Gromov-Hausdorff limits in quantum field theory

Daniele Guido

Università di Roma Tor Vergata
-
work in progress with H. Bostelmann and L. Suriano

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Outline

1. Quantum metric spaces and qGH distance
2. Nuclearity and compactness in QFT
3. Lip-von Neumann Algebras
4. Ultraproducts and qGH limits
5. Scaling limits
   - Assumptions and Results
   - Relations with the Buchholz-Verch construction

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Some references

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Order unit spaces

Order unit space $X \cong$ Space of affine real-valued functions on a compact convex set [Kadison].

- Order structure: pointwise order of functions,
- Norm: sup-norm,
- Unit I: the constant function 1.

Lip-seminorm L on X: densely defined semi-norm such that

(*i*) $L(x) = 0$ *iff* $x = \lambda I$, $\lambda \in \mathbb{R}$,

(*ii*) $\{x \in X : L(x) \leq 1, \|x\| \leq 1\}$ is norm-compact.

Quantum Metric Space := Order-unit space with a Lip-norm.
L induces a distance d^L on the states $\mathcal{S}(X)$:
$d^L(\varphi, \psi) = \sup\{|\langle \varphi - \psi, x\rangle| : L(x) \leq 1\}$.
$(ii) \cong (ii')$ d^L induces the w^*-topology on $\mathcal{S}(X)$.

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Quantum Gromov-Hausdorff distance

Given two quantum metric spaces $(X, L_X), (Y, L_Y)$, let $\mathcal{L}(X, Y)$ be the family of Lip-norms on $X \oplus Y$ which induce L_X, resp. L_Y, on X, resp. Y, and set

$$d_{qGH}(X, Y) = \inf_{\tilde{L} \in \mathcal{L}(X,Y)} d_H^{\tilde{L}}(\mathcal{S}(X), \mathcal{S}(Y)),$$

where $d_H^{\tilde{L}}(\mathcal{S}(X), \mathcal{S}(Y))$ denotes the Hausdorff distance between the natural embeddings of $\mathcal{S}(X), \mathcal{S}(Y)$ in $(\mathcal{S}(X \oplus Y)), \tilde{L})$.

Theorem (Rieffel, 04)

d_{qGH} is a complete distance on the space of quantum metric spaces modulo isomorphisms.

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qGH distance for C^*-algebras

The self-adjoint part of a C^*-algebra is an order unit space, but isomorphism as order unit spaces does not imply isomorphism as C^*-algebras. Therefore let us define

$$d^2_{qGH}(\mathcal{A},\mathcal{B}) = \inf_{\tilde{L}\in\mathcal{L}(\mathcal{A},\mathcal{B})} d_H^{\tilde{L}}(\mathcal{S}_2(\mathcal{A}),\mathcal{S}_2(\mathcal{A})),$$

where $\mathcal{S}_2(\mathcal{A})$ denotes the u.c.p. maps from $\mathcal{A}$ to $\mathcal{M}_2(\mathbb{C})$ and $d^L(\varphi,\psi) := \sup_{L(a)\leq 1} \|\langle \varphi - \psi, a\rangle\|$, for $\varphi,\psi \in \mathcal{S}_2(\mathcal{A})$.

Theorem (Kerr, 03)

d^2_{qGH} is a distance on the space of Lip-C^*-algebras modulo isomorphisms.

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d^2_{qGH} is a distance on the space of Lip-C^*-algebras modulo isomorphisms.

d^2_{qGH} is not complete on C^*-algebras [Isola-DG, 06]: essentially because $L(x), L(y) < \infty$ does not imply $L(xy) < \infty$. However if L has the Leibniz property $L(ab) \leq \|a\|L(b) + L(a)\|b\|$ then a Cauchy sequence is converging to a C^*-algebra [Li, 03].

In general, if ones considers a complete distance, taking into account $n \times n$ matrices for all n, we get that the limit of a Cauchy sequence of C^*-algebras always exists as an operator system.

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Compactness for local algebras

The compactness property for local algebras takes the following general form: there is a map $T_{\mathcal{O}} : \mathcal{R}(\mathcal{O}) \to B$, for a suitable target Banach space B, which is compact, or nuclear.

