The scaling and mass expansion

Michael Dütsch

Abstract. The scaling and mass expansion (shortly 'sm-expansion') is a new axiom for causal perturbation theory, which is a stronger version of a frequently used renormalization condition in terms of Steinmann's scaling degree [8, 1].

If one quantizes the underlying free theory by using a Hadamard function (which is smooth in $m \geq 0$), one can reduce renormalization of a massive model to the extension of a minimal set of mass-independent, almost homogeneously scaling distributions by a Taylor expansion in the mass m. The sm-expansion is a generalization of this Taylor expansion, which yields this crucial simplification of the renormalization of massive models also for the case that one quantizes with the Wightman two-point function, which contains a $\log(-(m^2(x^2 - ix^0 0))$-term.

We construct the general solution of the new system of axioms (i.e. the usual axioms of causal perturbation theory completed by the sm-expansion), and illustrate the method for a divergent diagram which contains a divergent subdiagram.

Keywords. Perturbative quantum field theory, causal perturbation theory.

1. Introduction

In the inductive Epstein-Glaser construction of time-ordered products [8, 16, 1, 2] renormalization amounts to the extension of numerical distributions $t^{(m)\,0} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ to $t^{(m)} \in \mathcal{D}'(\mathbb{R}^k)$, where we assume translation invariance. By the upper index m we denote the mass of the underlying free theory. In the extension $t^{(m)\,0} \to t^{(m)}$ one wants to maintain the property that $t^{(m)\,0}$ scales almost homogeneously under $(x, m) \to (\rho x, m/\rho)$ with a degree $D \in \mathbb{N}$, i.e.

$$\Big(\sum_r x_r \partial_r - m\partial_m + D\Big)^N t^{(m)\,0}(x) = 0 \tag{1.1}$$

for a sufficiently large $N \in \mathbb{N}$. For an m-independent distribution $u^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$, which scales almost homogeneously under $x \to \rho x$ (i.e. u^0 fulfils (1.1) without the $m\partial_m$-term), quite a lot is known about the extension to an $u \in \mathcal{D}'(\mathbb{R}^k)$ such that the almost homogeneous scaling is preserved (see e.g. proposition A.1 and [12, 10, 4, 11]). To profit from these knowledges, one wants to expand $t^{(m)\,0}(x)$ in terms of such distributions $u^0(x)$ (as done in [10, 4]). If $t^{(m)\,0}$ is smooth in $m \geq 0$, this expansion is simply the Taylor expansion in m [4]:

$$t^{(m)\,0}(x) = \sum_{l=0}^{L} m^l \, u_l^0(x) + \mathfrak{r}_{L+1}^{(m)\,0}(x) \ . \tag{1.2}$$

Choosing L sufficiently large, the remainder $\mathfrak{r}_{L+1}^{(m)\,0} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ can easily be extended and one is left with the almost homogeneous extension of the u_l^0-distributions. This procedure maintains the scaling property (1.1) and it also fulfils the renormalization condition $\mathrm{sd}(t^{(m)}) = \mathrm{sd}(t^{(m)\,0})$, which is frequently used in causal perturbation theory. ('sd' means Steinmann's scaling degree (A.1), which is a measure for the UV-behavior of the distribution.)

If one quantizes the underlying free theory by using a *Hadamard function* (which is smooth in $m \geq 0$), one can require smoothness in $m \geq 0$ as a renormalization condition for the time-ordered products and with that one can proceed as just described, see [4].

However, mostly the *Wightman two-point function* Δ_m^+ is used for the quantization. In even dimensions d, Δ_m^+ is not smooth in m at $m = 0$; for $d = 4$ it is of the form

$$\Delta_m^+(x) = \frac{-1}{4\pi^2(x^2 - ix^0 0)} + m^2 \, f(m^2x^2) \log(-m^2(x^2 - ix^0 0)) + m^2 \, F(m^2x^2), \tag{1.3}$$

with f and F being certain analytic functions. To reduce renormalization to the extension of a minimal set of m-independent, almost homogeneously scaling distributions also for time-ordered products based on quantization with Δ_m^+, we generalize (1.2) to

$$t^{(m)\,0}(x) = \sum_{l=0}^{L} m^l \sum_{p=0}^{P_l} \big(\log \tfrac{m}{M}\big)^p \, u_{l,p}^0(x) + \mathfrak{r}_{L+1}^{(m)\,0}(x) \ , \quad L, P_l \in \mathbb{N}_0, \tag{1.4}$$

where $M > 0$ is a fixed mass scale and $u_{l,p}^0, \mathfrak{r}_{L+1}^{(m)\,0} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$. We call (1.4) the 'scaling and mass expansion'. This name refers to the following two possibilities to interpret (1.4): on the one hand it is an expansion in terms of m-independent, almost homogeneously scaling distributions $u_{l,p}(x)$ and on the other hand it is a "Taylor expansion in the mass m modulo $\log m$".

We require the sm-expansion for the $t^{(m)}$-distributions as a new axiom for causal perturbation theory [sect. 3]. We will construct the general solution of the so modified system of axioms [sect. 4].

The sm-expansion (1.4) is strongly related to the 'scaling expansion' of Hollands and Wald for time-ordered products on curved space-times [10]. A main conceptual difference is that we require the structure (1.4) directly as an axiom, whereas the 'scaling expansion' in [10] is a non-trivial *consequence* of the system of axioms used there.

Working with a dimensionally regularized Feynman propagator as introduced in [5], the sm-expansion (1.4) is of a different form: $t^{(m)\,0}(x) = \sum_i (\frac{m}{M})^{z_i}\, u_i^0(x)$ plus a remainder, where $z_i \in \mathbb{C}$ and the m-independent distributions $u_i^0 \in \mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ scale even *homogeneously* with a degree $\kappa_i \in \mathbb{C}\setminus\mathbb{Z}$ [sect. 5].

We assume that the reader is familiar with the formalism for causal perturbation theory introduced in [4].

2. Axioms for causal perturbation theory

2.1. General axioms

For simplicity we study a real scalar field φ on d-dimensional Minkowski space $\mathbb{M}$, $d > 2$. On the space $\mathcal{F}$ of observables (defined in [4, formulas (2.1-2)][1]) we introduce an m-dependent star product $\star_m : \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ [3] by

$$F\star_m G := \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\int dx_1\cdots dx_n\, dy_1\cdots dy_n \cdot \frac{\delta^n F}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\prod_{l=1}^{n} H_m(x_l-y_l)\,\frac{\delta^n G}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\,, \tag{2.1}$$

where

- either $H_m = \Delta_m^+$ is the Wightman two-point function (1.3),
- or $H_m = H_m^\mu$ is a *Hadamard function*, which depends on an additional mass parameter $\mu > 0$. In even dimensions d, H_m^μ is related to Δ_m^+ by
  $$H_m^{\mu\,(d)}(x) := \Delta_m^{+\,(d)}(x) - m^{d-2}\, f^{(d)}(m^2x^2)\log(m^2/\mu^2)\ , \tag{2.2}$$
  where $f^{(d)}$ is an analytic function which agrees for $d = 4$ with the function f in (1.3) (see [4, Appendix A]). Thus, the $\log(-m^2(x^2 - ix^00))$ factor in (1.3) is replaced by $\log(-\mu^2(x^2 - ix^00))$, due to that H_m^μ is smooth in $m \geq 0$.

In both cases H_m is a Lorentz invariant solution of the Klein-Gordon equation; the antisymmetric part of H_m is fixed by $H_m(x) - H_m(-x) = i\,\Delta_m(x)$ (where Δ_m is the commutator function).

Let $\mathcal{P}$ be the space of polynomials in $\partial^\beta\varphi$, for $\beta \in \mathbb{N}_0^d$. Following [4], a **time-ordered product** $T^{(m)} \equiv T = (T_n)_{n\in\mathbb{N}}$ (m denotes the mass of the underlying star product) is a sequence of maps $T_n\colon \mathcal{P}^{\otimes n} \to \mathcal{D}'(\mathbb{M}^n,\mathcal{F})$,[2] which are **linear**; and satisfy

(a) **Initial value:** $T_1(A(x)) = A(x)$ for any $A \in \mathcal{P}$;
(b) **Permutation symmetry:** $T_n(A_{\pi(1)}(x_{\pi(1)}),\dots,A_{\pi(n)}(x_{\pi(n)}))$
$$= T(A_1(x_1),\dots,A_n(x_n)) \quad \forall\pi\in S_n\ ;$$

[1]Note that the elements of $\mathcal{F}$ are polynomials in $(\partial^\beta)\varphi$ and they are formal power series in $\hbar$. The generalization to non-polynomial observables is given in [2].
[2]Note that both the arguments and the values of T_n are *off-shell fields*, i.e. not restricted by any field equation.

(c) **Causality:** $T_n(A_1(x_1),\dots,A_n(x_n))$

$$= T_k(A_1(x_1),\dots,A_k(x_k)) \star_m T_{n-k}(A_{k+1}(x_{k+1}),\dots,A_n(x_n)) \tag{2.3}$$

$$\text{whenever} \quad \{x_1,\dots,x_k\} \cap \big(\{x_{k+1},\dots,x_n\} + \overline{V}_-\big) = \emptyset.$$

These are the basic axioms. In the inductive step $\{T_1,...,T_{n-1}\} \to T_n$ of the construction of the sequence T, these axioms determine

$$T_n^0(A_1(x_1),...) := T_n(A_1(x_1),...)|_{\mathcal{D}(\mathbb{M}^n\setminus\Delta_n)} \tag{2.4}$$

uniquely, where $\Delta_n := \{(x_1,\dots,x_n) \in \mathbb{M}^n : x_1 = \dots = x_n\}$ is the thin diagonal.

The further axioms (called 'renormalization conditions') restrict only the *extension* to $\mathcal{D}'(\mathbb{M}^n,\mathcal{F})$.

(d) **Field independence**:

$$\frac{\delta}{\delta\varphi(x)}T_n(A_1(x_1),...,A_n(x_n)) = \sum_{l=1}^n T_n\Big(A_1(x_1),...,\frac{\delta A_l(x_l)}{\delta\varphi(x)},...,A_n(x_n)\Big). \tag{2.5}$$

Using this property in a (finite) Taylor expansion of $T_n(A_1(x_1),\dots)$ w.r.t. $\varphi = 0$, one obtains the **causal Wick expansion**: for *monomials* $A_1,\dots,A_n \in \mathcal{P}$ it holds

$$T_n\big(A_1(x_1),\dots,A_n(x_n)\big) = \sum_{\underline{A}_l\subseteq A_l} \omega_0\big(T_n\big(\underline{A}_1(x_1),\dots,\underline{A}_n(x_n)\big)\big) \cdot \overline{A}_1(x_1)\cdots\overline{A}_n(x_n), \tag{2.6}$$

where $\omega_0 : F \mapsto \omega_0(F) := F|_{\varphi=0}$ denotes the vacuum state. In addition, each *submonomial* $\underline{A}$ of a given monomial A and its *complementary submonomial* $\overline{A}$ are defined by

$$\underline{A} := \frac{\partial^k A}{\partial(\partial^{\beta_1}\varphi)\cdots\partial(\partial^{\beta_k}\varphi)} \neq 0\ , \quad \overline{A} := C_{\beta_1\dots\beta_k}\,\partial^{\beta_1}\varphi\cdots\partial^{\beta_k}\varphi \tag{2.7}$$

(no sum over $\beta_1,...,\beta_k$), where each $C_{\beta_1\dots\beta_k}$ is a certain combinatorial factor and the range of the sum $\sum_{\underline{A}\subseteq A}$ are all allowable k and $\beta_1,...,\beta_k$. (For $k=0$ we have $\underline{A} = A$ and $\overline{A} = 1$.)

(e) **Translation invariance**: the $\mathbb{C}$-valued distributions

$$t^{(m)}(A_1,\dots,A_n)(x_1-x_n,\dots,x_{n-1}-x_n) := \omega_0\big(T_n^{(m)}\big(A_1(x_1),\dots,A_n(x_n)\big)\big) \tag{2.8}$$

depend *only* on the relative coordinates.

