# The impact of the algebraic approach on perturbative quantum field theory

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# Introduction

Quantum field theory is quantum theory together with the locality principle (Haag).

Incorporation of the locality principle by association of $C^*$-algebras to regions of spacetime.

Close relation between algebraic and geometric structures

Algebraic Quantum Field Theory = Local Quantum Physics

Axiomatic approach: Clear principles, rigorous mathematics, qualitative agreement with particle physics

Successes:

- Explanation of the multiparticle structure of quantum field theory (Haag-Ruelle) as a consequence of
  - Existence of single particle states (eigen states of the mass operator)
  - Local commutativity of observables
  - Translation invariance
- Structure of superselection sectors as the dual object to inner symmetries (Doplicher-Haag-Roberts)
- Origin of particle statistics
- Thermal equilibrium states and modular structure

Construction of models:

- Free fields
- Superrenormalizable models (Glimm-Jaffe)
- Conformal nets in 2 dimensions (Kawahigashi-Longo-Rehren)
- Integrable models in 2 dimensions (Lechner)

But:

- No interacting model in 4 dimensions
- No realistic model of elementary particles

Developments of quantum field theory separate from AQFT:

- Perturbative renormalization
- Renormalization group
- Relation to statistical mechanics by Wick rotation
- Relation to classical field theory
- Anomalies, asymptotic freedom
- Path integral

Questions:

- How is AQFT related to other formulations of QFT?
- Can algebraic field theory be reformulated such that perturbative and semiclassical techniques can be applied?
- Will this contribute to field theory in general?

Early work:

- Epstein-Glaser renormalization: Inductive construction of time ordered products as operator valued distributions on Fock space.
    - Rigorous and conceptually clear
    - Interacting fields are constructed as formal power series of free fields
    - Problems with gauge theories
    - Removal of spacetime cutoff (adiabatic limit) complicated
    - Role of the renormalization group unclear (no divergences)
- Steinmann renormalization: Inductive construction of retarded products and of Wightman functions
    - Essentially equivalent to Epstein-Glaser
    - No spacetime cutoff

New challenge: Quantum field theory on Lorentzian spacetime

Needed: Improved incorporation of the principle of locality

Conventional approach to perturbation theory: Not applicable

- No vacuum
- No particles
- No S matrix
- No distinguished Feynman propagator
- No generally covariant path integral
- No associated euclidean version

Program: Elimination of nonlocal features in the foundations of the theory

1st step: Microlocal spectrum condition (Radzikowski 1995)

Immediate consequences:

- Construction of the algebra of composite fields associated to free fields (Brunetti, F, Köhler 1996)
- Epstein-Glaser renormalization on a fixed globally hyperbolic spacetime (Brunetti, F 2000)

Unsolved: Generally covariant renormalization conditions

2nd step: Principle of local covariance (Brunetti, F, Hollands, Verch, Wald 2001-3)

Haag-Kastler net is generalized to a functor from the category of globally hyperbolic spacetimes with isometric embeddings as morphisms to the category of C*- algebras.

Conditions on locality and covariance are subsumed in the concept of natural transformations between functors.

Example: Locally covariant fields

$$\varphi = (\varphi_M)_M \ , \chi : M \to N \ , \alpha_\chi : \mathfrak{A}(M) \to \mathfrak{A}(N)$$

Condition on local covariance

$$\varphi_N(\chi(x)) = \alpha_\chi(\varphi_M(x))$$

Next steps:

- Generally covariant renormalization (Hollands-Wald 2001,2)
- Renormalization group (Hollands-Wald 2003, Dütsch-F 2004, Brunetti-Dütsch-F 2008)
- Gauge theories (Dütsch-F 1999, Dütsch et al 1990-2008, Hollands 2008)
- Operator product expansion
  - Axiomatic (Bostelmann 2005)
  - Perturbative (Hollands 2007)

# Algebraic structure of perturbative renormalization

Restriction of the locally covariant formalism to a scalar field on Minkowski space

$\mathfrak{E}$ space of smooth field configurations

$\mathfrak{F}$ space of observables $F : \mathfrak{E} \to \mathbb{C}$

Operator product: (off shell Wick theorem)

$$F \star G(\varphi) = e^{\langle \Delta_+, \frac{\delta^2}{\delta\varphi\delta\varphi'} \rangle}(F(\varphi)G(\varphi'))|_{\varphi'=\varphi}$$

