Boundaries in relativistic quantum field theory

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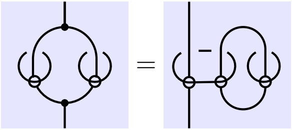


Literature

Based on joint work with M. Bischoff, Y. Kawahigashi, R. Longo:

arXiv:1405.7863

and

arXiv:1410.8848

Owing much to the work of

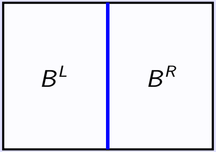
Fuchs-Fröhlich-Kong-Runkel-Schweigert etal (1998 – now)

HEURISTIC INTRODUCTION

Relativistic Quantum Physics is Algebra.

- Covariance = symmetry = commutation relations.
- Noether: generators are integrals over densities
- $\Rightarrow$ covariance = local commutation relations with densities.
- In particular:
- Dynamics = time evolution = local commutation relations.
- Einstein causality = vanishing of commutators at spacelike distance.

Imagine a **timelike boundary** in spacetime, with the physics on either side described by a different local relativistic QFT (or the "same" QFT in a different phase).



We shall assume that the boundary is "transparent" to energy and momentum (see below). In particular, energy and momentum are conserved at the boundary.

Moreover, we do not admit additional degrees of freedom "living at the boundary".

- This situation is to be described by algebras B^L and B^R of local quantum observables on a common Hilbert space (the state space of the combined system).
- The Hilbert space is a common representation of B^L and B^R.

Covariance, conservation laws, inner symmetries (if present), and causality constitute algebraic constraints.

What are the **possible algebraic relations** between B^L and B^R at the boundary?

- A decent QFT has a conserved **stress-energy tensor** $T_{\mu\nu}$ (SET). The QFT is an extension of its SET subtheory.
- We assume the boundary to be **"transparent" to energy and momentum**, in the sense that $T^L_{\mu\nu} = T^R_{\mu\nu}$.

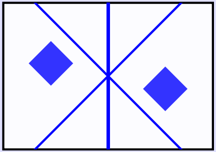
The latter property implies, and in 2D CFT is equivalent to the **conservation of energy and momentum** at the boundary:

$$T^L_{01}(t, 0_-) \stackrel{!}{=} T^R_{01}(t, 0_+) \quad \text{and} \quad T^L_{11}(t, 0_-) \stackrel{!}{=} T^R_{11}(t, 0_+)$$

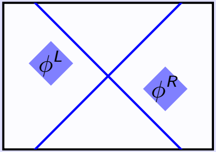
$\Leftrightarrow$ (using the chiral decomposition $T_{01} = T_+ - T_-$, $T_{11} = T_+ + T_-$)

$$T^L_+(t + x) = T^R_+(t + x) \quad \text{and} \quad T^L_-(t - x) = T^R_-(t - x).$$

- The boundary cannot “outwit” **Einstein Causality**.
- Hence local observables of B^L and of B^R must also commute with each other at spacelike separation:



- One can use the covariance under the common SET to (fictitiously) extend both B^L and B^R to all of Minkowski spacetime.
- Thus we have two local QFT on all spacetime, represented on the same Hilbert space, such that B^L and B^R commute with each other whenever



- We call this property **"left-locality"** of B^L w.r.t. B^R.

The boundary can be freely moved around – for which reason it is also called "topological".

TASK

- **Classify algebraic realizations** of this situation, whenever quantum field theories B^L and B^R are separately given, each on their own vacuum Hilbert space.

- **Understand the joint, left-local representations** on a common Hilbert space with a unique vacuum state.

By energy conservation (identification of the common subalgebra of the SET), such a representation cannot be a tensor product.

- Both QFT are extensions of a common SET subtheory A.
- We therefore need to construct (and classify)

$$A \quad \begin{matrix} \subset & B^L & \subset \\ \subset & B^R & \subset \end{matrix} \quad C$$

where by definition, C is generated by B^L and B^R.

C will not be local because B^L is only left-local w.r.t. B^R, but not also right-local. But C is relatively local w.r.t. A.

