satisfies locality.*

If, further, $\Delta_W^{it} \in \mathcal{J}$ for all $W \in \mathcal{W}$, $t \in \mathbb{R}$, then modular covariance is satisfied and the state ω is a vacuum state.

Only the vacuum state has the properties stated in the conclusion of the Bisognano–Wichmann theorem. And this theorem provides an intrinsic characterization of the vacuum state.

So, from a suitable state and net of observable algebras on Minkowski space one can derive a representation of the isometry group of the space–time acting covariantly upon the observables *etc*. But the space–time itself and its isometries are not expressed in terms of states and observables.

Can one derive space–time itself from a suitable state and collection of observable algebras?

Deriving Space–Time From States and Observables

Theorem 4 (S. & White)**.** *Let ω be a state on a net $\mathcal{R}_i$, $i \in I$, of von Neumann algebras such that Ω_ω is cyclic and separating for each $\mathcal{R}_i$, $i \in I$. Let $\mathcal{J}$ be the group on $\mathcal{H}_\omega$ generated by $\{J_i \mid i \in I\}$. Assume that the CGMA is satisfied (for each $i \in I$, $\mathrm{ad} J_i$ leaves the set $\{\mathcal{R}_i\}_{i \in I}$ invariant). Then if certain purely algebraic relations in $\mathcal{J}$ hold, there exists a model of three dimensional Minkowski space on which each J_i, $i \in I$, acts adjointly as the reflection about the edge of some wedge. $\mathcal{J}$ is then isomorphic to $\mathcal{P}_+$ and forms a strongly continuous (anti)unitary representation U of $\mathcal{P}_+$. Moreover, there exists a bijection $\chi : I \to \mathcal{W}$ such that after defining $\mathcal{R}(\chi(i)) = \mathcal{R}_i$, the resultant net $\{\mathcal{R}(\chi(i))\}$ of wedge algebras on Minkowski space is covariant under the action of the representation $U(\mathcal{P}_+)$ and satisfies Haag duality.*

If, further, $\Delta_j^{it} \in \mathcal{J}$ for all $j \in I$, $t \in \mathbb{R}$, then modular covariance is satisfied and the state ω is a vacuum state on the net $\{\mathcal{R}(\chi(i))\}$.

Hence, a net of observable algebras $\mathcal{A}_i$ and a state ω determine a space–time, a strongly continuous representation of the isometry group of the space–time, and an identification of each i with a suitable region of the space–time, such that the original net is re-interpreted as a local Poincaré covariant quantum field theory on the space–time.

Similar (purely algebraic) conditions have been found which entail that the resultant space–time on which the re-interpreted observable algebras are then localized is four dimensional Minkowski space, resp. three dimensional de Sitter space. A given structure $(\mathcal{J}, \{J_i \mid i \in I\})$ can satisfy at most one of these sets of conditions.

Work is in progress on four dimensional de Sitter and anti–de Sitter spaces.

The modular involutions J_W (together with the modular unitaries Δ_W^{it}) for wedge algebras in the **vacuum only** encode the following information.

- the isometry group of the space–time
- a strongly continuous unitary representation of this isometry group (acting covariantly upon the net of observable algebras and leaving the vacuum invariant)
- the dynamics of the quantum systems
- the scattering theory of the quantum systems
- the locality, *i.e.* the Einstein causality, of the quantum systems
- the spin–statistics connection in the quantum systems (Kuckert; Guido & Longo)
- the stability of the quantum systems
- the thermodynamic behavior of the quantum systems

- characterization of the vacuum state
- the space–time itself

A Convenient Reference

- S.J. Summers, Yet More Ado About Nothing: The Remarkable Relativistic Vacuum State, arXiv:0802.1854