Vacuum:

$$\omega_0(F) = F(0)$$

## Time ordering operator

$$(TF)(\varphi) = e^{\langle \Delta_F, \frac{\delta^2}{\delta\varphi^2} \rangle} F(\varphi) \equiv \int d\mu_{\Delta_F}(\Phi) F(\varphi - \Phi)$$

($\mu_{\Delta_F}$ Gaussian "measure" with covariance $\Delta_F$)

## Time ordered product:

$$F \cdot_T G = T(T^{-1}F \cdot T^{-1}G)$$

(equivalent to pointwise product, but not everwhere well defined)

$$F \cdot_T G(\varphi) = e^{\langle \Delta_F, \frac{\delta^2}{\delta\varphi\delta\varphi'} \rangle} (F(\varphi)G(\varphi'))|_{\varphi'=\varphi}$$

$$F \star G = F \cdot_T G$$

if supp $F$ later than supp $G$.

Formal S-matrix:

$$S = T \circ \exp \circ T^{-1}$$

Connection with path integral:

$$\omega_0(S(V)) = S(V)(\varphi = 0) = \int d\mu_{\Delta_F} e^{:V:}$$

Retarded interacting fields: Møller operators $R_V : \mathfrak{F} \to \mathfrak{F}$

$$S(V) \star R_V(F) = S(V) \cdot_T F$$

If

$$\omega_0(S(V) \star R_V(F)) \to \omega_0(S(V))\omega_0(R_V(F))$$

(adiabatic limit, unique vacuum) $\Longrightarrow$ (Gell-Mann Low)

$$\omega_0(R_V(F)) = \frac{\int d\mu_{\Delta_F} e^{:V:} : F :}{\int d\mu_{\Delta_F} e^{:V:}}$$

Up to now: Functionals $F$ on $\mathfrak{E}$ must have smooth functional derivatives.

Excludes nonlinear local functionals

Local functionals can be characterized by the additivity condition

$$F(\varphi + \psi + \omega) = F(\varphi + \psi) - F(\psi) + F(\psi + \omega)$$

if $\operatorname{supp}\varphi \cap \operatorname{supp}\omega = \emptyset$, $\varphi, \psi, \omega \in \mathfrak{E}$.

Consequence: Derivatives of local functionals have support on the thin diagonal $D$.

Smoothness condition: A local functional is smooth if its derivatives exist as distributions whose wavefront set is orthogonal to the tangent bundle of $D$.

Example: $F(\varphi) = \int dx\, f(x)\varphi(x)^2$

$$\frac{\delta^2 F}{\delta\varphi(x)\delta\varphi(y)} = 2f(x)\delta(x-y)$$

where $f$ has to be smooth.

Theorem 0 of Epstein-Glaser, slightly extended:

- $\star$-products of smooth local functionals exist and generate an associative *-algebra.
- Time ordered products of smooth local functionals exist under conditions on the support and generate a partial algebra (allows Euclidean Epstein Glaser renormalization (Keller 2009)).

Renormalization=Extension of $S$ to smooth local functionals

$n$-fold time ordered product: formal series of $n$-linear differential operators:

$$S^{(n)} = \sum_{\alpha} \langle S_\alpha, \delta^\alpha \rangle$$

$$\delta^\alpha (F_1 \otimes \cdots \otimes F_n)(\varphi) = \frac{\delta^{\sum \alpha_i}}{\prod_i \delta \varphi_i^{\alpha_i}} F(\varphi_1) \cdots F(\varphi_n)|_{\varphi_1 = \ldots = \varphi_n = \varphi}$$

$S_\alpha$ extension of

$$\sum_{G \in \mathcal{G}_\alpha} c_G \bigotimes_{l \in E(G)} \Delta_F \ ,$$

$\mathcal{G}_\alpha$ set of graphs with vertices $\{1, \ldots, n\}$ and $\alpha_i$ lines at $i$, pairing $\langle , \rangle$ determined by $G$, $c_G$ combinatorial factor

$F_1, \ldots, F_n$ smooth and local $\Longrightarrow$

$$\delta^\alpha(F_1 \otimes \cdots \otimes F_n)(\varphi) \in \mathfrak{D}(M^n) \otimes \mathcal{V}$$

where

$$\mathcal{V} = \bigotimes_{i=1}^{n} \mathfrak{D}'_0(M^{n_i-1})$$

$\mathfrak{D}'_0$ space of distributions with support $\{0\}$
(separation in center of mass and relative coordinates at each vertex)

Grading of $\mathcal{V}$ by number of derivatives in front of the $\delta$-function:

$$\mathcal{V} = \bigoplus_{k=0}^{\infty} \mathcal{V}_k$$

Causal factorization: $S^{(n)}$ is uniquely determined on $\mathfrak{D}(M^n \setminus D) \otimes \mathcal{V}$ by $S^{(k)}$, $k < n$.