In particular, for Buchholz-Pormann compactness, T has a predual map $T_* : \mathcal{B}(\mathcal{H})_* \to \mathcal{R}(\mathcal{O})_*$ with dense range, given by $\xi \in \mathcal{B}(\mathcal{H})_* \to e^{-H}\xi e^{-H}|_{\mathcal{R}(\mathcal{O})}$.

As a consequence, $\mathcal{R}(\mathcal{O})_*$ has a Lip-norm L on $Rg(T_*)$, $L(\omega) = \inf\{\|\xi\|, \xi \in \mathcal{B}(\mathcal{H})_*, T_*\xi = \omega\}$ such that $\{\omega \in \mathcal{R}(\mathcal{O})_* : L(\omega) \leq 1\}$ is norm-compact,

$\mathcal{R}(\mathcal{O})$ has a dual Lip-norm $L'(a) := \sup_\omega \dfrac{|\langle \omega, a \rangle|}{L(\omega)} = \|Ta\|$ such that $\mathcal{R}(\mathcal{O})_1$ is L'-compact.

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Lip-norms for local algebras

Therefore we get a framework which is opposite to that of Rieffel: the Lip-norm is on the normal state side, while the dual Lip-norm is on the algebraic side.
Hence we should develop a framework of (dual) Lip-vNA's and of a corresponding qGH distance.

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Definition

A Lip-von Neumann algebra (M, L') is a von Neumann algebra with a (dual-Lip) $*$-invariant norm L' which induces the w^*-topology on the bounded subsets of M. Equivalently, M_* has a densely defined norm L such that $\{\omega \in M_* : L(\omega) \leq 1\}$ is norm compact.

Extend L' to $\mathcal{M}_2(M)$ as $L'((a_{ij})) = \max_{ij} L(a_{ij})$. For a pair of Lip-vNA (M_1, L'_1), (M_2, L'_2), $\mathcal{L}(M_1, M_2)$ is the set of dual Lip-norms on $M_1 \oplus M_2$ which restrict to L'_1, resp. L'_2 on M_1, resp. M_2. Then $d_{qGH}(M_1, M_2) = \inf_{\tilde{L} \in \mathcal{L}(M_1, M_2)} d_H^{\tilde{L}}(\mathcal{M}_2(M_1)_{1+}, \mathcal{M}_2(\mathcal{M})_{1+})$.

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Theorem

d_{qGH} is a distance on the space of Lip-vNA modulo isomorphisms.

Remark

- *the non trivial property is $d_{qGH}(M_1, M_2) = 0 \Rightarrow M_1 \cong M_2$ as vNA.*
- *d_{qGH} appears not to be complete, essentially because, as above, we do not have an estimate for $L'(ab)$. Also in this case, it should be possible to develop a theory for dual operator systems, and show that a Cauchy sequence of Lip-vNA is always converging to a dual operator system.*

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Let (M_n, L'_n) be a sequence of Lip-vNA, with corresponding preduals (M_{n*}, L_n), and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. We consider
$\ell^\infty(M_{n*}) = \{\{\omega_n\} : \omega_n \in M_{n*}, \|\{\omega_n\}\| = \sup_n \|\omega_n\| < \infty\}$,
$\ell^\infty_R(M_{n*}) = \{\{\omega_n\} \in \ell^\infty(M_{n*}) : L(\{\omega_n\}) = \sup_n L_n(\omega_n) < \infty\}^{-\|\cdot\|}$,
the Banach subspace $K_\mathcal{U} = \{\{\omega_n\} \in \ell^\infty(M_{n*}) : \lim_\mathcal{U} \|\omega_n\| = 0\}$
and the quotient projection $p_\mathcal{U} : \ell^\infty(M_{n*}) \to \ell^\infty(M_{n*})/K_\mathcal{U}$.

Definition

We define $M_{\mathcal{U}*}$ as $p_\mathcal{U}(\ell^\infty_R(X_n))$ in $\ell^\infty(X_n)/K_\mathcal{U}$, with the quotient norms $\|\cdot\|_\mathcal{U}$, and $L_\mathcal{U}$ (restricted ultraproduct).

