The Quantum Sine Gordon in pAQFT

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Mathematics of interacting QFT models
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Upshot

- Algebraic Quantum Field Theory (AQFT)
  and perturbative AQFT (pAQFT)

- The Sine Gordon model in pAQFT
  - A representation of the massless scalar field in 2 D
  - Convergence and unitarity of the S-matrix in this rep

- Construction of relative S-matrices and the Haag-Kaster net as operators in a suitable representation (not the vacuum).

Setting: Quantization in pAQFT

- Free (=linear) theory → unique Pauli-Jordan commutator "function"

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  Regular functionals $\mathcal{F}_{reg}$: n-th functional derivatives $F^{(n)}(\phi) \in C_c^\infty(M^n)$

  Ex.: $F(\phi) = \int f(x)\phi(x)dx$, $f \in C_c^\infty(M)$, $F^{(1)}(\phi) = f$.

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- Formal deformation quantization (star product, formal power series)

$$(F \star G)(\phi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \langle F^{(n)}(\phi), (\tfrac{i}{2}\Delta^{PJ})^{\otimes n} G^{(n)}(\phi)\rangle$$

Quantization II: Algebra of free fields and Normal ordering

- Let H be a bisolution (of the free theory), s.t. $W := \frac{i}{2}\Delta^{PJ} + H$ satisfies the Hadamard condition (condition on W's singular support). Existence H (even on glob. hyp. mfd): deformation argument.
- New star product on $\mathcal{F}_{\mu c}[[\hbar]]$,

$$(F \star_H G)(\phi) := \sum \frac{\hbar^n}{n!} \langle F^{(n)}(\phi), (\tfrac{i}{2}\Delta^{PJ} + H)^{\otimes n} G^{(n)}(\phi)\rangle$$

microcausal functionals $\mathcal{F}_{\mu c}$, i.e. F with $WF(F^{(n)}) \subset M^n \times \overline{V_+^n}$.
- On $\mathcal{F}_{reg} \subset \mathcal{F}_{\mu c}$, the products $\star_H$ and $\star$ are equivalent,

$$\alpha_H^{-1}(F) \star \alpha_H^{-1}(G) = \alpha_H^{-1}(F \star_H G) \qquad (1)$$

linear, invertible map α_H on $\mathcal{F}_{reg}$ explicitly known (as a formal p.s.). Different choices of $H \to$ equivalent star products on $\mathcal{F}_{\mu c}[[\hbar]]$.
- Certain completion/extension process of $\mathcal{F}_{reg}$ using (1) leads to algebra $(\mathfrak{A}[[\hbar]], \star)$.
- Ordinary setting: $\alpha_H^{-1} \leftrightarrow$ normal ordering. Write $:F:_H = \alpha_H^{-1}(F)$ for $F \in \mathcal{F}_{reg}$, and $:F:_H = \lim_n \alpha_H^{-1}(F_n) \in \mathfrak{A}[[\hbar]]$ for $F \in \mathcal{F}_{\mu c}$, $F_n \to F$.

More formal power series: Interaction

S-matrix: Dyson's series (formal p.s., Feynman graphs), formalized in pAQFT on the level of functionals. No need for a representation (yet). Building block are time ordered products.

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Def: Time-ordering operator $\mathcal{T}$

$$(\mathcal{T}F)(\phi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(2n)}(\phi), (iE_D)^{\otimes n} \right\rangle \ , \quad F \in \mathcal{F}_{reg}$$

with the Dirac propagator (mean) $E_D = \frac{1}{2}(E_R + E_A)$.

Def: Time-ordered product (inverse taken in the sense of f.p.s.)

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G) \ , \quad F, G \in \mathcal{F}_{reg}$$

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Def: formal S-matrix (for an interaction $F \in \mathcal{F}_{reg}$)

$$\mathcal{S}(\lambda F) \doteq \mathcal{T}\left(e^{i\mathcal{T}^{-1}(\lambda F)/\hbar}\right) = \sum_{n=0}^{\infty} \left(\frac{i\lambda}{\hbar}\right)^n \frac{1}{n!} \, F^{\cdot_{\mathcal{T}} n}$$

with a second formal parameter λ (coupling constant)

The Feynman propagator and normal ordering

Feynman propagator: Interaction needs normal ordering.

Time ordered products (Dirac propagator E_D)
combined with
normal ordering the interaction (choice of H)

leads to formal power series containing tensor powers of the Feynman propagator

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Renormalization: For nonregular functionals (even with normal ordering), the time ordered products are only defined on $M^n \setminus \Delta$. Extension of distributions (microlocal analysis). Not necessary here.

The Sine Gordon model

The model:

- Free massless scalar field on 2-D Minkowski space

$$\Box\phi = 0$$

- Interaction potential

$$\cos(a\phi)$$

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The trouble with ϕ...