(f) **Action Ward Identity (AWI)**:

$$\partial_{x_k^\mu}T_n\big(A_1(x_1),...,A_k(x_k),...\big) = T_n\big(A_1(x_1),...,\partial_\mu A_k(x_k),...\big)\ . \tag{2.9}$$

The axioms (d) and (e) simplify the extension $T_n^0(A_1,...) \to T_n(A_1,...)$ to the problem of extending the $\mathbb{C}$-valued distributions $t^0(A_1,\dots)(x_1-x_n,\dots) := \omega_0\big(T_n^0(A_1(x_1),\dots)\big) \in \mathcal{D}'(\mathbb{R}^{d(n-1)}\setminus\{0\})$ to $t(A_1,\dots)(x_1-x_n,\dots) \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$, $\forall A_1,...,A_n \in \mathcal{P}$.

The AWI can be fulfilled by using that there exists a subspace $\mathcal{P}_{\rm bal}\subset\mathcal{P}$ (called 'balanced fields') such that every $A\in\mathcal{P}$ can *uniquely* be written as a finite sum

$$A=\sum_k \partial^{\beta_k}B_k\quad\text{where}\quad B_k\in\mathcal{P}_{\rm bal}\ ,\quad \beta_k\in\mathbb{N}_0^d$$

(see [4, Sect. 3.2] for the definition of $\mathcal{P}_{\rm bal}$). Since t^0 fulfills the AWI by induction, one can proceed as follows: one constructs the extension $t(B_1,...,B_n)$ first only for all balanced fields $B_1,...,B_n\in\mathcal{P}_{\rm bal}$. Then, using linearity of T_n and writing arbitrary $A_1,...,A_n\in\mathcal{P}$ as $A_i=\sum_{k_i}\partial^{\beta_{ik_i}}B_{ik_i}$ (where $B_{ik_i}\in\mathcal{P}_{\rm bal}$), the definition

$$t(A_1,\dots,A_n)(x_1-x_n,\dots):=\sum_{k_1,\dots,k_n}\partial^{\beta_{1k_1}}\cdots\partial^{\beta_{nk_n}}t(B_{1k_1},\dots,B_{nk_n})(x_1-x_n,\dots)\ . \tag{2.10}$$

yields indeed an extension of $t^0(A_1,\dots,A_n)$ which satisfies the AWI.

(g) **Scaling**: The mass dimension of a field monomial is defined by

$$\dim\prod_{j=1}^J\partial^{\beta_j}\varphi:=J\,\tfrac{d-2}{2}+\sum_{j=1}^J|\beta_j|\ . \tag{2.11}$$

Let $\mathcal{P}_{\rm hom}$ be the set of "homogeneous" polynomials, i.e. an $A\in\mathcal{P}_{\rm hom}$ is a linear combination of monomials which have the *same mass dimension.*

The scaling axioms requires that for $A_1,\dots,A_n\in\mathcal{P}_{\rm hom}$ the numerical distributions (2.8) scale almost homogeneously under $(x,m)\to(\rho x,m/\rho)$,[3] that is

$$0=(\rho\,\partial_\rho)^N\Big(\rho^D t^{(m/\rho)}\big(A_1,\dots,A_n\big)(\rho x)\Big) \tag{2.12}$$

for a sufficiently large $N\in\mathbb{N}$, where the degree D is given by $D:=\sum_{k=1}^n\dim A_k\in\mathbb{N}$. That D is a natural number follows from the observation that $t^{(m)}(A_1,\dots,A_n)$ is non-vanishing only if the number of basic fields $\partial^\beta\varphi$ in $\{A_1,\dots,A_n\}$ is even.

By the 'power' of the almost homogeneous scaling we mean $N-1$ for the minimal $N\in\mathbb{N}$ fulfilling (2.12) (or equivalently (1.1)).

(h) The axioms **Lorentz covariance, unitarity, off-shell field equation** and **symmetries** are not relevant for our purposes, hence, we do not explain them here.

2.2. Axioms for quantization with a Hadamard function

In this subsection we assume that quantization is done by a Hadamard function H_m^μ. Then the star product $\star_{m,\mu}$ and, via the causality axiom, the time-ordered product $T^{(m,\mu)}$ depend on μ. We complete the system of axioms as follows [4]:

[3]When quantizing with a Hadamard function H_m^μ, the mass parameter μ is not scaled.

(i) **Smoothness in the mass** $m \geq 0$: Since H^μ_m is smooth in $m \geq 0$, we may require that the functions

$$0 \leq m \longmapsto \langle t^{(m,\mu)}(A_1, ..., A_n)\,,\, g\rangle \quad \text{be smooth} \quad \forall A_1, ..., A_n \in \mathcal{P} \tag{2.13}$$

and $\forall g \in \mathcal{D}(\mathbb{R}^{d(n-1)})$.

(j) μ**-covariance**: Let

$$\Gamma := \int dx\, dy\, m^{d-2}\, f^{(d)}(m^2(x-y)^2)\, \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} \quad \text{and}$$
$$r^\Gamma := 1 + \sum_{k=1}^\infty \frac{1}{k!}\,((\log r)\Gamma)^k, \tag{2.14}$$

where $r > 0$ and the function $f^{(d)}$ is the one that appears in the definition (2.2) of the Hadamard function. With that the operator $(\frac{\mu_2}{\mu_1})^\Gamma$ intertwines the different star products for μ_1 and μ_2:

$$F \star_{m,\mu_2} G = (\tfrac{\mu_2}{\mu_1})^\Gamma \Big(((\tfrac{\mu_2}{\mu_1})^{-\Gamma} F) \star_{m,\mu_1} ((\tfrac{\mu_2}{\mu_1})^{-\Gamma} G)\Big)\ . \tag{2.15}$$

We require the same relation for the time-ordered products:

$$T_n^{(m,\mu_2)}\big(A_1(x_1), ..., A_n(x_n)\big) =$$
$$(\tfrac{\mu_2}{\mu_1})^\Gamma \Big(T_n^{(m,\mu_1)}\big((\tfrac{\mu_2}{\mu_1})^{-\Gamma} A_1(x_1), ..., (\tfrac{\mu_2}{\mu_1})^{-\Gamma} A_n(x_n)\big)\Big). \tag{2.16}$$

2.3. Modification of the axioms such that the Wightman two-point function is admitted

Smoothness in $m \geq 0$, axiom (i), excludes the Wightman two-point function Δ^+_m in even dimensions d. However, a time-ordered product $(T_n^{(m)})_{n\in\mathbb{N}}$ based on quantization with Δ^+_m can be axiomatically defined by using that the operator $(\frac{\mu}{m})^\Gamma$ intertwines the star products $\star_m$ (based on Δ^+_m) and $\star_{m,\mu}$ (based on H^μ_m). (This statement is obtained by inserting $H^m_m = \Delta^+_m$ into (2.15).) Due to that one may replace axiom (i) by the requirement that the transformed time-ordered product

$$(\tfrac{\mu}{m})^\Gamma \Big(T_n^{(m)}\big((\tfrac{\mu}{m})^{-\Gamma} A_1(x_1), ..., (\tfrac{\mu}{m})^{-\Gamma} A_n(x_n)\big)\Big) \tag{2.17}$$

be smooth in $m \geq 0$, as done in [4, 5]. (That is, the vacuum expectation values $t^{(m,\mu)}(A_1, ..., A_n) := \omega_0\big((2.17)\big)$ fulfil (2.13).) In addition, the μ-covariance, axiom (j), is unnecessary, it has to be omitted; all other axioms remain unchanged.

Since smoothness in $m \geq 0$ is very helpful for the construction of the time-ordered products (by means of the Taylor expansion (1.2)), the obvious way to construct a solution of the so modified system of axioms is, to construct first the time-ordered product $(T_n^{(m,\mu)})_{n\in\mathbb{N}}$ (which is based on H^μ_m), and then $(T_n^{(m)})_{n\in\mathbb{N}}$ is obtained by the inverse transformation of (2.17).

Following essentially [4], we explain why this construction fulfils the axiom (g) (scaling). First, for $t_H^{(m,\mu)} := \omega_0(T^{(m,\mu)})$ this is obtained as follows: using causality and the inductive assumption one shows that $t_H^{(m,\mu)\,0} \in$

$\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ fulfils (1.1), this is analogous to our procedure in sect. 4.1. It follows that the pertinent distributions $u_l^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ in the Taylor expansion (1.2) scale almost homogeneously with degree $D-l$. The extension $u_l^0 \to u_l \in \mathcal{D}'(\mathbb{R}^k)$ is done such that this property is maintained. Therefore, inserting u_l into (1.2), we obtain that the resulting $t_H^{(m,\mu)} \in \mathcal{D}'(\mathbb{R}^k)$ fulfils (1.1) (or equivalently (2.12)).[4]

The second step is to verify that the axiom (g) is preserved in the inverse transformation of (2.17):[5] we use that $t^{(m)} := \omega_0(T^{(m)})$ can be written as $t^{(m)} = t_H^{(m,m)}$ (due to $H_m^m = \Delta_m^+$). With that, the assertion

$$0 = (\rho\,\partial_\rho)^N\Big(\rho^D t^{(m/\rho)}(A_1,\dots)(\rho x)\Big) = (\rho\,\partial_\rho)^N\Big(\rho^D t_H^{(m/\rho,m/\rho)}(A_1,\dots)(\rho x)\Big)$$

can equivalently be written as

$$0 = \Big(x\,\partial_x - m\partial_m + D - \mu\,\partial_\mu\Big)^N\, t_H^{(m,\mu)}(A_1,\dots)(x)|_{\mu=m}\ , \tag{2.18}$$

where $x\,\partial_x := \sum_r x_r\,\partial_{x_r}$. From (2.16) we see that

$$(\mu\,\partial_\mu)^K t_H^{(m,\mu)}(A_1,\dots) = (\mu\,\partial_\mu)^K \omega_0\Big((\tfrac{\mu}{\mu_0})^\Gamma \circ T^{(m,\mu_0)}\big((\tfrac{\mu}{\mu_0})^{-\Gamma}A_1,\dots\big)\Big) = 0 \tag{2.19}$$

for $K \in \mathbb{N}$ sufficiently large; namely, since our functionals $F \in \mathcal{F}$ are *polynomials* in $(\partial^\beta)\varphi$, an expression $r^\Gamma F$ (2.14) is a polynomial in $\log r$. Now writing the r.h.s. of (2.18) as

$$\sum_{K=0}^N \binom{N}{K}(x\,\partial_x - m\,\partial_m + D)^{N-K}(-\mu\,\partial_\mu)^K\, t_H^{(m,\mu)}(A_1,\dots)(x)|_{\mu=m}\ ,$$

we see that this expression vanishes indeed for $N \in \mathbb{N}$ sufficiently large.

Example. We illustrate for the setting sun diagram in $d=4$ dimensions how $t^{(m)}(\varphi^3,\varphi^3)$ (based on Δ_m^+) can be obtained from $T^{(m,\mu)}$-terms in practice. From (2.2) we know that the Feynman(-like) propagators fulfil

$$\Delta_m^F(x) = H_m^{F,\mu}(x) + d_m^\mu(x) \quad \text{with} \quad d_m^\mu \in C^\infty\ ,$$

where $H_m^{F,\mu}(x) := \theta(x^0)H_m^\mu(x) + \theta(-x^0)H_m^\mu(-x)$. Inserting this into $t^{(m)}(\varphi^3,\varphi^3)(x) = 6\,\hbar^3\,(\Delta_m^F(x))^3$ we obtain

$$\begin{aligned} t^{(m)}(\varphi^3,\varphi^3)(x) =& t^{(m,\mu)}(\varphi^3,\varphi^3)(x) + 9\,\hbar\, t^{(m,\mu)}(\varphi^2,\varphi^2)(x)\ d_m^\mu(x) \\ &+ 18\,\hbar^3\, H_m^{F,\mu}(x)\,(d_m^\mu(x))^2 + 6\,\hbar^3\,(d_m^\mu(x))^3\ . \end{aligned}$$

Since d_m^μ is smooth, all appearing pointwise products exist.

[4] The remainders in the Taylor expansion (1.2) are teated in the same way as in our construction in sect. 4.2, hence we neglect them here.