Power counting ($\omega = |\alpha|(d-2)/2 - (n-1)d$):

$S_\alpha$ can be uniquely extended to

$$\mathfrak{D}_\omega = \bigoplus_k \mathfrak{D}_{\omega+k}(M^n) \otimes \mathcal{V}_k \ ,$$

$\mathfrak{D}_\omega$ set of test functions which vanish at order $\omega$ at the thin diagonal $D$.

Renormalization: Choice of projection

$$(1-W) : \mathfrak{D} \to \mathfrak{D}_\omega$$

$$S_\alpha^{\text{ren}} := S_\alpha \circ (1-W)$$

Main Theorem of Renormalization:

$S, \hat{S}$ renormalized S-matrices $\Longrightarrow$ There exists a unique map $Z$ which maps the set of smooth local functionals into itself, such that

$$\hat{S} = S \circ Z$$

Structure of $Z$:

$$Z(V) = V + \sum_{n=2}^{\infty} Z^{(n)}(V^{\otimes n})$$

$Z^{(n)}$ $n$-linear differential operator,

$$Z^{(n)} = \sum_{\alpha} \langle Z_\alpha, \delta^\alpha \rangle$$

$\operatorname{supp} Z_\alpha \subset D$ (produces finite local counterterms)

The set of maps $Z$ forms the Renormalization Group.

# Adiabatic limit

Generalized Lagrangian: Map $\mathcal{L}$ from test functions to smooth local functionals such that

$$\mathcal{L}(f+g+h) = \mathcal{L}(f+g) - \mathcal{L}(g) + \mathcal{L}(g+h)$$

if $\mathrm{supp} f \cap \mathrm{supp} h = \emptyset$.

Proposition: The space of generalized Lagrangians is invariant under the renormalization group.

Definition: $\mathcal{L} \sim \mathcal{L}'$ if $\mathrm{supp}\,(\mathcal{L}(f) - \mathcal{L}'(f)) \subset \mathrm{supp}\, df$.

Proposition: $\mathcal{L} \sim \mathcal{L}' \implies Z \circ \mathcal{L} \sim Z \circ \mathcal{L}'$

Observables of the interacting theory:

$$S_V(F) := S(V)^{-1} \star S(V+F)$$

is the generating functional for time ordered products of the interacting (retarded) observable $R_V(F)$ (Bogoliubov).

Causal factorization:

supp $V_1 - V_2 \cap \overline{\mathcal{O}} = \emptyset \Longrightarrow \exists\, U$ invertible with

$$R_{V_2}(F) = U\, R_{V_1}(F)\, U^{-1}$$

for all $F$ with support $F \subset \mathcal{O}$.

Algebraic adiabatic limit: Let

$$\mathcal{V}_{\mathcal{L}}(\mathcal{O}) = \{V, \mathrm{supp}\,(V - \mathcal{L}(f)) \cap \overline{\mathcal{O}} = \emptyset, f \equiv 1 \text{ on } \mathcal{O}\}$$

Algebraic structure within $\mathcal{O}$ is independent of the choice of $V \in \mathcal{V}_{\mathcal{L}}(\mathcal{O})$.

$\Longrightarrow$ Algebra of local observables $\mathfrak{A}_{\mathcal{L}}(\mathcal{O})$ can be generated by the families

$$S_{\mathcal{L}}^{\mathcal{O}}(F) = (S_V(F))_{V \in \mathcal{V}_{\mathcal{L}}(\mathcal{O})}$$

with $\mathrm{supp}\, F \subset \mathcal{O}$.