We then consider the space $\ell^\infty(M_n) = \{\{a_n\} : a_n \in M_n, \|\{a_n\}\| = \sup_n \|a_n\| < \infty\}$, with $L'(\{a_n\}) = \sup_n L'_n(a_n)$, the subspace $K'_{L,\mathcal{U}} = \{\{a_n\} \in \ell^\infty(M_n) : \lim_{\mathcal{U}} L'(a_n) = 0\}$ and the quotient projection $p'_{\mathcal{U}} : \ell^\infty(X'_n) \to \ell^\infty(X'_n)/K'_{L,\mathcal{U}}$.

Definition

We define $M_{\mathcal{U}}$ as $p'_{\mathcal{U}}(\ell^\infty(M_n))$, with the quotient norms $\|\cdot\|_{\mathcal{U}}$, and $L'_{\mathcal{U}}$ (dual restricted ultraproduct).

N.B.: $K'_{L,\mathcal{U}}$ is not an ideal!

Theorem

qGH-lim $M_n = M \Rightarrow M = M_{\mathcal{U}}$ *and* $M_* = M_{\mathcal{U}*}, \forall\mathcal{U}$ *ultrafilter.*

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A condition for completeness

Definition

A family (M_i, L'_i), $i \in I$, of Lip-vNA is uniform if

(a) Uniform compactness: The family $\{((M_i)_1, L'_i)\}$ is uniformly totally bounded.

(b) Uniform normalizer condition: $\forall \varepsilon > 0\ \exists K > 0 : \forall i \in I$, $a \in (M_i)_1$, $\exists b \in (M_i)_1 : N_i(b) \leq K$ and $L'_i(a - b) \leq \varepsilon$, where $N(a) = \sup\{\max(L'(ab), L'(ba)) : L'(b) \leq 1\}$.

Theorem

A uniform sequence (M_n, L'_n) of Lip-vNA is qGH-precompact. For any ultrafilter $\mathcal{U}$, $M_{\mathcal{U}} = \text{qGH-lim}_{\mathcal{U}} M_n$ is a Lip-vNA.

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A description of the ultraproduct vNA

- $N(a)$ may be infinite;
- $N(ab) \leq N(a)N(b)$ and $N(a) = N(a^*)$, i.e. $\{N(a) < \infty\}$ is a $*$ algebra.
- $L'(ab) \leq N(a)L(b)$, $L'(ab) \leq N(b)L(a)$.

Given a sequence (M_n, L'_n) of Lip-vNA, let us consider the space $\underline{\mathcal{A}} = \{\{a_n\} \in \ell^\infty(M_n) : \sup_n N_n(a_n) < \infty\}^{-\|\cdot\|}$. Then $\mathcal{A}_\mathcal{U} := p'_\mathcal{U}(\mathcal{A})$ is a C^*-algebra. If (M_n, L'_n) is uniform, we have

$$M_\mathcal{U} = \bigoplus_{\underline{\omega} \in \underline{\mathcal{S}}} \pi_{\underline{\omega}}(\underline{\mathcal{A}})'',$$

where $\underline{\mathcal{S}}$ is the set of states of $\underline{\mathcal{A}}$ represented by sequences $\{\omega_n\}$ of normal states of M_n for which $\sup_n L_n(\omega_n) < \infty$.

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Nuclearity conditions

Let $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ a local net of von Neumann algebras in the vacuum representation, and let H be the generator of the time translations.

Assumption (Uniform nuclearity)

$\forall r_0 > 0\ \exists d > 0 : \forall r \leq r_0, r/\beta \leq d$, the maps

$$\Xi_{\beta,r} : \mathcal{R}(\mathcal{O}_r) \to \mathcal{B}(\mathcal{H}), \quad a \mapsto e^{-\beta H} a e^{-\beta H}$$

are nuclear, uniformly in r/β.

Theorem

Uniform nuclearity holds for the free real scalar field of mass $m \geq 0$ in $s \geq 3$ spatial dimensions.

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$\forall r_0 > 0\ \exists d > 0 : \forall r \leq r_0, r/\beta \leq d$, the maps

$$\Xi_{\beta,r} : \mathcal{R}(\mathcal{O}_r) \to \mathcal{B}(\mathcal{H}), \quad a \mapsto e^{-\beta H} a e^{-\beta H}$$

are nuclear, uniformly in r/β.