[5] In [4] this verification is done in terms of a scaling transformation σ_ρ, which is an algebra isomorphism from $(\mathcal{F}, \star_{(\rho^{-1}m,\rho^{-1}\mu)})$ to $(\mathcal{F}, \star_{(m,\mu)})$. To minimize the mathematical tools, we do not introduce σ_ρ in this paper.

However, in view of a *direct* construction of $(T_n^{(m)})_{n\in\mathbb{N}}$, we are searching a direct axiomatic definition of these objects. We want to keep almost homogeneous scaling (with degree D) of the distributions $t \equiv t^{(m)}(A_1,\dots,A_n)$, $A_1,\dots A_n \in \mathcal{P}_{\text{hom}}$, see (2.12). This axiom admits the addition of a term

$$t(x_1-x_n,\dots,x_{n-1}-x_n)+\sum_{|\beta|+l=D-d(n-1)} m^l\, C_{l,\beta}^{(m)}\,\partial^\beta\delta(x_1-x_n,\dots,x_{n-1}-x_n), \tag{2.20}$$

where $l\in\mathbb{Z}$ (since $D\in\mathbb{N}$) and the numbers $C_{l,\beta}^{(m)}\in\mathbb{C}$ are, as functions of m, polynomials in $\log(m/M)$, where $M>0$ is some renormalization mass scale. But to fulfil the usual requirement $\mathrm{sd}(t)=\mathrm{sd}(t^0)$ on extensions t of t^0, we need a substitute for smoothness in $m\geq 0$, which excludes negative values of l. Such a candidate is:

(i′) **Continuity in the mass** $m\geq 0$: We require that the functions

$$0\leq m\longmapsto \langle t^{(m)}(A_1,...,A_n)\,,\,g\rangle \quad \text{be continuous} \quad \forall A_1,...,A_n\in\mathcal{P} \tag{2.21}$$

and $\forall g\in\mathcal{D}(\mathbb{R}^{d(n-1)})$.

With that, the Wightman two-point function Δ_m^+ is admitted also in even dimensions d. (Recall that Δ_m^+ is actually C^1 in $m\geq 0$.)

Remark 2.1 (central solution and mass-shell renormalization). If all fields are massive (i.e., $m>0$), any admissible extension $t^{(m)}\in\mathcal{D}'(\mathbb{R}^k)$ of a given $t^{(m)\,0}\in\mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ has the property that its Fourier transformed[6] distribution $\hat{t}^{(m)}(p)$ is analytic in a neighbourhood of $p=0$ (see [8]). Therefore, the so-called "central solution $t_c^{(m)}$" of the extension problem exists, which is defined by

$$\partial^\beta\hat{t}_c^{(m)}(0)=0\quad\forall|\beta|\leq\omega\ ,\quad \omega:=\mathrm{sd}(t^{(m)\,0})-k\ .$$

It can be obtained from any extension $t^{(m)}$ with $\mathrm{sd}(t^{(m)})=\mathrm{sd}(t^{(m)\,0})$, by Taylor subtraction:

$$\hat{t}_c^{(m)}(p)=\hat{t}^{(m)}(p)-\sum_{|\beta|\leq\omega}\frac{p^\beta}{\beta!}\,\partial^\beta\hat{t}^{(m)}(0)\ ,$$

which corresponds to "BPHZ-subtraction at $p=0$". We conclude: if there exists an extension $t^{(m)}$ which fulfills the scaling axiom (2.12) with degree $D=\omega+k\in\mathbb{N}$ and power $(N-1)$, i.e.

$$(\rho\partial_\rho)^N\left(\rho^\omega\,\hat{t}^{(m/\rho)}(p/\rho)\right)=0\ ,$$

then, this holds also for $t_c^{(m)}$ with the same degree and the same power. But, it is well known that the limit $\lim_{m\downarrow 0} t_c^{(m)}$ diverges in general,[7] i.e. the central solution is in conflict with continuity in $m\geq 0$ and, hence, also with the sm-expansion axiom (which is treated in the following sections).

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[6]Fourier transformation is meant w.r.t. the relative coordinates $x\equiv(x_1-x_n,...,x_{n-1}-x_n)$.
[7]This holds e.g. for the fish diagram in $d=4$ dimensions.

To discuss mass-shell renormalization we study a φ^4-interaction in $d = 4$ dimensions (or φ^3 in $d = 6$). Let

$$\Sigma_n^m(p^2) := \hat{t}^{(m)}(\varphi^3, \varphi^4, ..., \varphi^4, \varphi^3)(p, 0, ..., 0)$$

(or the same for $\hat{t}(\varphi^5, \varphi^6, ..., \varphi^6, \varphi^5)$ in the $(d = 6)$-case) be the self-energy contribution to n-th order; it has $\omega = 2$ to all orders. The inner momenta p_j $(j = 2, ..., n-1)$ are set to $p_j = 0$, due to integrating out the inner vertices x_j with $g(x_j) \equiv 1$ ("partial adiabatic limit", see e.g. [6]). We use the notation $\Sigma_n^{m\,\prime}(p^2) := \frac{\partial}{\partial p^2}\Sigma_n^m(p^2)$. In addition, let m_0 be the physical mass. The mass-shell renormalization $\Sigma_{n,m_0}^m(p^2)$ is uniquely defined by

$$\Sigma_{n,m_0}^m(m_0^2) = 0 \quad \text{and} \quad \Sigma_{n,m_0}^{m\,\prime}(m_0^2) = 0\ , \quad \forall n \geq 2\ ,$$

and is obtained by Taylor subtraction ("BPHZ-subtraction at $p^2 = m_0^2$"):

$$\Sigma_{n,m_0}^m(p^2) = \Sigma_n^m(p^2) - \Sigma_n^m(m_0^2) - (p^2 - m_0^2)\,\Sigma_n^{m\,\prime}(m_0^2)\ .$$

If $\Sigma_n^m(p^2)$ scales almost homogeneously with power $N_n - 1$, i.e.

$$(\rho\partial_\rho)^{N_n}\left(\rho^2\,\Sigma_n^{m/\rho}(p/\rho)\right) = 0\ , \text{ then generally } (\rho\partial_\rho)^N\left(\rho^2\,\Sigma_{n,m_0}^{m/\rho}(p/\rho)\right) \neq 0$$

$\forall N \in \mathbb{N}$, because the subtraction point m_0^2 is not scaled. However, usually one sets $m := m_0$ ("mass renormalization") and, if m_0 is also scaled, we obtain

$$(\rho\partial_\rho)^{N_n}\left(\rho^2\,\Sigma_{n,m_0/\rho}^{m_0/\rho}(p/\rho)\right) = 0\ .$$

But, the limit $\lim_{m_0\downarrow 0}\Sigma_{n,m_0}^{m_0}$ diverges in general, because the central solution $\Sigma_{n,0}^m(p^2) = \hat{t}_c^{(m)}(p, 0, ..., 0)$ generally does not exist for $m = 0$.

3. The scaling and mass expansion

The difficult question is: how to construct a solution of the just proposed system of axioms (a)-(h) and (i′)? We solve the problem in an indirect way, by replacing the almost homogeneous scaling, axiom (g), and the continuity in $m \geq 0$, axiom (i′), by the following new axiom:

(k) **Scaling and mass expansion**: For all field *monomials* $A_1, \ldots, A_n \in \mathcal{P}$, the vacuum expectation values $t^{(m)}(A_1, \ldots, A_n)(x_1 - x_n, \ldots, x_{n-1} - x_n)$ (2.8) fulfil the sm-expansion with degree $D := \sum_{k=1}^n \dim A_k$, where the following definition is used:

Definition 3.1. A distribution $f^{(m)} \in \mathcal{D}'(\mathbb{R}^k)$ or $f^{(m)} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$, depending on $m \geq 0$, fulfils the sm-expansion with degree D, if for all $l, L \in \mathbb{N}_0$ there exist distributions $u_l^{(m)}$, $\mathfrak{r}_{L+1}^{(m)} \in \mathcal{D}'(\mathbb{R}^k[\setminus\{0\}])$ such that

$$f^{(m)}(x) = \sum_{l=0}^{L} m^l\, u_l^{(m)}(x) + \mathfrak{r}_{L+1}^{(m)}(x) \quad \forall L \in \mathbb{N}_0\ , \tag{3.1}$$

and

(A) $u_0 \equiv u_0^{(m)}$ is independent of m and $u_0 = f^{(0)}$;

(B) For $l \geq 1$ the m-dependence of $u_l^{(m)}(x)$ is a polynomial in $\log \frac{m}{M}$, where $M > 0$ is a fixed mass scale. Explicitly, there exist m-independent distributions $u_{l,p} \in \mathcal{D}'(\mathbb{R}^k[\setminus\{0\}])$ such that

$$u_l^{(m)}(x) = \sum_{p=0}^{P_l} \big(\log \tfrac{m}{M}\big)^p \, u_{l,p}(x), \quad P_l < \infty \ . \tag{3.2}$$

(Of course, the distributions $u_{l,p}$ depend on M.)

(C) $u_l^{(m)}(x)$ scales almost homogeneously in x with degree $D - l$ and, hence, this holds also for all $u_{l,p}$ (3.2);

(D) $\mathfrak{r}_{L+1}^{(m)}(x)$ is almost homogeneous with degree D under the scaling $(x, m) \mapsto (\rho x, m/\rho)$;

(E) $\mathfrak{r}_{L+1}^{(m)}$ is smooth in m for $m > 0$ and

$$\lim_{m \downarrow 0} \big(\tfrac{m}{M}\big)^{-(L+1)+\varepsilon} \ \mathfrak{r}_{L+1}^{(m)} = 0 \qquad \forall \varepsilon > 0 \ .$$

(All properties are meant in the weak sense, e.g. (E) holds for $\langle \mathfrak{r}_{L+1}^{(m)}, h \rangle$ $\forall h \in \mathcal{D}(\mathbb{R}^k[\setminus\{0\}])$.)

As explained after (2.12), the degree $D = \sum_k \dim A_k$ is a natural number.

One easily verifies that, in $d = 4$ dimensions, the Wightman two-point function Δ_m^+ (1.3) fulfils the sm-expansion with degree $D = 2$. For arbitrary $d \geq 3$, $\Delta_m^{+\,(d)}$ fulfils the sm-expansion with degree $D = d - 2$. (If d is odd, $\Delta_m^{+\,(d)}$ is smooth in $m \geq 0$, hence the sm-expansion is simply the Taylor expansion.) Taking additionally $\dim \varphi = \frac{d-2}{2}$ into account, we find that

$$t^{(m)}(\varphi(x_1), \varphi(x_2)) = \hbar\, \Delta_m^F(y) = \hbar \left(\theta(y^0) \Delta_m^+(y) + \theta(-y^0) \Delta_m^+(-y) \right)$$

(where $y \equiv x_1 - x_2$) fulfils the new axiom (k).

The following lemma gives basic properties of distributions fulfilling the sm-expansion.

Lemma 3.2. *We assume that* $f^{(m)} \in \mathcal{D}'(\mathbb{R}^k[\setminus\{0\}])$, $f_1^{(m)} \in \mathcal{D}'(\mathbb{R}^{pd}[\setminus\{0\}])$ *and* $f_2^{(m)} \in \mathcal{D}'(\mathbb{R}^{qd}[\setminus\{0\}])$ *satisfy the definition* 3.1 *with degree* D, D_1 *or* D_2, *respectively. Then the following statements hold true:*

(1) $f^{(m)}$ *is smooth in* m *for* $m > 0$ *and* $\lim_{m \downarrow 0} f^{(m)} = u_0 = f^{(0)}$.

(2) $f^{(m)}(x)$ *is almost homogeneous with degree* D *under the scaling* $(x, m) \mapsto (\rho x, m/\rho)$.

(3) $\partial_x^\beta f^{(m)}(x)$ *(where* β *is a multi-index) fulfils the sm-expansion with degree* $D + |\beta|$.

(4) *We assume that the product of distributions* $f_1^{(m)}(x) f_2^{(m)}(y)$, *which may be a (partly) pointwise product*[8], *exists. Then,* $f_1^{(m)}(x)\, f_2^{(m)}(y)$ *fulfils also the sm-expansion with degree* $D = D_1 + D_2$.