Net structure:

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{V}_{\mathcal{L}}(\mathcal{O}_1) \supset \mathcal{V}_{\mathcal{L}}(\mathcal{O}_2),$$

hence embedding

$$\mathfrak{A}_{\mathcal{L}}(\mathcal{O}_1) \to \mathfrak{A}_{\mathcal{L}}(\mathcal{O}_2)$$

induced by restriction of the family $S_{\mathcal{L}}^{\mathcal{O}_1}(F)$ to $\mathcal{V}(\mathcal{O}_2)$.

$$\mathfrak{A}_{\mathcal{L}}(M)$$

is defined as the inductive limit of the net of algebras

$$\mathfrak{A}_{\mathcal{L}}(\mathcal{O})$$

for relatively compact $\mathcal{O} \subset M$.

Action of the renormalization group on observables: $\hat{S} = S \circ Z$

$$\hat{S}_V(F) := \hat{S}(V)^{-1} \star \hat{S}(V + F)$$

$\Longrightarrow$

$$\hat{S}_V(F) = S_{Z(V)}(Z_V(F))$$

with

$$Z_V(F) = Z(V + F) - Z(V)$$

("field strength renormalization").

Observation: $Z_V(F) = Z_{V'}(F)$ if $\mathrm{supp}\,(V - V') \cap \mathrm{supp}\, F = \emptyset$.

Definition: $Z_{\mathcal{L}}(F) = Z_{\mathcal{L}(f)}(F)$, $f \equiv 1$ on $\mathrm{supp}\, F$.

Action of the renormalization group in the adiabatic limit

$$\hat{S}^{\mathcal{O}}_{\mathcal{L}}(F) = \left(\hat{S}_V(F)\right)_{V \in \mathcal{V}_{\mathcal{L}}(\mathcal{O})} = \left(S_{Z(V)}(Z_V(F))\right)_{V \in \mathcal{V}_{\mathcal{L}}(\mathcal{O})}$$

Let $\hat{\mathfrak{A}}$ denote the net of algebras of observables obtained by using $\hat{S}$ instead of $S$.
Then the renormalization group element $Z$ induces an isomorphism $\alpha_Z$ between the nets $\hat{\mathfrak{A}}_{\mathcal{L}}$ and $\mathfrak{A}_{Z\circ\mathcal{L}}$ by

$$\alpha_Z : \hat{S}^{\mathcal{O}}_{\mathcal{L}}(F) \mapsto S^{\mathcal{O}}_{Z\circ\mathcal{L}}(Z_{\mathcal{L}}(F))$$

(Algebraic Renormalization Group Equation)

Application to scaling on Minkowski space:

Scaled net:

$$\mathfrak{A}^{\rho}_{\mathcal{L}}(\mathcal{O}) = \mathfrak{A}_{\mathcal{L}}(\rho^{-1}\mathcal{O})$$

Scaled Lagrangian:

$$\mathcal{L}^{\rho} = \sigma_{\rho} \circ \mathcal{L} \circ \sigma_{\rho}^{-1}$$

Scaled S-matrix:

$$S^{\rho} = \sigma_{\rho} \circ S \circ \sigma_{\rho}^{-1} = S \circ Z(\rho)$$

Theorem: The scaled net $\mathfrak{A}^{\rho}_{\mathcal{L}}$ is isomorphic to the net $\mathfrak{A}_{Z(\rho)\circ\mathcal{L}^{\rho}}$ .

(Algebraic Callan Symanzik Equation).

# Conclusions and Outlook

- Nets of algebras of observables can be constructed for generic interactions in the sense of formal power series.
- The renormalization group is realized as a group of isomorphisms of algebraic quantum field theories.
- The algebraic construction is purely local and does not suffer from any infrared problem.
- The construction does not depend on the choice of distinguished states.
- The construction is explicit and can be performed by exploiting standard techniques, thereby providing proofs for the validity of popular recipes, e.g. dimensional regularization.
- All structures are generally covariant and remain meaningful on globally hyperbolic spacetimes.

Open questions:

- What is the structure of the state space on generic spacetimes? Holographic ideas might turn out to be fruitful (see, e.g. Dappiaggi-Moretti-Pinamonti).
- Can the structural analysis of the axiomatic theory be applied to the perturbative setting? E.g., one may try to use the Buchholz-Verch concept of an intrinsic renormalization group and compare it with the perturbative renormalization group.
- Can the construction be extended beyond formal series? (Answer is yes for the classical theory (see the talk of Romeo Brunetti); probably also for finite loop order.)
- The described formalism for perturbation theory is very general and might be applied also to 2d conformal theories, to integrable models and even to quantum gravity. Will this improve our understanding of these models?