Theorem

Uniform nuclearity holds for the free real scalar field of mass $m \geq 0$ in $s \geq 3$ spatial dimensions.

Compactness conditions

Assumption (Uniform compactness)

The maps $\Theta_\lambda : \mathcal{R}(\mathcal{O}_\lambda) \to \mathcal{B}(\mathcal{H})$, $a \mapsto (I + \lambda H)^{-1} a (I + \lambda H)^{-1}$ *are uniformly compact.*

Theorem

Uniform nuclearity implies uniform compactness.

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Assume the net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ *satisfies uniform compactness. Then, setting* $L'_\lambda(a) = \|(I + \lambda H)^{-1} a (I + \lambda H)^{-1}\|$, $(\mathcal{R}(\mathcal{O}_\lambda), L'_\lambda)$ *is a Lip-vNA, with* $N_\lambda(a) \le \|a\| + \lambda \|[H, a]\|$.

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qGH-scaling limit

Assumption (Uniform inner regularity)

$\forall \varepsilon > 0 \, \exists r < 1 : \forall \lambda > 0, a \in \mathcal{R}(\mathcal{O}_\lambda)_1 \, \exists a' \in \mathcal{R}(\mathcal{O}_{r\lambda})_1$ *such that* $L'_\lambda(a - a') = \|(I + \lambda H)^{-1}(a - a')(I + \lambda H)^{-1}\| < \varepsilon.$

Theorem

Let $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ *be a local net as above, satisfying uniform compactness and uniform inner regularity. Then, for any* $\mathcal{O}$, *the family* $\lambda\mathcal{O} \to \mathcal{R}(\lambda\mathcal{O})$ *is a uniform family of Lip-von Neumann algebras, therefore, for any ultrafilter* $\mathcal{U}$, *the Lip-vNA's* $\mathcal{R}(\mathcal{O})_\mathcal{U} = \text{qGH-lim}_\mathcal{U} \mathcal{R}(\lambda\mathcal{O})$ *form a local net, which we call the qGH-scaling limit net.*

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The scaling algebra

qGH-scaling algebra:

As mentioned above, $N_\lambda(a) \leq \|a\| + \lambda \|[H, a]\|$, therefore

$$\begin{aligned}\underline{\mathcal{A}}(\mathcal{O}) &= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda N_\lambda(a_\lambda) < \infty\}^{-\|\cdot\|} \\ &= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \lambda\|[H, a]\| < \infty\}^{-\|\cdot\|} \\ &= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \|\alpha_{\lambda t}(a_\lambda) - a_\lambda\| \to 0,\ t \to 0\}.\end{aligned}$$

Buchholz-Verch scaling algebra:

$\underline{\mathcal{A}}(\mathcal{O})_{BV} = \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \|\alpha_{\lambda x}(a_\lambda) - a_\lambda\| \to 0,$ $x \to 0\}$.

N.B.: the two C^*-algebras coincide.

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The representation

qGH-representation:

$$\mathcal{R}(\mathcal{O})_{\mathcal{U}} = \bigoplus_{\underline{\omega} \in \underline{\mathcal{S}}} \pi_{\underline{\omega}}(\underline{\mathcal{A}}(\mathcal{O}))'',$$

where $\underline{\mathcal{S}} \ni \underline{\omega}$ if $\langle \underline{\omega}, \underline{a} \rangle = \lim_{\mathcal{U}} \langle \omega_\lambda, a_\lambda \rangle$, with $\{\omega_\lambda\}$ in $\mathcal{B}(\mathcal{H})_*$ and $\sup_\lambda \|(I + \lambda H)\omega_\lambda(I + \lambda H)\| < \infty$.

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Corollary

With the assumptions above, any scaling limit net of the original theory embeds as a subrepresentation in the qGH-scaling limit net associated with some ultrafilter.

Outlook

- Clarify the relations between BV-representations and qGH representations of the scaling algebra.
- qGH-limits may help to prove some structural properties.
- Use qGH-limits of Lip-vNA in different contexts.