- We use a general framework ("Algebraic quantum field theory", **DHR theory** of positive-energy representations). [Doplicher-Haag-Roberts, 1969ff]

- A may be any common local subtheory, assumed to possess finitely many inequivalent positive-energy representations (sectors).

This applies in particular to **rational conformal QFT**, but the setup is more general.

TAKE-AWAY MESSAGES

- Boundary conditions cannot simply be "imposed" (as usual in classical field theory), but the possible boundary behaviour is **implicitly constrained by the apriori algebraic relations**, in particular covariance of B^L and B^R, locality of B^L and B^R, and left-locality of B^L w.r.t. B^R.

- DHR representation theory provides the tools for classification.

- DHR theory allows a **"universal construction"** C of all transparent boundary conditions (TBC), of which the individual TBC arise by **central decomposition**.
- Every TBC is characterized by a system of sesquilinear algebraic relations between the generating fields of B^L and of B^R.
- Only in distinguished cases, some of these become linear relations

$$\Phi^R = \alpha(\Phi^L),$$

where α is an automorphism.

- In some cases of interest, the TBC classification problem (= central decomposition of C) can be explicitly solved.
- TBC of **modular invariant 2D conformal QFT models** have the mathematically "most interesting" classification.

- Instead of transparent boundaries, one may study, e.g., **"hard boundaries"** (no physics on the other side, violation of momentum conservation).

- Hard boundaries in 2D CFT are "holographic".

- There is an (unexplored) range of intermediate cases.
  Example (in a 2D CFT with two chiral currents):

$$T^L_+ = j_1^2, \qquad T^R_+ = (\cos\alpha\, j_1 - \sin\alpha\, j_2)^2,$$
$$T^L_- = (\sin\alpha\, j_1 + \cos\alpha\, j_2)^2, \qquad T^R_- = j_2^2,$$

  which ensure energy conservation:

$$T^L_{01}(t, 0_-) \overset{!}{=} T^R_{01}(t, 0_+).$$

DHR THEORY

- Representations of a local QFT A
  $\cong$ localized endomorphisms of the quasilocal algebra.
- $=$ the objects of a **C* tensor category** $DHR(A)$.

- C*TC $=$ operators intertwining between repn's, equipped with two multiplications: operator product and "tensor product".
- Equivalence classes of repn's $=$ sectors $=$ general notion of "charge".
- "Tensor product" $=$ composition of DHR endomorphisms $=$ **fusion product** of charges.

- Locality equips the DHR category with a **braiding**
- = intrinsic description of **“statistics”**.

- In 4D (no invariant distinction between “left” and “right”): The braiding is a permutation symmetry (= **maximally degenerate**): ⇒ Bose-Fermi alternative, para-statistics, Spin-Statistics Theorem, duality with global gauge symmetry.
- In 2D: braid-group statistics (anyonic, plektonic).

- In many CFT models, the braiding is “modular” (= **maximally non-degenerate**).

Unlike braided tensor categories in general, the objects of the DHR category of QFT carry a geometric tag “localization” (that can be freely moved within their unitary equivalence class).

In particular, whenever σ is localized to the left of ρ, then the braiding operators $\varepsilon_{\rho,\sigma}$ trivialize:

$$\varepsilon_{\rho,\sigma} = \mathbf{1}.$$

This feature allows to turn many abstract braided-C*TC results [FFRS+] into geometric results about QFT [BKLR].

EXTENSIONS

Definition:

A QFT B is an **extension** of a local QFT A, iff A is covariantly ("same SET") contained in B, and B is relatively local w.r.t. A.

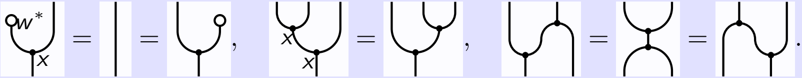
- B may or may not be local.
  (E.g., extensions by Fermi fields are only graded-local.)

The vacuum representation of B is a reducible positive-energy representation of A, hence a DHR endomorphism θ of A.