Standard approach (annihilation/creation operators on Fock space) does not work. Wightman axioms cannot be satisfied (!) for the free field. Explanation in the algebraic picture: algebra of free fields does exist, but there is no vacuum state. There are other states (but we know only few...).

pAQFT Ingredients and 1st Example

In our setting (2D Minokwski massless field):

$$E_A, E_R, \Delta^{PJ} \quad \text{linear combination of step functions}$$

Choose

$$H_\mu(x,y) = \tfrac{-1}{4\pi} \ln(\mu^2 |(x-y)^2|) \qquad \text{with a scale } \mu > 0$$

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Example: Vertex operators: For $g \in C_c(M)$, $a \in \mathbb{R}$, let

$$v_a(g) \in \mathcal{F}_{\mu c}(M) \ , \qquad v_a(g)[\phi] = \int e^{ia\phi(x)} g(x) dx$$

The vertex operator $V_a(g)[\phi]$ is the normal ordered version of $v_a(g)[\phi]$ with star product

$$V_a(g)[\phi] \star V_b(h)[\phi] = v_a(g)[\phi] \star_{H_1} v_b(h)[\phi] \ .$$

It converges for $b = \pm a$, $\hbar |a|^2 < 4\pi$.

States

Def: Quasifree state on $\mathfrak{A}$ given by a bisolution H, s.t.

$$\langle f, Hf\rangle \geq 0 \quad \text{and} \quad \langle f, \Delta^{PJ} g\rangle^2 \leq 4\langle f, Hf\rangle\langle g, Hg\rangle$$

via (with normal ordering defined w.r.t. any bisolution $\widetilde{H}$)

$$\omega_{H,\phi}(:F:_{\widetilde{H}}) = \alpha_{H-\widetilde{H}}(F)[\phi]$$

Rem. Observe: $H - \widetilde{H}$ is smooth!

Massless field in 2D: $\langle f, H_\mu f\rangle \geq 0$ only for test functions with vanishing integral... Schubert's modification [Diplom thesis HH, 2013]:

$$H(x,y) = H_\mu(x,y) + \text{extra terms}$$

Extra terms depend on H_μ, Δ^{PJ}, the choice of a test function ψ with $\int \psi = 1$ and a parameter $r > 0$. Satisfies quasi free state condition!

States II

Schubert's state can be understood in terms of a representation of the free scalar massless field Φ in 2D Minkowski used in string theory and formalized by Derezinski and Meissner 06:

- Carrier space: $\mathcal{H} = \mathcal{H}_0 \otimes L^2(\widehat{\mathbb{R}})$ with $\mathcal{H}_0$ ordinary massless Fock space.
- Representation: Choose test function ψ with $\int \psi = 1$, and set

$$\pi_\psi \phi(g) = \Phi_c \left(g - \psi \int g \right) \otimes 1 + 1 \otimes \left(\int g \right) q - 1 \otimes \left(\int g \Delta \psi \right) p$$

  with Φ_c the free field on Fock space $\mathcal{H}_0$ (well defined because $g - \psi \int g$ has vanishing integral), p and q momentum and position operator (in momentum space).
- Fact: Schubert's H equals

$$H(x, y) = \langle \underline{\Omega}, \pi_\psi \phi(x) \pi_\psi \phi(y) \underline{\Omega} \rangle$$

  where $\underline{\Omega} = \Omega_0 \otimes \Omega_r$, with Ω_0 Fock vacuum, Ω_r harmonic oscillator ground state (freq dep on r).

Derezinski Meissner rep'n (ctd.)

Theorem [BFR19]

Considered as a rep'n of the exponentiated CCR (Weyl algebra) of time zero fields, the DM rep π_ψ (any ψ) is locally normal w.r.t. to the rep induced by the massive (any $m > 0$) scalar 2D vacuum state, i.e. these rep's are locally quasiequivalent.

- Ad Proof: Schubert state and massive vacuum are quasifree. Use Araki-Yamagami criterion (equivalence of the topology induced locally by the respective scalar products, and kernels of the resp. 2-point functions w.r.t. one of the scal. prod. differ by a Hilbert Schmidt operator).
- Observe: states for different non-zero masses are locally normal w.r.t. each other, but not to the massless vacuum [Eckman-Fröhlich 74]. Deeper reason for e.g. auxiliary mass, ... in older approaches.

S matrix

Theorem [B-Fredenhagen-Rejzner19]
Let $\hbar a^2 < 4\pi$ (model's ultraviolet finite regime). Then the S-matrix of the Sine Gordon model, calculated within the framework of pAQFT converges strongly in the DM representation on a dense domain $D \subset \mathcal{H}$. The resulting operator is in fact unitary.

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Idea of proof: For $g, f \in C_c^\infty(\mathbb{R}^2)$

$$\|\pi_\psi(S_n(\lambda V(g)))\,\pi_\psi(e^{i\Phi(f)})\,\Omega_0 \otimes \xi\| \leq C^n (n!)^{\frac{1-p}{p}}$$

$1 < p < \frac{4\pi}{ha^2}$. Estimate itself relies on [BR17] (finiteness of S-matrix in the vacuum rep), which in turn relies on older estimates in Euclidean approach [Fröhlich et al 70's]. Tied to 2 dimensions.