[8]More precisely: let $(x_1, ..., x_p)$ and $(y_1, ..., y_q)$ (where $x_i, y_j \in \mathbb{R}^d$) be the linearly independent components of $x \in \mathbb{R}^{pd}$ and $y \in \mathbb{R}^{qd}$, respectively. Then, the set $\{x_1, ..., x_p, y_1, ..., y_q\}$ may be linearly dependent.

(5) *The sm-expansion is unique, i.e. if we know that a given $f^{(m)}$ has such an expansion, then the "coefficients" $u_l^{(m)}$ (and, hence, also the "remainders" $\mathfrak{r}_{L+1}^{(m)}$) are uniquely determined.*

(6) *The scaling degree of the remainder is bounded by* $\mathrm{sd}(\mathfrak{r}_{L+1}^{(m)}) \leq D-(L+1)$.

Proof. Part (1) follows immediately from (3.1) and properties (A),(B) and (E).

Part (2): we have to show that $m^l\, u_l^{(m)}(x)$ has the asserted scaling property. This can be done as follows:

$$(x\,\partial_x + D - m\,\partial_m)^N\, m^l\, u_l^{(m)}(x) = m^l\,(x\,\partial_x + (D-l) - m\,\partial_m)^N\, u_l^{(m)}(x)$$
$$= m^l \sum_{k=0}^{N} \binom{N}{k} (x\,\partial_x + D - l)^k (-m\,\partial_m)^{N-k}\, u_l^{(m)}(x)\ ,$$

where $x\,\partial_x := \sum_{i=1}^k x_i \partial_{x_i}$. Now, choosing N sufficiently large, at least one of the operators $(x\,\partial_x + D - l)^k$ or $(-m\,\partial_m)^{N-k}$ yields zero when applied to $u_l^{(m)}(x)$, due to properties (C) and (B), respectively.

Part (3): we show that $\partial_x^\beta u_l^{(m)}(x)$ and $\partial_x^\beta \mathfrak{r}_{L+1}^{(m)}(x)$ satisfy the properties (A)-(E) with degree $D+|\beta|$. To verify (D) let $N \in \mathbb{N}$ be such that $(x\,\partial_x + D - m\,\partial_m)^N\, \mathfrak{r}_{L+1}^{(m)}(x) = 0$. It follows that

$$0 = \partial_x^\beta\,(x\,\partial_x + D - m\,\partial_m)^N\, \mathfrak{r}_{L+1}^{(m)}(x) = (x\,\partial_x + D + |\beta| - m\,\partial_m)^N\, \partial_x^\beta\, \mathfrak{r}_{L+1}^{(m)}(x).$$

(C) can be shown analogously. To verify (A), (B) and (E) we use that these properties hold for $\langle g^{(m)}, h\rangle$, where $g^{(m)} = u_l^{(m)}$ or $g^{(m)} = \mathfrak{r}_{L+1}^{(m)}$, for all $h \in \mathcal{D}(\mathbb{R}^k[\setminus\{0\}])$. Hence, they hold for $(-1)^{|\beta|}\,\langle g^{(m)}, \partial^\beta h\rangle = \langle \partial^\beta g^{(m)}, h\rangle\ \forall h$.

Part (4): by a straightforward calculation we obtain

$$f_1^{(m)}(x)\, f_2^{(m)}(y) = \sum_{l=0}^{L} m^l\, u_l^{(m)}(x,y) + \mathfrak{r}_{L+1}^{(m)}(x,y)\ ,$$

where

$$u_l^{(m)}(x,y) := \sum_{k=0}^{l} u_{1,k}^{(m)}(x)\, u_{2,l-k}^{(m)}(y), \quad (0 \leq l \leq L)$$

$$\mathfrak{r}_{L+1}^{(m)}(x,y) := \mathfrak{r}_{1,L+1}^{(m)}(x)\,\mathfrak{r}_{2,L+1}^{(m)}(y) + \mathfrak{r}_{1,L+1}^{(m)}(x) \sum_{l=0}^{L} m^l\, u_{2,l}^{(m)}(y)$$
$$+ \Big(\sum_{l=0}^{L} m^l\, u_{1,l}^{(m)}(x)\Big)\, \mathfrak{r}_{2,L+1}^{(m)}(y) + \sum_{l=L+1}^{2L} m^l \sum_{k=l-L}^{L} u_{1,k}^{(m)}(x)\, u_{2,l-k}^{(m)}(y)\ .$$

With that, it is an easy task to verify that $u_l^{(m)}(x,y)$ and $\mathfrak{r}_{L+1}^{(m)}(x,y)$ satisfy the properties (A)-(E) with degree $D = D_1 + D_2$, by using that $u_{j,l}^{(m)}$ and $\mathfrak{r}_{j,L+1}^{(m)}$ fulfil these properties with degree D_j (where $j = 1, 2$).

Part (5): the determination of u_0 is given in part (1). For $l \geq 1$ we assume that $u_k^{(m)}$ is known for $k < l$ and we determine the coefficients $u_{l,p}$ of $u_l^{(m)}$ (3.2) as follows: for $\mathbb{N} \ni P > P_l$ the limit

$$\lim_{m\downarrow 0} \left(\log \tfrac{m}{M}\right)^{-P} m^{-l} \left(f^{(m)}(x) - \sum_{k=0}^{l-1} m^k\, u_k^{(m)}(x)\right) \tag{3.3}$$

gives zero, for $P = P_l$ it gives u_{l,P_l} and for $P < P_l$ it diverges. Since P_l is unknown, we start with a P which is sufficiently high that the limit exists, if it vanishes we lower P by 1 etc.. Having determined P_l and u_{l,P_l} in this way, we compute

$$\lim_{m\downarrow 0} \left(\log \tfrac{m}{M}\right)^{-(P_l-1)} m^{-l} \left(f^{(m)}(x) - \sum_{k=0}^{l-1} m^k\, u_k^{(m)}(x) - m^l \left(\log \tfrac{m}{M}\right)^{P_l} u_{l,P_l}(x)\right)$$
$$= u_{l,P_l-1}\ ;$$

and so on.

Part (6): from property (E) we know that the distribution

$$t^{(m)}(x) := m^{-(L+1)}\, \mathfrak{r}_{L+1}^{(m)}(x) \quad \text{fulfils} \quad \lim_{m\downarrow 0} (\tfrac{m}{M})^{\varepsilon}\, t^{(m)} = 0 \qquad \forall \varepsilon > 0\ .$$

From (D) we conclude that

$$\rho^{D-(L+1)}\, t^{(m)}(\rho x) = t^{(\rho m)}(x) + \sum_{k=1}^{N} l_k^{(\rho m)}(x)\, (\log \rho)^k \qquad \forall \rho > 0 \tag{3.4}$$

with some $l_k^{(m)} \in \mathcal{D}'(\mathbb{R}^k[\setminus\{0\}])$. Multiplying the latter equation by $(\rho m)^{\varepsilon}$ and performing the limit $m \downarrow 0$, we conclude that

$$\lim_{m\downarrow 0} (\tfrac{m}{M})^{\varepsilon}\, l_k^{(m)} = 0 \qquad \forall \varepsilon > 0\ ,\ \ k = 1, ..., N\ .$$

It follows that

$$\lim_{\rho\downarrow 0} \rho^{D-(L+1)+\varepsilon}\, \mathfrak{r}_{L+1}^{(m)}(\rho x) = m^{L+1} \Big(\lim_{\rho\downarrow 0} \rho^{\varepsilon}\, t^{(\rho m)}(x)$$
$$+ \sum_{k=1}^{N} \big(\lim_{\rho\downarrow 0} \rho^{\varepsilon/2}\, l_k^{(\rho m)}(x)\big) \big(\lim_{\rho\downarrow 0} \rho^{\varepsilon/2}\, (\log \rho)^k\big)\Big) = 0 \quad \forall \varepsilon > 0\ .$$

□

From parts *(1)* and *(2)* we see that the new axiom (k), sm-expansion, is *sufficient* for the above proposed axioms (i′), continuity in $m \geq 0$, and (g), almost homogeneous scaling. We will see that (k) is even *equivalent* to the combination of (i′) and (g), in the sense that the set of solutions of the axioms (a)-(f), (h) and (k) is *equal* to the set of solutions of (a)-(h) and (i′).

4. Construction of a solution of the new system of axioms

In this section we use the inductive Epstein-Glaser construction [8], to obtain the general solution of the system of axioms (a)-(f), (h) and (k). More precisely we work with Stora's extension of distributions [16, 1] instead of Epstein and Glaser's distribution splitting method.

4.1. Inductive step, off the thin diagonal

We use that $T^0_n(A_1(x_1),...) \in \mathcal{D}'(\mathbb{M}^n \setminus \Delta_n, \mathcal{F})$ (2.4) is uniquely determined by causal factorization (2.3), see [1]. Due to the uniqueness of the sm-expansion, we only have to show that for every configuration $(x_1,...,x_n) \in \mathbb{M}^n \setminus \Delta_n$ there **exists** such an expansion; in particular, the resulting expansion does not depend on the way we split $\{x_1,...,x_n\}$ into two nonempty subsets such that one is later than the other.

Without restricting generality, we may assume that $\{x_1,...,x_l\} \cap (\{x_{l+1},...,x_n\}+\bar{V}_-) = \emptyset$, in addition let $A_1,...,A_n$ be field *monomials*. Inserting the causal Wick expansion (2.6) into (2.3), we see that $t^0(A_1,...,A_n)(x_1-x_n,...) := \omega_0\big(T^0_n(A_1(x_1),...)\big)$ is a linear combination of products

$$t(\underline{A}_1,...,\underline{A}_l)(x_1-x_l,...)\, t(\underline{A}_{l+1},...,\underline{A}_n)(x_{l+1}-x_n,...) \cdot \omega_0\Big(\big(\overline{A}_1(x_1)\cdots\overline{A}_l(x_l)\big) \star_m \big(\overline{A}_{l+1}(x_{l+1})\cdots\overline{A}_n(x_n)\big)\Big) \ . \tag{4.1}$$

The $\omega_0(...)$-factor is, if it does not vanish, a linear combination of products

$$\prod_{k=1}^{K} \partial^{\beta_k}\Delta^+_m(x_{i_k}-x_{j_k}) \quad \text{with} \quad K(d-2)+\sum_{k=1}^{K}|\beta_k| = \sum_{i=1}^{n}\dim\overline{A}_i \ , \tag{4.2}$$

where $i_k \in \{1,...,l\}$ and $j_k \in \{l+1,...,n\}$. By induction $t(\underline{A}_1,\underline{A}_l)$ and $t(\underline{A}_{l+1},\underline{A}_n)$ fulfil the sm-expansion with degree $D_{(i)} := \sum_{i=1}^{l}\dim\underline{A}_i$ and $D_{(ii)} := \sum_{j=l+1}^{n}\dim\underline{A}_j$, respectively; in addition $\partial^{\beta_k}\Delta^+_m$ satisfies this expansion with degree $D_k := d-2+|\beta_k|$ (due to part *(3)* of the lemma). By means of part *(4)* of the lemma, we conclude that (4.1) fulfils the sm-expansion with degree

$$D_{(i)}+D_{(ii)}+\sum_{k=1}^{K}D_k = \sum_{i=1}^{n}\dim A_i \ ,$$

where we use that $\dim\underline{A}+\dim\overline{A} = \dim A$ (which follows immediately from (2.7)). Hence, T^0_n fulfils the new axiom (k).

4.2. Extension to the thin diagonal

To maintain the sm-expansion of $t^{(m)\,0}_n \in \mathcal{D}'(\mathbb{R}^{d(n-1)}\setminus\{0\})$,

$$t^{(m)\,0}_n(x) = u^0_0(x)+\sum_{l=1}^{L} m^l \sum_{p=0}^{P_l}\big(\log\tfrac{m}{M}\big)^p\, u^0_{l,p}(x)+\mathfrak{r}^{(m)\,0}_{L+1}(x) \ , \tag{4.3}$$

we extend each distribution u^0_0, $u^0_{l,p}$, $\mathfrak{r}^{(m)\,0}_{L+1} \in \mathcal{D}'(\mathbb{R}^{d(n-1)}\setminus\{0\})$ individually.