B is generated by A and finitely many "charged fields" $\Phi_\rho \in B$ for $\rho \prec \theta$, that intertwine to the charged representations of the "neutral" observables A:

$$\Phi_\rho a = \rho(a)\Phi_\rho \quad (a \in A), \qquad \Phi_\rho^* \Phi_\rho = \mathbf{1}.$$

The precise algebraic position of A within B specifies a pair of intertwiners $w \in Hom(id, \theta)$ and $x \in Hom(\theta, \theta^2)$ [= two vectors in finite-dimensional intertwiner spaces of $DHR(A)$], satisfying relations



Benefit: finitely many relations control the operator algebra of all observable fields.

Theorem [LR]:

- The data (θ, w, x) form a Q-system (= standard C* Frobenius algebra) in $DHR(A)$.
- Every Q-system in $DHR(A)$ allows to (re)construct an extension B in its vacuum representation.
- B is local iff the Q-system is commutative w.r.t. the DHR braiding.

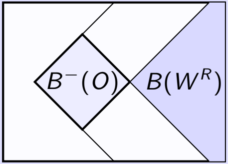
Proposition [FFRS+]:

Every Q-system in a braided tensor category contains two maximal commutative sub-Q-systems ("left centre, right centre").

Proposition [BKLR]:

In 2D QFT, the left/right centres correspond to maximal local intermediate extensions, defined by relative commutants of local algebras of left/right wedge regions:

$$B^{-}(O) = B(W^{R})' \cap B(W^{R} - a)$$



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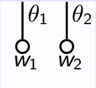
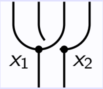
Proposition [FFRS+]:

- Q-systems in a braided tensor category admit two **“braided products”**, via

$w =$ 

$, \quad x^+ =$ 

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- In QFT, the braided products of Q-systems define braided products $B_1 \times^{\pm} B_2$ of extensions.

Proposition [BKLR]:

$B_1 \times^{\pm} B_2$ is an extension of A which contains both B_1 and B_2 as intermediate extensions, such that B_2 is left/right-local w.r.t. B_1.

Thus, the braided product of extensions $C := B^L \times^{-} B^R$ solves the task to construct a boundary representation of B^L and B^R on a common Hilbert space.

Theorem [BKLR]:

$C := B^L \times^- B^R$ is a "universal boundary representation". Its irreducible decomposition yields the individual boundary conditions.

Lemma:

$A' \cap C$ is spanned by products $\Phi_\rho^{L*}\Phi_\rho^R$, where ρ is a common subsector of θ^L and θ^R, and $\Phi_\rho \in B$ are the charged intertwiners (of either extension):

$$\Phi_\rho^{L*}\Phi_\rho^R a = \Phi_\rho^{L*}\rho(a)\Phi_\rho^R = a\Phi_\rho^{L*}\Phi_\rho^R \qquad (a \in A).$$

Proposition [BKLR]:

If both B^L and B^R are local, then the irreducible decomposition equals the central decomposition:

$$A' \cap C = C' \cap C.$$

- Thus, the boundary conditions correspond to minimal central projections in $A' \cap C = C' \cap C$.
- In the range of a minimal central projection e, each operator $\Phi^{L*}_\rho \Phi^R_\rho$ has a numerical value:

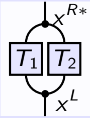
$$\pi_e\left(\Phi^{L*}_\rho \Phi^R_\rho\right) = S_{e;\rho} \cdot \mathbf{1}.$$

To compute the minimal central projections of C ($=$ boundary conditions), one has to diagonalize the finite-dimensional algebra spanned by $\Phi_\rho^{L*}\Phi_\rho^R$.

Lemma:

This amounts to diagonalize the commutative "convolution product" in $Hom(\theta^L, \theta^R)$

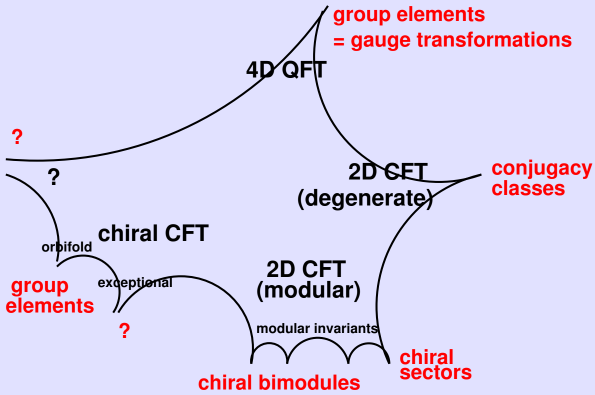
$$T_1 * T_2 :=$$



within the C* tensor category $DHR(A)$.