Improvement of our original estimate [BR17] for convergence in massless vacuum (where a condition on the test function g (IR behaviour) was needed): consistent with local quasiequivalence of DM rep to massive vacuum.

Haag Kastler net – general pAQFT setting

General setting within pAQFT:

- Test objects $\underline{\mathcal{D}}$ (compactly supported functions on M)
- $S(\underline{f})$, $\underline{f} \in \underline{\mathcal{D}}$, the timeordered exponential of normal ordered L, $S(0) = 1$, satisfies Bogolubov's factorization relation

$$S(\underline{f} + \underline{g} + \underline{h}) = S(\underline{f} + \underline{g})S(\underline{g})^{-1}S(\underline{g} + \underline{h})$$

if $J_{-}(\mathrm{supp}\underline{h}) \cap \mathrm{supp}\underline{f} = \emptyset$ (causality)

Def Given $\underline{g} \in \underline{\mathcal{D}}$, S as above. The relative S-matrix $S_{\underline{g}}$ is:

$$\underline{\mathcal{D}} \ni \underline{f} \mapsto S_{\underline{g}}(\underline{f}) = S(\underline{g})^{-1}S(\underline{g} + \underline{f})$$

Def Haag Kastler net (pAQFT) $\mathfrak{A}_{\underline{g}}$ is defined by the local algebras $\mathfrak{A}(\mathcal{O})$ (i.g. formal p.s.) generated by the relative S-matrices $S_{\underline{g}}(\underline{f})$, where $\mathrm{supp}\underline{f} \subset \mathcal{O}$.

Haag Kastler net – adiabatic limit (pAQFT setting)

How to remove the dependence on the cutoff function $\underline{g}$?

Let G be smooth (no support restriction) and set

$$[G]_{\mathcal{O}} := \{\underline{g} \in \underline{\mathcal{D}} \mid \underline{g} = G \text{ on nhb of } J_+(\mathcal{O}) \cap J_-(\mathcal{O})\}$$

The local algebra $\mathfrak{A}_G(\mathcal{O})$ is defined as the algebra generated by $S_{G,\mathcal{O}}(\underline{f})$, $\mathrm{supp}\underline{f} \subset \mathcal{O}$, where

$$S_{G,\mathcal{O}}(\underline{f}) : [G]_{\mathcal{O}} \ni \underline{g} \mapsto S_{\underline{g}}(\underline{f}) \in \mathfrak{A}$$

Thm [Fredenhagen-Rejzner 15]: The net $\mathcal{O} \mapsto \mathfrak{A}_G(\mathcal{O})$ with G =constant satisfies the Haag-Kastler axioms in the sense of formal power series.

Haag Kastler net – Sine Gordon

Observables: the field, the interaction Lagrangian (Cosine), the interaction in the field equation (Sine). Hence, the test objects are $\underline{\mathcal{D}} = C_c^\infty(M, \mathbb{C}) \oplus C_c^\infty(M, \mathbb{R})$, and the Lagrangian

$$L(g, h) = v_a(g) + v_{-a}(\bar{g}) + \Phi(h)$$

with $v_a(g) = \int e^{ia\Phi(x)} g(x)$. Observe: $: L(0, h) :$ is the field and $: L(g, 0) :$ is the Sine Gordon theory's interaction term for g real-valued.

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Thm [BFR17] The formal power series $S(g, h)$ defined as usual in pAQFT (time ordered exponential of normal ordering $L(g, h)$) in the DM representation converges strongly on the dense domain $D \subset \mathcal{H}$ and gives rise to unitary operators on $\mathcal{H}$. The Bogoliubov factorization property is satisfied.
Completing each $\mathfrak{A}_G(\mathcal{O})$ with G =constant w.r.t. the family of seminorms $\|A\|_{\Psi,\underline{g}} = |\langle \Psi, A(\underline{g})\Psi\rangle|$ (independent of $\underline{g} \in G_{\mathcal{O}}$), yields the net of local von Neumann algebras.

Conclusion and Outlook

- Original motivation [B-Rejzner17]: pAQFT put to the test.
- [BFR19]: Construction of the net of local von Neumann algebras in the DM representation. No Wick rotation needed, no auxiliary mass.
- Comparison with traditional constr QFT: needs vacuum state!
  - 1st step: thermal states (ongoing jt work w/ Rejzner+Pinamonti)
  - Understanding of the representation theory (DM rep'n vs. massless with auxiliary mass vs. solitonic quantization vs. "true vacuum"?) interesting in its own right.
  - Long term goal: better understanding of "equivalence" of Osterwalder-Schrader and Wightman axioms
- Understand how exactly our net fits into the universal algebra of Buchholz and Fredenhagen.
- Further work in progress: integrability and infinitely many conserved currents; quantum inequalities (we now have a model to play with) – dependence on representation?