Due to part *(6)* of the lemma, the remainders

$$\mathfrak{r}^{(m)\,0}_{L+1} \quad \text{with} \quad L \geq L_0 := D - d(n-1)$$

can be extended by the direct extension (A.3).

The distributions $u^0_{l,p}$ $(l \geq 1)$ and u^0_0 $(l = 0)$ scale almost homogeneously in x with degrees $(D-l)$. Thus, by proposition A.1, there exist extensions $u_{l,p} \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$ and $u_0 \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$, respectively, which scale almost homogeneously with the same degree as the corresponding $u^0_{...}$-distributions. For $l > L_0$ the almost homogeneous extension is unique and agrees with the direct extension (A.3). For $0 \leq l \leq L_0$ the extension needs a mass scale $M_1 > 0$; we choose M_1 independent of m, such that $\partial_m u_{l,p} = 0$ and $\partial_m u_0 = 0$. One may choose $M_1 = M$.

We have to maintain the relation

$$\mathfrak{r}^{(m)\,0}_{L_1+1}(x) = \mathfrak{r}^{(m)\,0}_{L_2+1}(x) + \sum_{l=L_1+1}^{L_2} m^l \sum_{p=0}^{P_l} \bigl(\log \tfrac{m}{M}\bigr)^p\, u^0_{l,p}(x)\ , \quad 0 \leq L_1 < L_2\ . \tag{4.4}$$

For $L_1 \geq L_0$ the extensions indeed satisfy this relation, because all distributions appearing in (4.4) are extended by the unique direct extension (A.3). For $L_1 < L_0$ we fulfil (4.4) by *defining* the extension of $\mathfrak{r}^{(m)\,0}_{L_1+1}$ by

$$\mathfrak{r}^{(m)}_{L_1+1}(x) := \mathfrak{r}^{(m)}_{L_0+1}(x) + \sum_{l=L_1+1}^{L_0} m^l \sum_{p=0}^{P_l} \bigl(\log \tfrac{m}{M}\bigr)^p\, u_{l,p}(x) \quad \text{for} \quad 0 \leq L_1 < L_0\ .$$

An extension $t^{(m)}_n \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$ of $t^{(m)\,0}_n$, which fulfils the sm-expansion (with the same degree D as $t^{(m)\,0}_n$), is obtained by inserting the constructed extensions of the various distributions into (4.3); it does not matter which L we use, since the extensions fulfil (4.4).

From the preceding subsection we only know that $t^0(A_1, ..., A_n)$ satisfies the sm-expansion for field *monomials* $A_1, ..., A_n$. Hence, we have to explain, how the just described construction matches with the procedure (2.10) (in which the extension is done first for *balanced fields*). To explain this, note that, due to linearity of the map $\otimes_{i=1}^n A_i \mapsto t^0(A_1, \ldots, A_n)$, the sm-expansion holds for $t^0(A_1, \ldots, A_n)$ for all $A_1, \ldots, A_n \in \mathcal{P}_{\rm hom}$ (and not only for field *monomials*). With that an extension $t(A_1, \ldots, A_n)$ which fulfills the sm-expansion can be constructed as just described for all $A_1, \ldots, A_n \in \mathcal{P}_{\rm bal} \cap \mathcal{P}_{\rm hom}$. Symmetrization w.r.t. permutations of $(A_1, x_1), ..., (A_n, x_n)$ does not violate the sm-expansion. Then, by means of (2.10), we construct $t(A_1, ..., A_n)$ for all $A_1, \ldots, A_n \in \mathcal{P}$. To complete the inductive step, we have to show that, on the level of the extensions, the sm-expansion holds for all *monomials* $A_1, \ldots, A_n$ (and not only for $A_1, \ldots, A_n \in \mathcal{P}_{\rm bal} \cap \mathcal{P}_{\rm hom}$). For this purpose we write arbitrary monomials A_i $(1 \leq i \leq n)$ as $A_i = \sum_{k_i} \partial^{\beta_{ik_i}} B_{ik_i}$ with $B_{ik_i} \in \mathcal{P}_{\rm bal} \cap \mathcal{P}_{\rm hom}$. Note that $\dim B_{ik_i} + |\beta_{ik_i}| = \dim A_i$, $\forall k_i$. Then, $t(A_1, ..., A_n)$ is given in terms of the distributions $t(B_{1k_1}, ..., B_{nk_n})$ by (2.10).

In this formula, each summand fulfils the sm-expansion with degree

$$\sum_{i=1}^{n} \dim B_{ik_i} + \sum_{i=1}^{n} |\beta_{ik_i}| = \sum_{i=1}^{n} \dim A_i \ ,$$

hence, this holds also for $t(A_1, \ldots, A_n)$.

The **most general solution** of the system of axioms is obtained by adding to a particular solution $t^{(m)}(A_1, ..., A_n)(x_1 - x_n, ...)$ a polynomial in derivatives of the delta distribution which fulfils the sm-expansion:

$$\sum m^l \, (\log \tfrac{m}{M})^p \, C_{l,p,\beta}(A_1, \ldots, A_n) \, \partial^\beta \delta(x_1 - x_n, \ldots, x_{n-1} - x_n) \ , \qquad (4.5)$$

where the sum runs over $l \in \mathbb{N}_0$, $p \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^{d(n-1)}$, with the restrictions

$$|\beta| + l = D - d(n-1) \quad \text{and} \quad p \leq P \ \text{ for some } \ P < \infty \ ; \qquad (4.6)$$

the numbers $C_{l,p,\beta}(A_1, \ldots, A_n) \in \mathbb{C}$ do not depend on m. In addition (4.5) has to be Lorentz covariant and invariant under permutations of $(A_1, x_1), \ldots,$ (A_n, x_n); the coefficients $C_{l,p,\beta}(A_1, \ldots, A_n)$ are also restricted by further axioms as e.g. unitarity.

We return to the assertion at the end of sect. 3: if we replace the axiom (k) by the (possibly weaker) axioms (g) and (i′), the freedom of (re)normalization (4.5)-(4.6) does not get bigger. (This follows from the discussion in (2.20)-(2.21).) Therefore, the two systems of axioms are indeed equivalent.

5. The scaling and mass expansion for a dimensionally regularized theory

In [5] dimensional regularization in position space is introduced by a change of the order of the Bessel functions defining the propagators: the regularized Feynman propagator is of the form

$$\Delta_m^{F\,\zeta}(x) = \sum_{l=0}^{\infty} h_l^\zeta \, M^{2\zeta} \, m^{2l} \, (-(x^2 - i\epsilon))^{l+1-\frac{d}{2}+\zeta} + \sum_{l=0}^{\infty} c_l^\zeta \, M^{2\zeta} \, m^{d-2+2l-2\zeta} \, (-x^2)^l \ , \qquad (5.1)$$

where $\zeta \in \Omega \setminus \{0\}$ for a neigborhood $\Omega \subset \mathbb{C}$ of 0; and $M > 0$ is a mass parameter, the factor $M^{2\zeta}$ is introduced to keep the mass dimension constant. The coefficients h_l^ζ, $c_l^\zeta \in \mathbb{C}$ do not depend on (x, m). In the limit $\zeta \to 0$, $\Delta_m^{F\,\zeta}(x)$ converges in a suitable sense to $\Delta_m^F(x)$. From (5.1) we see that $\Delta_m^{F\,\zeta}(x)$ is homogeneous under $(x, m) \to (\rho x, m/\rho)$:

$$\rho^{d-2-2\zeta} \, \Delta_{\rho^{-1}m}^{F\,\zeta}(\rho x) = \Delta_m^{F\,\zeta}(x) \ . \qquad (5.2)$$

To find the sm-expansion for the so regularized theory, we study a product of derivated, regularized Feynman propagators – with different ζ_{ij} for different arguments $(x_i - x_j)$, since the Epstein-Glaser forest formula requires the ability to vary the regularization parameters independently in this way, see [5]. We only treat the *even dimensional* case.[9] For $x_i \neq x_j\ \forall i < j$, we obtain the structure

$$\prod_{k=1}^{Q} \partial^{\beta_k} \Delta_{F,\,m}^{\zeta_{i_k j_k}}(x_{i_k} - x_{j_k}) = \sum_{|\mathbf{c}|+|\mathbf{h}|=Q} \sum_{p=0}^{\infty} (\tfrac{m}{M})^{2p-2\mathbf{c}\boldsymbol{\zeta}}\, u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(x)\ , \tag{5.3}$$

where $x := (x_1 - x_n, ..., x_{n-1} - x_n)$, and $h_{ij} \in \mathbb{N}_0$ ($c_{ij} \in \mathbb{N}_0$ resp.) is the number of h-lines (c-lines resp.) (i.e. the propagator is given by a h^{ζ}_l-term (c^{ζ}_l-term resp.)) connecting the vertices x_i and x_j, and

$$\boldsymbol{\zeta} := (\zeta_{ij})_{i<j}\ , \quad \mathbf{c} := (c_{ij})_{i<j}\ , \quad |\mathbf{c}| := \sum_{i<j} c_{ij}\ , \quad \mathbf{c}\boldsymbol{\zeta} := \sum_{i<j} c_{ij}\zeta_{ij}\ ,$$

and $\mathbf{h}$, $|\mathbf{h}|$ and $\mathbf{h}\boldsymbol{\zeta}$ are similarly defined. In addition the m-independent distributions $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(x)$ are homogeneous:

$$\rho^{\kappa}\, u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(\rho x) = u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(x) \quad \text{with} \quad \kappa := Q(d-2) - 2p - 2\mathbf{h}\boldsymbol{\zeta} + \sum_k |\beta_k|\ . \tag{5.4}$$

It follows that on the r.h.s. of (5.3) the sum $\sum_p (\frac{m}{M})^{2p-2\mathbf{c}\boldsymbol{\zeta}}\, u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}$ is homogeneous under $(x,m) \to (\rho x, m/\rho)$ with degree

$$\kappa + 2p - 2\mathbf{c}\boldsymbol{\zeta} = Q(d-2) + \sum_k |\beta_k| - 2\,(\mathbf{h}+\mathbf{c})\boldsymbol{\zeta}\ .$$

This motivates to require the following version of the sm-expansion axiom for the $\boldsymbol{\zeta}$-dependent regularized time-ordered product $T^{(m)\,\boldsymbol{\zeta}} \equiv (T^{(m)\,\boldsymbol{\zeta}}_n)$: for a field monomial $A = \prod_{j=1}^{J} \partial^{\beta_j}\varphi$ let $|A| := J$ and, similarly to (2.8), we define the vacuum expectation values $t^{(m)\,\boldsymbol{\zeta}}(A_1, \ldots, A_n) \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$. In addition let $N := \binom{n}{2}$.