CLASSIFICATIONS

Classification of boundary conditions by . . .



Depending on the "amount of modularity", the fusion of transparent boundary conditions (upon juxtaposition of boundaries) may be again a tensor category, or a fusion ring, or just a ring.

Proposition:

In 4D QFT, let B be the canonical field algebra such that A are the fixed points under a global gauge group G acting on B.

The B-B boundary conditions are **classified by the global gauge transformations** $g \in G$:

$$\Phi^R = \alpha_g(\Phi^L).$$

Speculation: local gauge transformations by juxtaposition of many boundaries.

Proposition:

In 2D conformal QFT with underlying chiral algebra $A \otimes A$ (A rational and $DHR(A)$ modular), let B be the canonical diagonal extension (with modular invariant coupling matrix $Z = \mathbf{1}$).

The B-B boundary conditions are **classified by the chiral sectors** of A. The values of $\Phi_\rho^{L*}\Phi_\rho^R$ are given by the Verlinde S-matrix:

$$\pi_\sigma\left(\Phi_\rho^{L*}\Phi_\rho^R\right) = S_{\sigma;\rho}.$$

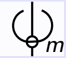
This includes the well-known three boundary conditions of the Ising model: trivial, fermionic and "dual" (in which the order and the disorder parameter coexist as local, but not mutually local fields; instead, one is left-local w.r.t. the other.

Proposition:

- In 2D conformal QFT with underlying chiral algebra $A \otimes A$ as before, all modular-invariant extensions B arise by the α-induction construction $=$ full centre of a chiral Q-system.
- The B^L-B^R boundary conditions between any two modular-invariant extensions are **classified by the irreducible bimodules** m between the corresponding chiral Q-systems.

The values $\pi_m\left(\Phi_\rho^{L*}\Phi_\rho^R\right)$ are given by generalized Verlinde S-matrices, which can be most efficiently computed by their property of diagonalizing the bimodule fusion rules.

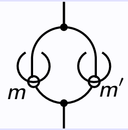
"Proof":

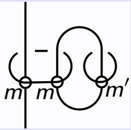
Claim: The intertwiners $E_m :=$ 

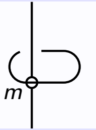
$\in Hom(\theta^L, \theta^R)$ diagonalize the convolution product:

Bimodule property (simple) $\Downarrow$

Nondegeneracy of braiding (tricky) $\Downarrow$



$=$ 

$= \delta_{mm'} \cdot$ 

q.e.d.

- As an "intermediate" instance between 2D and 4D, let a chiral CFT A be the fixed points of a local chiral CFT $\widetilde{A}$ under a finite group G ("orbifold"). The corresponding untwisted sectors of A form a subcategory of $DHR(A)$ isomorphic to the dual $\widehat{G}$ (with completely degenerate braiding as in 4D).
- Let B be the diagonal extension of $A \otimes A$ using only the untwisted sectors.

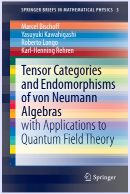
Proposition:

Then, the B-B boundary conditions are **classified by the conjugacy classes** of G. The values $\pi_c\left(\Phi_\rho^{L*}\Phi_\rho^R\right)$ are given by the character table $\chi_\rho(c)$ of G.

Further reading:

M. Bischoff, Y. Kawahigashi, R. Longo, KHR:

SpringerBriefs in Mathematical Physics 3, 2015:



(on display at the book exhibition)

. . . developping the relevant calculus with Q-systems = Frobenius algebras in braided C* tensor categories (here: the DHR category of positive-energy representations of a given QFT), and its interpretation in terms of QFT extensions.