- **Scaling and mass expansion ($d > 2$ even):** There exists an open neighborhood $\Omega_n \subset \mathbb{C}^N$ of the origin such that for all field *monomials* $A_1, \ldots, A_n \in \mathcal{P}$, the distributions $t^{(m)\,\boldsymbol{\zeta}}(A_1, \ldots, A_n)(x_1 - x_n, \ldots, x_{n-1} - x_n)$ fulfil for $\boldsymbol{\zeta} \in \Omega_n \setminus \{0\}$ the regularized sm-expansion with degree $D = \sum_{k=1}^{n} \dim A_k \in \mathbb{N}_0$ and $l = \frac{1}{2}\sum_{k=1}^{n} |A_k| \in \mathbb{N}_0$ lines; where the following definition is used:

Definition 5.1. Let $\Lambda \subset \mathbb{C}^N$ be an open set. A distribution $f^{(m)\,\boldsymbol{\zeta}} \in \mathcal{D}'(\mathbb{R}^{d(n-1)}[\setminus\{0\}])$, depending on $m \geq 0$, fulfils for $\boldsymbol{\zeta} \in \Lambda$ the *regularized sm-expansion* with degree D and $l \in \mathbb{N}_0$ lines, if it is analytic in $\boldsymbol{\zeta} \in \Lambda$, and if for all $p, P \in \mathbb{N}_0$ and $\mathbf{c}, \mathbf{h} \in \mathbb{N}_0^N$ with $|\mathbf{c}|, |\mathbf{h}| \leq l$, there exist m-independent distributions $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}} \in \mathcal{D}'(\mathbb{R}^{d(n-1)}[\setminus\{0\}])$ and remainders $\mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}} \in \mathcal{D}'(\mathbb{R}^{d(n-1)}[\setminus\{0\}])$, such that

$$f^{(m)\,\boldsymbol{\zeta}}(x) = \sum_{|\mathbf{c}|+|\mathbf{h}|=l} \Big[\sum_{p=0}^{P} (\tfrac{m}{M})^{2p-2\mathbf{c}\boldsymbol{\zeta}}\, u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(x) + \mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}(x)\Big]\ , \quad \forall P \in \mathbb{N}_0 \tag{5.5}$$

[9] In odd dimensions, $(\frac{m}{M})^{2p-2\mathbf{c}\boldsymbol{\zeta}}$ is replaced by $(\frac{m}{M})^{p-2\mathbf{c}\boldsymbol{\zeta}}$, $p \in \mathbb{N}_0$.

and $\forall\boldsymbol{\zeta}\in\Lambda$; in addition

(A) for $p=0$ and $\mathbf{c}\neq\mathbf{0}$ we have $u^{\boldsymbol{\zeta}}_{0,\mathbf{c},\mathbf{h}}\equiv 0\ \forall\mathbf{h}$, and for $m=0$ it holds $f^{(0)\,\boldsymbol{\zeta}}=\sum_{|\mathbf{h}|=l}u^{\boldsymbol{\zeta}}_{0,\mathbf{0},\mathbf{h}}$;

(B) for $\mathbf{h}=\mathbf{0}$ we have $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{0}}\in C^\infty$;

(C) $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}(x)$ is *homogeneous* (not only almost homogeneous) in x with degree

$$\kappa^{\boldsymbol{\zeta}}_{p,\mathbf{h}}:=D-2p-2\mathbf{h}\boldsymbol{\zeta}\ ; \tag{5.6}$$

(D) $\mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}(x)$ is homogeneous under $(x,m)\to(\rho x,m/\rho)$ with degree

$$D^{\boldsymbol{\zeta}}_{\mathbf{c},\mathbf{h}}=D-(\mathbf{h}+\mathbf{c})\boldsymbol{\zeta}\ ; \tag{5.7}$$

(E) $\mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}(x)$ is smooth in m for $m>0$ and

$$\lim_{m\downarrow 0}\big(\tfrac{m}{M}\big)^{-2(P+1)+2\mathbf{c}\boldsymbol{\zeta}+\epsilon}\ \mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}=0\qquad\forall\epsilon>0\ . \tag{5.8}$$

Similarly to (C) and (D), the properties (A) and (B) are motivated by their validity for (5.3). (B) is important for the extension of the distributions $u^{\boldsymbol{\zeta}\,0}_{p,\mathbf{c},\mathbf{h}}\in\mathcal{D}'(\mathbb{R}^{d(n-1)}\setminus\{0\})$: for almost all values of $\boldsymbol{\zeta}\in\Lambda$ we have $\kappa^{\boldsymbol{\zeta}}\notin d(n-1)+\mathbb{N}_0$ (i.e. we are in the much simpler case (i) of proposition A.1).

Suitably modified, all statements of lemma 3.2 hold true also for the regularized sm-expansion. The modifications are:[10] let (D,l) be the degree and the number of lines in the regularized sm-expansion of the distribution $f^{(m)\,\boldsymbol{\zeta}}\in\mathcal{D}'(\mathbb{R}^{d(n-1)}[\setminus\{0\}])$.

(1′) (No change for $m>0$.) In order that the limit $m\downarrow 0$ exists, we assume that $\Re(\zeta_{ij})<\frac{1}{l}\ \forall i,j$ (which implies $\Re(\mathbf{c}\boldsymbol{\zeta})<1$). With that it holds

$$\lim_{m\downarrow 0}f^{(m)\,\boldsymbol{\zeta}}=\sum_{|\mathbf{h}|=l}u^{\boldsymbol{\zeta}}_{0,\mathbf{0},\mathbf{h}}=f^{(0)\,\boldsymbol{\zeta}}\ . \tag{5.9}$$

(2′) Only the expression in the $\big[...\big]$-bracket of (5.5) (and not the complete $f^{(m)\,\boldsymbol{\zeta}}$) is *homogeneous* under $(x,m)\to(\rho x,m/\rho)$, with degree $D^{\boldsymbol{\zeta}}_{\mathbf{c},\mathbf{h}}$ (5.7).

(3′) $\partial^\beta_x f^{(m)\,\boldsymbol{\zeta}}(x)$ fulfils the regularized sm-expansion with $(D+|\beta|,l)$.

(4′) We formulate the statement in the form in which it is used in the inductive step of the construction of $T^{(m)\,\boldsymbol{\zeta}}$: let $\Delta^{+\,\zeta}_m$ be the regularized two-point function belonging to $\Delta^{F\,\zeta}_m$.[11] We assume that $f_1^{(m)\,\boldsymbol{\zeta}_1}(x_1-x_s,...)\in\mathcal{D}'(\mathbb{R}^{d(s-1)})$ and $f_2^{(m)\,\boldsymbol{\zeta}_2}(x_{s+1}-x_n,...)\in\mathcal{D}'(\mathbb{R}^{d(n-s-1)})$ fulfil the regularized sm-expansion with (D_1,l_1) and (D_2,l_2), respectively. Then,

$$f_1^{(m)\,\boldsymbol{\zeta}_1}(x_1-x_s,...)\,f_2^{(m)\,\boldsymbol{\zeta}_2}(x_{s+1}-x_n,...)\prod_{k=1}^K\partial^{\beta_k}\Delta_m^{+\,\zeta_{i_kj_k}}(x_{i_k}-x_{j_k}) \tag{5.10}$$

[10] For shortness we do not specify the domain for $\boldsymbol{\zeta}$.

[11] That is $\Delta^{F\,\zeta}_m(x)=\theta(x^0)\Delta^{+\,\zeta}_m(x)+\theta(-x^0)\Delta^{+\,\zeta}_m(-x)$.

(where $i_k \in \{1,...,s\}$ and $j_k \in \{s+1,...,n\}$ $\forall k$), which is an element of $\mathcal{D}'(\mathbb{R}^{d(n-1)} \setminus \{0\})$, satisfies the regularized sm-expansion with

$$\boldsymbol{\zeta} := \left(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, (\zeta_{ij})_{i\in\{1,...,s\}}^{j\in\{s+1,...,n\}}\right) , \quad D = D_1 + D_2 + K(d-2) + \sum_{k=1}^{K} |\beta_k| \quad (5.11)$$

and $l = l_1 + l_2 + K$.

(5′) If we know that a given $f^{(m)\,\boldsymbol{\zeta}}$ fulfils the regularized sm-expansion with given numbers (D, l), then the coefficients $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}$ are uniquely determined.

(6′) $\mathrm{sd}(\mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}) \leq \Re(\kappa^{\boldsymbol{\zeta}}_{P+1,\mathbf{h}}) = D - 2(P+1) - 2\Re(\mathbf{h}\boldsymbol{\zeta})$.

Proof. *(1′)*, *(2′)* and *(6′)* are easy. (Note that *(2′)* and *(6′)* are simpler to prove than the corresponding statements in lemma 3.2, since $u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}$ and $\mathfrak{r}^{(m)\,\boldsymbol{\zeta}}_{P+1,\mathbf{c},\mathbf{h}}$ scale even *homogeneously.*)

(3′) can be verified in the same way as in lemma 3.2.

(4′) can be proved by proceeding analogously to the unregularized theory (see part *(4)* of lemma 3.2 and sect. 4.1) and by using that $\Delta^{+\,\zeta}_m$ is also of the form (5.1) (one only has to replace $(x^2 - i\varepsilon)$ by $(x^2 - ix^0\varepsilon)$).

To prove *(5′)* let $\boldsymbol{\zeta} \in \Lambda$ be such that

$$p - \Re(\mathbf{c}\boldsymbol{\zeta}) \neq p' - \Re(\mathbf{c}'\boldsymbol{\zeta}) \quad \forall (p,\mathbf{c}) \neq (p',\mathbf{c}') \quad \text{and} \quad \mathbf{h}\boldsymbol{\zeta} \neq \mathbf{h}'\boldsymbol{\zeta} \quad \forall \mathbf{h} \neq \mathbf{h}' \ . \tag{5.12}$$

This excludes only a set of measure zero – this is no harm, due to analyticity in $\boldsymbol{\zeta}$. The first condition implies that $f^{(m)\,\boldsymbol{\zeta}}$ is of the form

$$f^{(m)\,\boldsymbol{\zeta}} = \sum_{i=1}^{K} U_i\,(\tfrac{m}{M})^{z_i} + \mathfrak{r}^{(m)}_{K+1} \quad \text{with} \quad \Re(z_i) < \Re(z_{i+1}) \quad \forall i \tag{5.13}$$

and $\lim_{m\downarrow 0}(\frac{m}{M})^{-z_K}\,\mathfrak{r}^{(m)}_{K+1} = 0$, where $K \in \mathbb{N}$ is arbitrary. The coefficients U_i can be determined inductively:

$$U_n = \lim_{m\downarrow 0}\Big(f^{(m)\,\boldsymbol{\zeta}} - \sum_{i=1}^{n-1} U_i\,(\tfrac{m}{M})^{z_i}\Big)\,(\tfrac{m}{M})^{-z_n} \ . \tag{5.14}$$

Finally from $U_i = \sum_{\mathbf{h}} u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}}$, where $z_i = 2(p - \mathbf{c}\boldsymbol{\zeta})$ and the sum is restricted by $|\mathbf{h}| = l - |\mathbf{c}|$, a single summand is obtained by the projection

$$u^{\boldsymbol{\zeta}}_{p,\mathbf{c},\mathbf{h}_0} = \frac{\prod_{\mathbf{h}\neq\mathbf{h}_0}\Big(D - 2p - 2\mathbf{h}\boldsymbol{\zeta} + \sum_r x_r\partial_{x_r}\Big)}{\prod_{\mathbf{h}\neq\mathbf{h}_0} 2\,(\mathbf{h}_0 - \mathbf{h})\boldsymbol{\zeta}}\; U_i \ . \tag{5.15}$$

□

Notice that for $f^{(m)\,\boldsymbol{\zeta}} = t^{(m)\,\boldsymbol{\zeta}}(A_1, ..., A_n)$ (where $A_1, ..., A_n$ are arbitrary field monomials) the property *(2′)* is an equivalent formulation of the axiom 'Scaling' in [5].

The system of axioms for the regularized time-ordered product $T^{(m)\,\boldsymbol{\zeta}}$ given in [5] can now be modified as follows: similarly to the procedure in sect. 3, we replace the axioms 'Smoothness in m^2' and 'Scaling' by the sm-expansion axiom. Essentially by the same construction as in sect. 4, one obtains the general solution of the so modified system of axioms.

6. Applications of the scaling and mass expansion

The sm-expansion is very helpful for *practical computations*: choosing $L = L_0 = D - d(n-1)$ it reduces the main problem – the extension from $\mathcal{D}'(\mathbb{R}^{d(n-1)} \setminus \{0\})$ to $\mathcal{D}'(\mathbb{R}^{d(n-1)})$ – to a minimal set of almost homogeneous scaling distributions (namely $\{u^0_{l,p} \,|\, 0 \leq l \leq L_0,\, 0 \leq p \leq P_l\}$); the direct extension (A.3) of the remainder gives no computational work. We illustrate this by the following examples.

Example (setting sun diagram). We study again the setting sun diagram in $d = 4$ dimensions. We have to extend

$$t^0(\varphi^3, \varphi^3)(x) = (\Delta^F_m(x))^3 \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) \ , \qquad (6.1)$$

where Δ^F_m is the Feynman propagator. Due to (1.3) its sm-expansion can be written as

$$\Delta^F_m(x) = \frac{a_0}{X} + m^2 \Big(\big(a_1 \log(M^2 X) + A_1\big) + 2a_1 \log \tfrac{m}{M} \Big) + R^{(m)}_4(x) \ , \qquad (6.2)$$

where $X := -(x^2 - i0)$, with constants $a_0,\, a_1,\, A_1 \in \mathbb{C}$. Due to $D = 6$, $n = 2$, we have $L_0 = 2$. Using that, we insert (6.2) into (6.1) and obtain

$$t^0(\varphi^3, \varphi^3)(x) = u^0_0(x) + m^2 \big(u^0_{2,0}(x) + u^0_{2,1}(x) \log \tfrac{m}{M}\big) + \mathfrak{r}^{(m)\,0}_4(x) \ ,$$

where

$$u^0_0(x) = \tfrac{a_0^3}{X^3} \ , \quad u^0_{2,0}(x) = \tfrac{3\, a_0^2\, (a_1 \log(M^2 X) + A_1)}{X^2} \ , \quad u^0_{2,1}(x) = \tfrac{6\, a_0^2\, a_1}{X^2} \ ,$$
$$\mathfrak{r}^{(m)}_4(x) = 3\, R^{(m)}_4(x)\, (\Delta^F_m(x))^2 + 3\, m^4\, (a_1 \log(m^2 X) + A_1)^2\, \tfrac{a_0}{X}$$
$$+\, m^6\, (a_1 \log(m^2 X) + A_1)^3 \ .$$

Note that $u^0_{2l+1} = 0 \;\forall l \in \mathbb{N}$.

The non-direct, almost homogeneous extensions of $u^0_0(x)$, $u^0_{2,0}$ and $u^0_{2,1}$ can be computed by using differential renormalization (see e.g. [4, Appendix B] and references cited there) – we use $M_1 = M$ as renormalization mass scale:

$$u_0(x) = a_0^3\, \Box_x \Box_x \overline{\left(\tfrac{-\log(M^2 X)}{32\, X}\right)} + C\, \Box_x \delta(x),$$
$$u_{2,0}(x) = 3\, a_0^2 \left[a_1\, \Box_x \overline{\left(\tfrac{(\log(M^2 X))^2 + 2 \log(M^2 X)}{8\, X}\right)} + A_1\, \Box_x \overline{\left(\tfrac{\log(M^2 X)}{4\, X}\right)} \right] + C_0\, \delta(x) \ ,$$
$$u_{2,1}(x) = 6\, a_0^2\, a_1\, \Box_x \overline{\left(\tfrac{\log(M^2 X)}{4\, X}\right)} + C_1\, \delta(x) \ , \qquad (6.3)$$

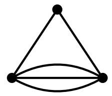
where $C,\, C_0,\, C_1 \in \mathbb{C}$ are arbitrary constants. These formulas have to be understood as follows: for $x \neq 0$ the derivatives can straightforwardly be computed and we obtain the corresponding $u^0_{\dots}$-distributions. However, the expressions in $\big(...\big)$-brackets have scaling degree $= 2$, hence, by the direct extension (A.3) (denoted by an over-line), they are uniquely defined as elements of $\mathcal{D}'(\mathbb{R}^4)$, and also their derivatives are in $\mathcal{D}'(\mathbb{R}^4)$. Therefore, the r.h. sides of (6.3) are indeed extensions of the corresponding $u^0_{...}$-distributions; and, obviously, they scale almost homogeneously.

We end up with

$$t(\varphi^3,\varphi^3)(x) = u_0(x)+m^2\big(u_{2,0}(x)+u_{2,1}(x)\,\log\tfrac{m}{M}\big)+\mathfrak{r}_4^{(m)}(x)\in\mathcal{D}'(\mathbb{R}^4)\ , \quad (6.4)$$

where $\mathfrak{r}_4^{(m)}$ is the direct extension of $\mathfrak{r}_4^{(m)\,0}$.

Example (setting sun with a hat). Again in $d = 4$ dimensions, we compute the "divergent" diagram



which contains the setting sun diagram as a "divergent" subdiagram.[12] That is we have to extend

$$t^0(x,y) = t(\varphi^3,\varphi^3)(x-y)\,\Delta^F_m(x)\,\Delta^F_m(y)\in\mathcal{D}'(\mathbb{R}^8\setminus\{0\})\ , \qquad (6.5)$$

to $\mathcal{D}'(\mathbb{R}^8)$, where $t(\varphi^3,\varphi^3)$ is given by (6.4). We have $D = 10$, $n = 3$ and, hence, $L_0 = 2$. The sm-expansion of $t^0(x,y)$ with $L = L_0 = 2$ is obtained by inserting (6.2) and (6.4) into (6.5):

$$t^0(x,y) = v^0_0(x,y) + m^2\big(v^0_{2,0}(x,y) + v^0_{2,1}(x,y)\,\log\tfrac{m}{M}\big) + \mathfrak{q}_4^{(m)\,0}(x,y)\ ,$$

where we use the letters $(v,\mathfrak{q})$ (instead of $(u,\mathfrak{r})$) to avoid confusion with the distributions appearing in the sm-expansion of the setting sun diagram. The $v^0_{...}$-distributions read:

$$v^0_0(x,y) = u_0(x-y)\,\tfrac{a_0^2}{XY}\ ,$$
$$v^0_{2,0}(x,y) = u_{2,0}(x-y)\,\tfrac{a_0^2}{XY} + u_0(x-y)\,a_0\Big(\tfrac{a_1\,\log(M^2Y)+A_1}{X} + \tfrac{a_1\,\log(M^2X)+A_1}{Y}\Big)\ ,$$
$$v^0_{2,1}(x,y) = u_{2,1}(x-y)\,\tfrac{a_0^2}{XY} + u_0(x-y)\,2a_0a_1\Big(\tfrac{1}{X}+\tfrac{1}{Y}\Big)\ ,$$

where Y is defined analogously to X (6.2).

Due to the choice $L = L_0 = 2$, the direct extension applies to the remainder $\mathfrak{q}_4^{(m)\,0}(x,y)$. The almost homogeneous extension of the $v^0_{...}$-distributions is more involved, we use an *analytic regularization* which respects the $(x\leftrightarrow y)$-symmetry, it is related to the methods in [9, 13, 14, 5] and [11, Sect.3.4]:

$$v^{\zeta\,0}(x,y) := v^0(x,y)\,(M^4XY)^\zeta\ , \qquad v = v_0,\, v_{2,0},\, v_{2,1}\ , \qquad (6.6)$$

where $\zeta\in\mathbb{C}\setminus\{0\}$, $|\zeta|$ sufficiently small. The factor $M^{4\zeta}$ is introduced for dimensional reasons.

For a general ζ, also $v^{\zeta\,0}$ cannot be renormalized by the direct extension. However, we gain by the regularization that $v^{\zeta\,0}$ scales almost homogeneously with a *non-integer degree* $D^\zeta = 8-4\zeta$ (for $v^{\zeta\,0}_{2,0}$, $v^{\zeta\,0}_{2,1}$) or $D^\zeta = 10-4\zeta$

[12] A diagram with n vertices is "divergent", iff its scaling degree (A.1) is greater or equal to $d(n-1)$, i.e. the direct extension (A.3) does not apply.

(for $v_0^{\zeta\,0}$). Due to that, the almost homogeneous extension $v^\zeta(x,y)$ is unique (proposition A.1) and can be computed by differential renormalization as follows:[13] writing $z := (x,y)$, $\partial_r z_r := \partial_{x^\mu} x^\mu + \partial_{y^\mu} y^\mu$ and $\eta := -4\zeta$, we obtain from

$$(\partial_r z_r + \eta)^2 v_{2,1}^{\zeta\,0}(z) = 0$$

the unique almost homogeneous extension

$$v_{2,1}^\zeta = \tfrac{-1}{\eta^2}\Big((2\eta-1)\,\partial_r \overline{(z_r\, v_{2,1}^{\zeta\,0})} + \partial_r\partial_s \overline{(z_r z_s\, v_{2,1}^{\zeta\,0})}\Big) \in \mathcal{D}'(\mathbb{R}^8)\ . \qquad (6.7)$$

Again, the over-line denotes the direct extension (A.3), which exists since $\mathrm{sd}(z_{r_1}...z_{r_l}\, v^{\zeta\,0}) = D^\zeta - l$. For $v_{2,0}^{\zeta\,0}$ the power of the almost homogeneous scaling is 2, hence we have

$$(\partial_r z_r + \eta)^3 v_{2,0}^{\zeta\,0}(z) = 0\ ,$$

which yields

$$\begin{aligned} v_{2,0}^\zeta = & \tfrac{-1}{\eta^3}\Big((3\eta^2-3\eta+1)\,\partial_r \overline{(z_r\, v_{2,0}^{\zeta\,0})} + (3\eta-3)\,\partial_r\partial_s \overline{(z_r z_s\, v_{2,0}^{\zeta\,0})} \\ & + \partial_p\partial_r\partial_s \overline{(z_p z_r z_s\, v_{2,0}^{\zeta\,0})}\Big)\ . \end{aligned} \qquad (6.8)$$

For $v_0^{\zeta\,0}$ we need at least $l=3$ factors z_{r_i} in order that the direct extension $\overline{z_{r_1}...z_{r_l}\, v_0^{\zeta\,0}}$ exists. Hence, we proceed as follows: from

$$(\partial_r z_r + 2 + \eta)^2 v_0^{\zeta\,0}(z) = 0$$

we obtain

$$v_0^{\zeta\,0} = \tfrac{-1}{(2+\eta)^2}\Big((3+2\eta)\,\partial_s(z_s\, v_0^{\zeta\,0}) + \partial_r\partial_s(z_r z_s\, v_0^{\zeta\,0})\Big)\ ,$$

analogously

$$(\partial_r z_r + 1 + \eta)^2 (z_s\, v_0^{\zeta\,0}(z)) = 0$$

gives

$$z_s\, v_0^{\zeta\,0} = \tfrac{-1}{(1+\eta)^2}\Big((1+2\eta)\,\partial_r(z_r z_s\, v_0^{\zeta\,0}) + \partial_p\partial_r(z_p z_r z_s\, v_0^{\zeta\,0})\Big)\ ,$$

and

$$(\partial_p z_p + \eta)^2 (z_r z_s\, v_0^{\zeta\,0}(z)) = 0$$

yields

$$z_r z_s\, v_0^{\zeta\,0} = \tfrac{-1}{\eta^2}\Big((2\eta-1)\,\partial_p(z_p z_r z_s\, v_0^{\zeta\,0}) + \partial_p\partial_q(z_p z_q z_r z_s\, v_0^{\zeta\,0})\Big)\ .$$

Inserting the lower equations into the upper ones and performing the direct extension we get

$$\begin{aligned} v_0^\zeta = & \tfrac{1}{\eta^2(1+\eta)^2(2+\eta)^2}\Big((2+2\eta-6\eta^2-4\eta^3)\,\partial_p\partial_r\partial_s \overline{(z_p z_r z_s\, v_0^{\zeta\,0})} \\ & - (2+6\eta+3\eta^2)\,\partial_p\partial_q\partial_r\partial_s \overline{(z_p z_q z_r z_s\, v_0^{\zeta\,0})}\Big)\ . \end{aligned} \qquad (6.9)$$

[13]For $v_{2,1}^\zeta$ and $v_{2,0}^\zeta$ we use the extension method given in [5, remark 4.9], for v_0^ζ we work with a further development of that method.

Obviously, the extensions v^ζ scale almost homogeneously with the same degree D^ζ and the same power as the initial $v^{\zeta\,0}$ (in agreement with proposition A.1); in addition, the maps $\zeta \mapsto \langle v^\zeta, f\rangle$ are meromorphic in ζ for all $f \in \mathcal{D}(\mathbb{R}^8)$, with a pole at $\zeta = 0$ of order 2 (for $v^\zeta_{2,1}$, v^ζ_0) or 3 (for $v^\zeta_{2,0}$). The latter shows explicitly that this extension method does not work for the unregularized theory (i.e. $\zeta = 0$).

According to definition 4.2 in [5], $v^\zeta \in \mathcal{D}'(\mathbb{R}^8)$ is a 'regularization' of $v^0 \in \mathcal{D}'(\mathbb{R}^8 \setminus \{0\})$ in the sense that

$$\lim_{\zeta\to 0}\langle v^\zeta, g\rangle = \langle v_\omega, g\rangle \qquad \forall g \in \mathcal{D}_\omega(\mathbb{R}^8)\ , \tag{6.10}$$

where v_ω is the unique extension of v^0 to $\mathcal{D}'_\omega(\mathbb{R}^8)$ (A.2) with $\mathrm{sd}(v_\omega) = \mathrm{sd}(v^0)$; and $\omega = 0$ (for $v^0_{2,0}$, $v^0_{2,1}$) or $\omega = 2$ (for v^0_0). Namely, using the functions χ_ρ (A.3) and that $\lim_{\zeta\to 0} v^{\zeta\,0} = v^0$ in $\mathcal{D}'(\mathbb{R}^8 \setminus \{0\})$, (6.10) can be verified as follows:

$$\begin{aligned}\langle v_\omega, g\rangle &= \lim_{\rho\to\infty}\langle v^0, \chi_\rho\, g\rangle = \lim_{\rho\to\infty}\ \lim_{\zeta\to 0}\langle v^{\zeta\,0}, \chi_\rho\, g\rangle \\ &= \lim_{\zeta\to 0}\ \lim_{\rho\to\infty}\langle v^{\zeta\,0}, \chi_\rho\, g\rangle = \lim_{\zeta\to 0}\langle v^\zeta, g\rangle\ .\end{aligned} \tag{6.11}$$

Turning to the limit $\zeta \to 0$, Corollary 4.4 in [5] states that the minimally subtracted distribution

$$v^{\mathrm{MS}} := \lim_{\zeta\to 0}\,(1 - \mathrm{pp})\, v^\zeta \tag{6.12}$$

(pp denotes the principle part) is an extension of v^0 with $\mathrm{sd}(v^{\mathrm{MS}}) = \mathrm{sd}(v^0)$.

Coming back to the explicit Laurent series $v^\zeta = \sum_{n=-L}^\infty \zeta^n\, v_{(n)}$ (where $L \in \mathbb{N}$) of our example, we have to compute the coefficients $v_{(0)} = v^{\mathrm{MS}}$. Expanding (in ζ) $(M^4XY)^\zeta$ and the rational functions of η appearing in (6.7), (6.8) and (6.9), we obtain the following results for the general, almost homogeneous and Lorentz invariant extensions $v = v^{\mathrm{MS}} + \sum_{|\beta|=\omega} C_\beta\, \partial^\beta\delta$, which are $(x \leftrightarrow y)$-invariant:

$$\begin{aligned}
v_{2,1} =&\partial_r\overline{\big(z_r\, v^0_{2,1}\,[\tfrac{1}{32}\,(\log(M^4XY))^2 + \tfrac{1}{2}\,\log(M^4XY)]\big)}\\
&- \partial_r\partial_s\overline{\big(z_r z_s\, v^0_{2,1}\,\tfrac{1}{32}\,(\log(M^4XY))^2\big)} + C_1\,\delta(x,y)\ ,\\
v_{2,0} =&\partial_r\overline{\big(z_r\, v^0_{2,0}\,[\tfrac{(\log(M^4XY))^3}{384} + \tfrac{3\,(\log(M^4XY))^2}{32} + \tfrac{3\,\log(M^4XY)}{4}]\big)}\\
&- \partial_r\partial_s\overline{\big(z_r z_s\, v^0_{2,0}\,[\tfrac{3\,(\log(M^4XY))^3}{384} + \tfrac{3\,(\log(M^4XY))^2}{32}]\big)}\\
&+ \partial_p\partial_r\partial_s\overline{\big(z_p z_r z_s\, v^0_{2,0}\,\tfrac{(\log(M^4XY))^3}{384}\big)} + C_0\,\delta(x,y)\ ,\\
v_0 =&\partial_p\partial_r\partial_s\overline{\big(z_p z_r z_s\, v^0_0\,[\tfrac{-1}{8} + \tfrac{1}{4}\,\log(M^4XY) + \tfrac{1}{64}\,(\log(M^4XY))^2]\big)}\\
&+ \partial_q\partial_p\partial_r\partial_s\overline{\big(z_q z_p z_r z_s\, v^0_0\,[\tfrac{7}{8} - \tfrac{1}{64}\,(\log(M^4XY))^2]\big)}\\
&+ C_2\,(\Box_x + \Box_y)\delta(x,y) + C_3\,\partial^x_\mu\partial^\mu_y\,\delta(x,y)\ .
\end{aligned} \tag{6.13}$$

We explicitly see that these extensions scale almost homogeneously with the same degree as the pertinent v^0-distributions. From proposition A.1 we know that the power of the log's may be increased at most by 1; therefore, terms of higher orders in $\log(M^2X)$, $\log(M^2Y)$ and $\log(M^2(X-Y))$ must cancel out in (6.13), by identities for the derivatives.

Remark 6.1 (treatment of subdivergences). There is an essential difference between the renormalization method used in this example and the one given in [5]: we insert for the divergent subdiagram (i.e. the setting sun) the *renormalized* expression and, hence, in the limit $\zeta \to 0$ we have to care only about the overall divergence located on the thin diagonal $x = 0 = y$. According to the method in [5], one inserts for the divergent subdiagram a *regularized* expression and, therefore, the limit which removes the regularization has to be done by means of the forest formula: one first subtracts the principle part of the divergent subdiagram (which is localized on the partial diagonal $x - y = 0$) and, after that, one subtracts the principle part of the overall diagram (which is localized on the thin diagonal).

7. Concluding remarks

In most papers dealing with causal perturbation theory (in particular in the original work [8]) the scaling degree axiom (shortly 'sd-axiom') is used, which restricts extensions $t \in \mathcal{D}'(\mathbb{R}^{d(n-1)})$ of $t^0 \in \mathcal{D}'(\mathbb{R}^{d(n-1)} \setminus \{0\})$ by the requirement $\mathrm{sd}(t) = \mathrm{sd}(t^0)$. In the system of axioms proposed by this paper (see sects. 3 and 4) one may replace the sm-expansion axiom by the weaker sd-axiom – this yields a reasonable system of axioms.

To illustrate that the sm-expansion axiom restricts the set of allowed time-ordered products truly stronger, we discuss the non-uniqueness of the inductive step $n = 2 \to n = 3$ for the example 'setting sun with a hat': taking also Lorentz invariance and $(x \leftrightarrow y)$-symmetry into account, the sd-axiom leaves the freedom to add a term of the form

$$\Big(f_2(\tfrac{m}{M})\,(\Box_x + \Box_y) + f_3(\tfrac{m}{M})\,\partial^x_\mu \partial^\mu_y + m^2\, f_1(\tfrac{m}{M})\Big)\,\delta(x,y)\ , \qquad (7.1)$$

where $M > 0$ is a fixed mass scale and f_1, f_2, f_3 are arbitrary functions $f_i : \mathbb{R} \to \mathbb{C}$ (the values are dimensionless). We have found that the sm-expansion axiom restricts these functions to

$$f_2(\tfrac{m}{M}) = C_2\ , \quad f_3(\tfrac{m}{M}) = C_3\ , \quad f_1(\tfrac{m}{M}) = C_0 + C_1\,\log(\tfrac{m}{M})\ , \qquad (7.2)$$

with arbitrary constants C_0, C_1, C_2, $C_3 \in \mathbb{C}$.

Such a reduction of the freedom of (re)normalization by a refinement of the sd-axiom is certainly desirable. As explained in (2.20), almost homogeneous scaling (axiom (g)) does not suffice, it needs to be supplemented, or replaced by a stronger condition. In [4] this problem is solved by quantizing with a Hadamard function and requiring as an additional axiom smoothness in $m \geq 0$. For time ordered products based on the Wightman two-point function, we have shown that the sm-expansion axiom is well suited for a stronger version of the sd-axiom.

As an outlook we mention that the sm-expansion axiom can be used to derive structural results about the renormalization group flow, see [7].

Appendix A. Extension of distributions from $\mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ to $\mathcal{D}'(\mathbb{R}^k)$

For the convenience of the reader we recall some main results about the extension of a given distribution $t^0\in\mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ to $t\in\mathcal{D}'(\mathbb{R}^k)$, proofs are given e.g. in [1, 4].

Steinmann's *scaling degree* [15] of a distribution $f\in\mathcal{D}'(\mathbb{R}^k)$ or $f\in\mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ is defined by

$$\mathrm{sd}(f):=\inf\{\,r\in\mathbb{R}:\lim_{\rho\downarrow 0}\rho^r\,f(\rho x)=0\,\}\ . \tag{A.1}$$

Let $\omega:=\mathrm{sd}(t^0)-k$ and introduce the subspace of test functions

$$\mathcal{D}_\omega(\mathbb{R}^k):=\{\,h\in\mathcal{D}(\mathbb{R}^k):\partial^\beta h(0)=0\ \text{for}\ |\beta|\leq\omega\,\}\ . \tag{A.2}$$

Then, t^0 has a *unique* extension t_ω to $\mathcal{D}'_\omega(\mathbb{R}^k)$ satisfying the condition $\mathrm{sd}(t_\omega)=\mathrm{sd}(t^0)$. t_ω is called the 'direct extension', it can be obtained by the limit

$$\langle t_\omega,h\rangle:=\lim_{\rho\to\infty}\langle t^0,\chi_\rho h\rangle\ ,\qquad h\in\mathcal{D}_\omega(\mathbb{R}^k)\ , \tag{A.3}$$

where $\chi_\rho(x):=\chi(\rho x)$ and $\chi\in C^\infty(\mathbb{R}^k)$ is such that $0\leq\chi(x)\leq 1$, $\chi(x)=0$ for $|x|\leq 1$ and $\chi(x)=1$ for $|x|\geq 2$.

In particular, for $\mathrm{sd}(t^0)<k$, the extension $t\in\mathcal{D}'(\mathbb{R}^k)$ is uniquely fixed by the requirement $\mathrm{sd}(t)=\mathrm{sd}(t^0)$ and it is given by the direct extension (A.3).

For $k\leq\mathrm{sd}(t^0)<\infty$, there are *several* extensions $t\in\mathcal{D}'(\mathbb{R}^k)$ fulfilling the condition $\mathrm{sd}(t)=\mathrm{sd}(t^0)$; the difference of two solutions is of the form $\sum_{|\beta|\leq\mathrm{sd}(t^0)-k}C_\beta\,\partial^\beta\delta(x)$ with $C_\beta\in\mathbb{C}$.

The main purpose of the sm-expansion is to reduce perturbative renormalization to the extension of almost homogeneously scaling distributions. The following proposition describes the possible homogeneities of the extensions [4, 10, 11, 12].

Proposition A.1. *Let $t^0\in\mathcal{D}'(\mathbb{R}^k\setminus\{0\})$ scale almost homogeneously with degree $D\in\mathbb{C}$ and power $N_0\in\mathbb{N}$ (see* (2.12) *with $m\equiv 0$, or* [4, definition 2.4]*). Then there exists an extension $t\in\mathcal{D}'(\mathbb{R}^k)$ which scales also almost homogeneously with degree D and power $N_1\geq N_0$:*

- (i) *if $D\notin\mathbb{N}_0+k$, then t is unique and $N_1=N_0$;*
- (ii) *if $D\in\mathbb{N}_0+k$, then t is non-unique and $N_1=N_0$ or $N_1=N_0+1$. In this case, two solutions differ by a term $\sum_{|\beta|=D-k}C_\beta\,\partial^\beta\delta(x)$ (where $C_\beta\in\mathbb{C}$ is arbitrary).*

In case (i) the unique t can be computed quite easily: if $\Re D<k$ it agrees with the direct extension of t^0 (A.3); otherwise it can be computed by differential renormalization, see [5, sect. 4.4] and sect. 6.

Acknowledgment

The author profited a lot from stimulating discussions with Philippe Blanchard, Klaus Fredenhagen, José M. Gracia-Bondía, Katarzyna Rejzner and Joseph C. Várilly. During working at this paper the author was mainly at the Max Planck Institute for Mathematics in the Sciences, Leipzig; he thanks Eberhard Zeidler for the invitations to Leipzig and for enlightening discussions.

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Michael Dütsch
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22
D-04103 Leipzig
Germany
e-mail: `michael.duetsch@theorie.physik.uni-goettingen.de`